Positive solutions for a class of Kirchhoff type problems in $\mathbb{R}^3$ involving critical Sobolev exponents

Gongbao Li
School of Mathematics and Statistics, CCNU

NUS
2014, 12.11

Based on a joint work with
Yi He

- **1. Introduction and main result**
- **2. The limiting problem**
- **3. Proof of the main result**

- **1. Introduction and main result**
- **2. The limiting problem**
- **3. Proof of the main result**

1. Introduction and main result

$$\begin{cases} -\left(\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla u|^2\right)\Delta u + V(x)u = \lambda |u|^{p-2}u + |u|^4 u \text{ in } \mathbb{R}^3, \\ u > 0, \ \ u \in H^1(\mathbb{R}^3), \end{cases} \tag{1}$$

where ε is a small positive parameter, $a, b > 0$, $\lambda > 0$, $2 < p \leq 4$.

The potential V satisfies:
(V_1) $V \in C(\mathbb{R}^3, \mathbb{R})$ **and** $\inf_{x \in \mathbb{R}^3} V(x) = \alpha > 0$**;**
(V_2) **There is a bounded domain Λ such that**

$$V_0 := \inf_{\Lambda} V < \min_{\partial\Lambda} V.$$

Set $\mathcal{M} := \{x \in \Lambda;\ V(x) = V_0\}$**. Without loss of generality, we may assume that** $0 \in \mathcal{M}$**.**

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2\right)\Delta u = f(x,u) \text{ in } \Omega, \\ u=0 \text{ on } \partial\Omega, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^3$ is a smooth domain. Such problems are often referred to be nonlocal because of the presence of the term $(\int_\Omega |\nabla u|^2)\Delta u$ which implies that the equation (2) is no longer a pointwise identity.

(2) is related to the stationary analogue of the equation

$$\begin{cases} u_{tt} - \left(a + b\int_\Omega |\nabla_x u|^2\right)\Delta_x u = f(x,u) \ (x \in \Omega, t > 0), \\ u(\cdot, t)\,|_{\partial\Omega} = 0 \ (t \geq 0), \end{cases} \tag{3}$$

(3) is proposed by Kirchhoff in [G. Kirchhoff, Mechanik, Teubner, Leipzig, 1883] as an existence of the classical D'Alembert's wave equations for free vibration of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. In (3), u **denotes the displacement,** $f(x,u)$ **the external force and** b **the initial tension while** a **is related to the intrinsic properties of the string (such as Young's modulus).**

Nonlocal problems also appear in other fields as biological systems, where u describes a process which depends on the average of itself (for example, population density). After the pioneer work of Lions [J. L. Lions, On some questions in boundary value problems of mathematical physics, 1977], where a functional analysis approach was proposed, the Kirchhoff type equations began to call attention of researchers.

★ **A. Arosio and S. Panizzi (Trans. Amer. Math. Soc. 1996, 305-330) considered the following equation**

$$\begin{cases} u_{tt} - m\left(\int_\Omega |\nabla_x u|^2 dx\right) \Delta_x u = f(x,t) \ \ (x \in \Omega, t > 0) \\ u(\cdot, t)\,|_{\partial\Omega} = 0 \ \ (t \geqslant 0), \end{cases}$$

where Ω is an open subset of $\mathbb{R}^n$ and m is a positive function of one real variable which is continuously differentiable. They proved the well-posedness in the Hadamard sense (existence, uniqueness and continuous dependence of the local solution upon the initial data) in Sobolev spaces of low order.

★ **K. Perera and Zhang Zhitao (J. Differential Equations 2006, 246-255) considered the equation**

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2\right)\Delta u = f(x,u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{4}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n = 1, 2, 3$, $a, b > 0$, f is a Caratheodory function on $\Omega \times \mathbb{R}$, s.t.

$$\lim_{t\to 0} \frac{f(x,t)}{at} = \lambda, \lim_{|t|\to\infty} \frac{f(x,t)}{bt^3} = \mu \text{ uniformly in } x.$$

Denote by $\lambda_1 < \lambda_2 \leqslant \lambda_3 \ldots$ **the eigenvalues of the operator** $-\Delta$ **and** $\mu_1 \leqslant \mu_2 \leqslant \mu_3 \ldots$ **the eigenvalues of the problem**

$$\begin{cases} -\|u\|^2_{H^1_0(\Omega)} \Delta u = \mu u^3 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

they proved that if $\lambda \in (\lambda_l, \lambda_{l+1})$ **and** $\mu \in (\mu_m, \mu_{m+1})$ **with** $l \neq m$, **then problem has a nontrivial solution by Yang index and critical group.**

★ **Chen C.Y., Kuo Y.C. and Wu T.F. (J. Differential Equations 2011, 1876-1908) studied**

$$\begin{cases} -\left(a\int_{\Omega}|\nabla u|^2dx+b\right)\Delta u=\lambda f(x)|u|^{q-2}u+g(x)|u|^{p-2}u \text{ in } \Omega \\ u=0 \text{ on } \partial\Omega \end{cases}$$

where $1<q<2<p<2^*$**,** $f,g\in C(\bar{\Omega})$**,** $f^+,g^+\neq 0$ **by using Nehari manifold and fibering map methods, and multiple positive solutions were obtained.**

Previous results

In recent years, the following Kirchhoff type equation

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\right)\Delta u+u=f(x,u) \text{ in } \mathbb{R}^3, \\ u\in H^1(\mathbb{R}^3) \end{cases} \tag{5}$$

has been studied extensively by many researchers where $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$, $a, b > 0$ are constants .

★ **X. He and W. Zou in (J. Differential Equations 2012, 1813-1834) studied (5) under the conditions that** $f(x,u) := f(u) \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ **satisfies the Ambrosetti-Rabinowitz condition ((AR) condition in short):**

$$\exists \mu > 4, 0 < \mu \int_0^u f(s)ds \leq f(u)u,$$

$\lim_{|u|\to 0} \frac{f(u)}{|u|^3} = 0$, $\lim_{|u|\to\infty} \frac{f(u)}{|u|^q} = 0$ **for some** $3 < q < 5$ **and** $\frac{f(u)}{u^3}$ **is strictly increasing for** $u > 0$**, i.e.** $f(u)$ **behaves like** $|u|^{p-2}u (4 < p < 6)$**. They showed that the Mountain Pass Theorem and the Nehari manifold can be used directly to obtain a positive ground state solution to (5).**

★ **Similarly, J. Wang, L. Tian, J. Xu and F. Zhang (J. Differential Equations 2012 2314-2351), Y. He, G. Li and S. Peng (Adv. Nonlinear Stud. 2014 441-468) and G. Li, H. Ye (Math. Meth. Appl. Sci. 2014 2570-2584) used the same arguments as X. He and W. Zou in (J. Differential Equations 2012, 1813-1834) to prove the existence of a positive ground state solution for (5) when** $f(x,u) := \lambda f(u) + |u|^4 u$**, which exhibits a critical growth, where** $\lim\limits_{|u|\to 0} \frac{f(u)}{|u|^3} = 0$, $f(u)u \geq 0$, $\frac{f(u)}{u^3}$ **is strictly increasing for** $u > 0$ **and** $|f(u)| \leq C(1 + |u|^q)$ **for some** $3 < q < 5$**, i.e.** $f(x,u) \sim \lambda |u|^{p-2}u + |u|^4 u (4 < p < 6)$**.**

★ **G. Li and H. Ye (J. Differential Equations 2014 566-600) studied (5) with $f(x,u) = |u|^{p-2}u (3 < p \leq 4)$. The corresponding energy functional of (5) is**

$$I(u) = \frac{1}{2}a\int_{\mathbb{R}^3}|\nabla u|^2 + \frac{b}{4}\Big(\int_{\mathbb{R}^3}|\nabla u|^2\Big)^2 + \frac{1}{2}\int_{\mathbb{R}^3}u^2 - \frac{1}{p}\int_{\mathbb{R}^3}|u|^p, \ \ u \in H^1(\mathbb{R}^3).$$

Inspired by Ruiz (J. Functional Analysis 2006 655-674), they observed that

$$\gamma(t) := I(tu(t^{-1}x)) = \frac{1}{2}at^3\int_{\mathbb{R}^3}|\nabla u|^2 + \frac{b}{4}t^6\Big(\int_{\mathbb{R}^3}|\nabla u|^2\Big)^2 + \frac{1}{2}t^5\int_{\mathbb{R}^3}u^2 - \frac{1}{p}t^{p+3}\int_{\mathbb{R}^3}|u|^p$$

has a unique critical point $t_0 > 0$ corresponding to its maximum.

Previous results

If u is a solution of (5), then $\gamma'(1) = 0$, i.e.

$$G(u) := \frac{3}{2}a\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{3}{2}b\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 + \frac{5}{2}\int_{\mathbb{R}^3} u^2 - \frac{p+3}{p}\int_{\mathbb{R}^3} |u|^p = 0,$$

where $G(u) = \langle I'(u), u\rangle + P(u)$ and $P(u) = 0$ is the corresponding Pohozaev's identity of (5). They used the constrained minimization on a new manifold M to get a positive ground state solution to (5) where

$$M := \{u \in H^1(\mathbb{R}^3)\backslash\{0\} | G(u) = 0\}.$$

Previous results

As far as we know, there is no result on the existence of positive ground state solutions for (5) under the condition $f(x,u) = \lambda|u|^{p-2}u + |u|^4u (2 < p \leq 4)$**. In this paper, we will fill this gap.**

Our motivation to study the concentration of solutions of (1) mainly comes from the results of perturbed Schrödinger equations, i.e.

$$-\varepsilon^2 \Delta u + V(x)u = |u|^{q-2}u, \;\; x \in \mathbb{R}^N, \tag{6}$$

where $2 < q < 2^*$**,** $N \geq 1$**.**

Many mathematicians proved the existence, concentration and multiplicity of solutions for (6).

★ **A. Floer and A. Weinstein (J. Functional Analysis 1986 397-408) studied (6) in the case where $N = 1$, $q = 4$, $V \in L^\infty$ with $\inf V > 0$. They construct a single peak solution which concentrates around any given non-degenerate critical point of the potential V.**

★ **Y. G. Oh (Commun. Partial Differential Equations 1988 1499-1519) extended the result of A. Floer and A. Weinstein (J. Functional Analysis 1986 397-408) in higher dimensions when** $2 < q < 2N/(N-2)$ **and the potential** V **belongs to a Kato class which means that** V **satisfies the following condition:**

$(V)_a : V \equiv a$ **or** $V > a$ **and** $(V-a)^{-\frac{1}{2}} \in \mathbf{Lip}(\mathbb{R}^N)$ **for some** $a \in \mathbb{R}$.

★ **Y. G. Oh (Commun. Math. Phys. 1990 223-253) proved the existence of multi-peak solutions to (6) which concentrate around any finite subsets of the non-degenerate critical points of V.**

★ **P. Rabinowitz (Z. Angew. Math. Phys. 1992 270-291) studied (6) under the conditions:**

(V_3) $V_\infty = \liminf_{|x|\to\infty} V(x) > V_0 = \inf_{x\in\mathbb{R}^N} V(x) > 0$.

Rabinowitz proved that (6) possesses a positive ground state solution for $\varepsilon > 0$ small by using the Mountain Pass Theorem.

★ **X. Wang (Commun. Math. Phys. 1993 229-244) proved that the positive ground state solutions of (6) must concentrate at global minima of V as $\varepsilon \to 0$.**

Under the same condition (V_3) **on** $V(x)$**, S. Cingolani and N. Lazzo (Topol. Methods Nonlinear Anal. 1997 1-13) proved the multiplicity of positive ground state solutions for (6) by using Ljusternik-Schnirelmann theory.**

★ **M. del Pino and P. L. Felmer (Calc. Var. Partial Differential Equations 1996 121-137) studied (6) with the conditions on** V**:**

(V_1) $V \in C(\mathbb{R}^3, \mathbb{R})$ **and** $\inf_{x \in \mathbb{R}^3} V(x) = \alpha > 0$**;**

(V_2) **There is a bounded domain** Λ **such that**

$$V_0 := \inf_{\Lambda} V < \min_{\partial\Lambda} V.$$

They proved that (6) possesses a positive bound state solution for $\varepsilon > 0$ **small which concentrates around the local minima of** V **in** Λ **as** $\varepsilon \to 0$ **via the penalization method.**

★ **Gui. C. (Commun. Partial Differential Equations 1996 787-820) studied (6) with the conditions on** V**:**

(V_1) $V \in C(\mathbb{R}^3, \mathbb{R})$ **and** $\inf_{x\in\mathbb{R}^3} V(x) = \alpha > 0$**;**

(V_2) **There exist** k **disjoint bounded regions** $\Omega_1, ..., \Omega_k$ **such that**

$$V_0 := \inf_{\Omega_i} V < \min_{\partial\Omega_i} V, \ \ i = 1, ...k.$$

Gui showed that (6) possesses a positive bound state solution u_ε for $\varepsilon > 0$ small with the following properties:
(1) u_ε has exactly k local maximum points $P_{\varepsilon,1}, \ldots, P_{\varepsilon,k}$ satisfying $P_{\varepsilon,i} \in \Omega_i$ and

$$\lim_{\varepsilon \to 0} V(P_{\varepsilon,i}) = \inf_{\Omega_i} V.$$

(2) There exist positive constants C, σ, (independent of x, ε), such that

$$|u_\varepsilon| \leq C \exp\left(-\frac{\sigma}{\varepsilon} \min |x - P_{\varepsilon,i}|\right).$$

Previous results

We note that Gui get the concentration result mainly by using a version of global compactness method which is different from del Pino and Felmer.

★ **X. He and W. Zou (J. Differential Equations 2012, 1813-1834) studied**

$$\begin{cases} -\left(\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla u|^2\right)\Delta u + V(x)u = f(u) \text{ in } \mathbb{R}^3, \\ u > 0, \ \ u \in H^1(\mathbb{R}^3), \end{cases} \tag{7}$$

with V **satisfies**

(V_3) $V_\infty = \liminf_{|x|\to\infty} V(x) > V_0 = \inf_{x\in\mathbb{R}^N} V(x) > 0$,

$f \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ **and satisfies the (AR) condition,** $\lim_{s\to 0} \frac{f(s)}{s^3} = 0$, $\lim_{|s|\to\infty} \frac{f(s)}{|s|^q} = 0$ **for some** $3 < q < 5$ **and** $\frac{f(s)}{s^3}$ **is strictly increasing for** $s > 0$.

Previous results

They obtained the existence, concentration and multiplicity of solutions for (7) by the same arguments as P. Rabinowitz (Z. Angew. Math. Phys. 1992 270-291), X. Wang (Commun. Math. Phys. 1993 229-244), S. Cingolani and N. Lazzo (Topol. Methods Nonlinear Anal. 1997 1-13).

★ **J. Wang, L. Tian, J. Xu and F. Zhang (J. Differential Equations 2012, 2314-2351) extended the result of X. He and W. Zou (J. Differential Equations 2012, 1813-1834) with the nonlinearity is of critical growth.**

★ **G. M. Figueiredo, N. Ikoma and J. R. Santos Junior (Arch. Rational Mech. Anal. 2014 931-979) obtained the existence of positive solutions of the following equation**

$$\begin{cases} \varepsilon^2 M\Big(\varepsilon^{2-N}\displaystyle\int_{\mathbb{R}^N}|\nabla u|^2\Big)\Delta u + V(x)u = f(u) \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N),\ \ u > 0 \text{ in } \mathbb{R}^N,\ \ N \geq 1 \end{cases} \tag{8}$$

concentrating around a local minima of V under the conditions that V satisfies (V_1) and (V_2),

(V_1) $V \in C(\mathbb{R}^3, \mathbb{R})$ **and** $\inf\limits_{x\in\mathbb{R}^3} V(x) = \alpha > 0$**;**

(V_2) **There is a bounded domain Λ such that**

$$V_0 := \inf_{\Lambda} V < \min_{\partial\Lambda} V.$$

f **satisfies**

(F_1) $f \in C(\mathbb{R}, \mathbb{R})$**,** $f(s) = 0$ **if** $s \leq 0$**;**

(F_2) $-\infty < \varliminf_{s \to 0^+} f(s)/s \leq \varlimsup_{s \to 0^+} f(s)/s < \hat{V}$**;**

(F_3) **When** $N \geq 3$**,** $f(s)/s^{(N+2)/(N-2)} \to 0$ **as** $s \to \infty$ **and when** $N = 2$**,** $f(s)/e^{\alpha s^2} \to 0$ **as** $s \to \infty$ **for any** $\alpha > 0$**;**

(F_4) **There exists an** $s_0 > 0$ **such that** $-V_0 s_0^2/2 + F(s_0) > 0$ **where** $F(s) := \int_0^s f(t)dt$ **when** $N \geq 2$**, and when** $N = 1$, $-V_0 s_0^2/2 + F(s_0) = 0, \ -V_0 s^2/2 + F(s) < 0$ **in** $(0, s_0), \ -V_0 s_0 + f(s_0) > 0.$

Generally, f **is of subcritical.**

$M \in C([0,\infty),\mathbb{R})$ **satisfies**

(M_1) **There exists** $m_0 > 0$ **such that** $M(t) \geq m_0 > 0$ **for any** $t \geq 2$**;**

(M_2) **Set** $\hat{M}(t) := \int_0^t M(s)ds$**, then there holds**

$$\varliminf_{t\to\infty} \{\hat{M}(t) - (1-2/N)M(t)t\} = \infty;$$

(M_3) $M(t)/t^{2/(N-2)} \to 0$ **as** $t \to \infty$**;**

(M_4) **The function** $M(t)$ **is nondecreasing in** $[0,\infty)$**;**

(M_5) **The function** $M(t)/t^{2/(N-2)}$ **is nonincreasing in** $[0,\infty)$**.**

★ **Theorem 1.1(Y.He-G.Li 2014) Let (V_1), (V_2) hold. There exist $\lambda^* > 0$ and $\varepsilon^* > 0$ such that for each $\lambda \in [\lambda^*, \infty)$ and $\varepsilon \in (0, \varepsilon^*)$, (1) possesses a positive solution $u_\varepsilon \in H^1(\mathbb{R}^3)$ such that**

(i) there exists a maximum point x_ε of u_ε such that

$$\lim_{\varepsilon \to 0} \textbf{dist}(x_\varepsilon, \mathcal{M}) = 0;$$

(ii) $\exists C_1, C_2 > 0$, such that

$$u_\varepsilon(x) \le C_1 \exp\Big(-\frac{C_2}{\varepsilon}|x - x_\varepsilon|\Big),$$

where C_1, C_2 are independent of ε.

The main difficulties:

★ **(i) The fact that the nonlinearity** $\lambda|u|^{p-2}u + |u|^4 u$ **with** $p \in (2, 4]$ **does not satisfy** $(\mathbf{AR})$ **condition and the fact that the function** $\frac{\lambda u^{p-1}+u^5}{u^3}$ **is not increasing for** $(u > 0)$ **prevent us from obtaining a bounded Palais-Smale sequence and using the Nehari manifold, respectively.**

★ **(ii) The unboundedness of the domain $\mathbb{R}^3$ and the nonlinearity $\lambda|u|^{p-2}u + |u|^4u(2 < p \leq 4)$ with the critical Sobolev growth lead to the lack of compactness.**

Firstly, we need to prove the existence of ground state solution to the following limiting equation of (1)

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\right)\Delta u + mu = \lambda|u|^{p-2}u + |u|^4 u \ \textbf{in} \ \mathbb{R}^3, \\ u>0, \ \ u\in H^1(\mathbb{R}^3) \end{cases} \tag{9}$$

for $m>0$**.**

The corresponding energy functional is

$$I_m(u) = \frac{a}{2}\int_{\mathbb{R}^3}|\nabla u|^2 + \frac{b}{4}\Big(\int_{\mathbb{R}^3}|\nabla u|^2\Big)^2 + \frac{m}{2}\int_{\mathbb{R}^3}u^2$$
$$-\frac{\lambda}{p}\int_{\mathbb{R}^3}(u^+)^p - \frac{1}{6}\int_{\mathbb{R}^3}(u^+)^6,\ \ u \in H^1(\mathbb{R}^3).$$

★ **J. Hirata, N. Ikoma and K. Tanaka (Topol. Methods Nonlinear Anal. 2010 253-276)**

$$-\Delta u = g(u), \ \ u \in H^1(\mathbb{R}^N)$$

with the corresponding energy functional

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - \int_{\mathbb{R}^N} G(u), \ \ u \in H_r^1(\mathbb{R}^N),$$

where $G(u) = \int_0^u g(s)ds$ **and** $g \in C(\mathbb{R}, \mathbb{R})$ **ia an odd function satisfying**

(g1) $-\infty < \varliminf_{s\to 0} \frac{g(s)}{s} \leqslant \varlimsup_{s\to 0} \frac{g(s)}{s} < 0;$

(g2) $\lim_{s\to\infty} \frac{g(s)}{s^{(N+2)/(N-2)}} = 0;$

(g3)$\exists \xi_0 > 0$ **s.t.** $G(\xi_0) = \int_0^{\xi_0} g(s)ds > 0.$

They applied the Mountain Pass theorem on the augmented functional $\tilde{I}(\theta, u) = I(u(e^{-\theta}x))$ to get a bounded (PS) sequence $\{u_n\}_{n=1}^{\infty}$ with an extra property $P(u_n) \to 0$ as $n \to \infty$ where $P(u) = 0$ is the corresponding Pohozaev's identity. Note that they have used the information of Pohozaev's identity.

★ **Ruiz (J. Functional Analysis 2006 655-674)**

$$\begin{cases} -\Delta u + u + \lambda \phi u = u^{p-1} \ \textbf{in}\ \mathbb{R}^3, \\ -\Delta \phi = u^2 \ \textbf{in}\ \mathbb{R}^3, 3 < p \leq 4 \end{cases}$$

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^3} u^2 + \frac{1}{16\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} dxdy$$
$$-\frac{\lambda}{p}\int_{\mathbb{R}^3} |u|^p, \ \ u \in H_r^1(\mathbb{R}^3).$$

★ **Ruiz obtained a positive radial nontrivial solution by using the constrained minimization method on a new manifold which is obtained by combining the usual Nehari manifold and the Pohozaev's identity** M**, where**

$$M = \{u \in H^1(\mathbb{R}^3)\backslash\{0\} | 2\left\langle I'(u), u\right\rangle - P(u) = 0\}$$

and $P(u) = 0$ **is the corresponding Pohozaev's identity.**

2. The limiting problem

$$I(t^2u(tx)) = \frac{1}{2}t^3\int_{\mathbb{R}^3}|\nabla u|^2 + \frac{1}{2}t\int_{\mathbb{R}^3}u^2$$
$$+\frac{1}{16\pi}t^3\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u^2(x)u^2(y)}{|x-y|}dxdy - \frac{\lambda}{p}t^{2p-3}\int_{\mathbb{R}^3}|u|^p$$

$$2p-3>3 \Rightarrow p>3$$

$\gamma(t) := I(t^2u(tx))$ **has a unique critical point** $t_0 > 0$ **corresponding to its maximum.**

★ **G. Li and H. Ye (J.Differential Equations 2014 566-600)**

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\right)\Delta u + u = |u|^{p-2}u \textbf{ in } \mathbb{R}^3, \\ u>0,\ u\in H^1(\mathbb{R}^3),\ 3<p\leqslant 4, \end{cases}$$

$$I(u)=\frac{a}{2}\int_{\mathbb{R}^3}|\nabla u|^2+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2\right)^2+\frac{1}{2}\int_{\mathbb{R}^3}u^2-\frac{1}{p}\int_{\mathbb{R}^3}|u|^p,\ u\in H^1(\mathbb{R}^3).$$

They obtained a positive ground state solution by using the constrained minimization method on a new manifold which is obtained by combining the usual Nehari manifold and the Pohozaev's identity M**, where**

$$M = \{u \in H^1(\mathbb{R}^3)\backslash\{0\} | \left\langle I'(u), u\right\rangle + P(u) = 0\}$$

and $P(u) = 0$ **is the corresponding Pohozaev's identity.**

$$I(tu(t^{-1}x)) = \frac{a}{2}t^3 \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{b}{4}t^6 \left(\int_{\mathbb{R}^3} |\nabla u|^2 \right)^2 + \frac{1}{2}t^5 \int_{\mathbb{R}^3} u^2 - \frac{1}{p}t^{p+3} \int_{\mathbb{R}^3} |u|^p$$

$$p + 3 > 6 \Rightarrow p > 3$$

$\gamma(t) := I(tu(t^{-1}x))$ **has a unique critical point** $t_0 > 0$ **corresponding to its maximum.**

2. The limiting problem

In view of P. Pucci and J. Serrin (Indiana Univ. Math. J. 1986, 681-703), if $u \in H^1(\mathbb{R}^3)$ is a weak solution to problem (9), then we have the following Pohozaev's identity:

$$\frac{d}{dt} I_m(u(\frac{x}{t}) \mid_{t=1} \equiv P_m(u) = \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{b}{2}\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 + \frac{3}{2} m \int_{\mathbb{R}^3} u^2 - \frac{3}{p}\lambda \int_{\mathbb{R}^3} (u^+)^p - \frac{1}{2}\int_{\mathbb{R}^3} (u^+)^6 = 0. \tag{10}$$

2. The limiting problem

$$I_m(tu(t^{-2}x)) = \frac{a}{2}t^4\int_{\mathbb{R}^3}|\nabla u|^2 + \frac{b}{4}t^8\left(\int_{\mathbb{R}^3}|\nabla u|^2\right)^2 + \frac{m}{2}t^8\int_{\mathbb{R}^3}u^2$$
$$-\frac{\lambda}{p}t^{p+6}\int_{\mathbb{R}^3}(u^+)^p - \frac{1}{6}t^{12}\int_{\mathbb{R}^3}(u^+)^6$$

$$p+6>8 \Rightarrow p>2$$

$\gamma(t) := I_m(tu(t^{-2}x))$ **has a unique critical point** $t_0 > 0$ **corresponding to its maximum.**

We introduce the following manifold:

$$M_m := \left\{u \in H^1(\mathbb{R}^3)\backslash\{0\} : G_m(u) = 0\right\},$$

where

$$G_m(u) = 2a\int_{\mathbb{R}^3}|\nabla u|^2 + 2b\left(\int_{\mathbb{R}^3}|\nabla u|^2\right)^2 + 4m\int_{\mathbb{R}^3}u^2$$
$$-\frac{p+6}{p}\lambda\int_{\mathbb{R}^3}(u^+)^p - 2\int_{\mathbb{R}^3}(u^+)^6.$$

It is clear that

$$G_m(u) = \langle I'{}_m(u), u\rangle + 2P_m(u).$$

★ **Lemma 3.2 For any** $u \in H^1(\mathbb{R}^3)\backslash\{0\}$**, there is a unique** $\tilde{t} > 0$ **such that** $u_{\tilde{t}} \in M_m$**, where** $u_{\tilde{t}}(x) := \tilde{t}u(\tilde{t}^{-2}x)$**. Moreover,** $I_m(u_{\tilde{t}}) = \max\limits_{t>0} I_m(u_t)$**.**

★ **Lemma 3.3** I_m **possesses the Mountain-Pass geometry.**
Fix $u \in H^1(\mathbb{R}^3)\setminus\{0\}$, **set** $u_t(x) := tu(t^{-2}x)$,

$$I_m(u_t) = \frac{a}{2}t^4 \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{b}{4}t^8\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 + \frac{1}{2}mt^8 \int_{\mathbb{R}^3} u^2 - \frac{\lambda}{p}t^{p+6} \int_{\mathbb{R}^3} (u^+)^p - \frac{1}{6}t^{12} \int_{\mathbb{R}^3} (u^+)^6 < 0$$

for $t > 0$ **large, then** $\exists t_0 > 0$, **set** $u_0 := u_{t_0}$, $I(u_0) < 0$.

The Mountain-Pass level of I_m:

$$c_m := \inf_{\gamma \in \Gamma_m} \sup_{t \in [0,1]} I_m(\gamma(t)),$$

where

$$\Gamma_m := \left\{ \gamma \in C([0,1], H^1(\mathbb{R}^3)) : \gamma(0) = 0 \textbf{ and } I_m(\gamma(1)) < 0 \right\}.$$

Although I_m **satisfies the Mountain-Pass geometry, but the nonlinearity** $g(u) := \lambda|u|^{p-2}u + |u|^4 u$ **with** $2 < p \leq 4$ **does not satisfy the Ambrosetti-Rabinowitz condition** **(**$\exists\mu > 4, 0 < \mu \int_0^u g(s)ds \leq g(u)u$**), the boundedness of (PS) sequence seems to be difficult to be proved. We need some more information for the (PS) sequence.**

★ **(General Minimax Principle) Let X be a Banach space. Let M_0 be a closed subspace of the metric space M and $\Gamma_0 \subset C(M_0, X)$. Define**

$$\Gamma := \{\gamma \in C(M, X) : \gamma|_{M_0} \in \Gamma_0\}.$$

If $\varphi \in C^1(X, \mathbb{R})$ satisfies

$$\infty > c := \inf_{\gamma \in \Gamma} \sup_{u \in M} \varphi(\gamma(u)) > a := \sup_{\gamma_0 \in \Gamma_0} \sup_{u \in M_0} \varphi(\gamma_0(u)),$$

then, for every $\varepsilon \in (0, (c-a)/2)$, $\delta > 0$ and $\gamma \in \Gamma$ such that $\sup_M \varphi \circ \gamma \leq c + \varepsilon$, there exists $u \in X$ such that

(a) $c - 2\varepsilon \leq \varphi(u) \leq c + 2\varepsilon$,

(b) **dist**$(u, \gamma(M)) \leq 2\delta$,

(c) $\|\varphi'(u)\| \leq 8\varepsilon/\delta$.

★ **Proposition 3.4 There exists a sequence** $\{u_n\}_{n=1}^{\infty}$ **in** $H^1(\mathbb{R}^3)$ **such that, as** $n \to \infty$**,**

$$I_m(u_n) \to c_m, \ \ {I'}_m(u_n) \to 0, \ \ G_m(u_n) \to 0.$$

Define the map

$$\Phi(\theta, v) := e^{\theta} v(e^{-2\theta} x),\ \mathbb{R} \times H^1(\mathbb{R}^3) \to H^1(\mathbb{R}^3).$$

$$I_m \circ \Phi(\theta, v) = \frac{a}{2} e^{4\theta} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{b}{4} e^{8\theta} \left(\int_{\mathbb{R}^3} |\nabla u|^2 \right)^2 + \frac{1}{2} m e^{8\theta} \int_{\mathbb{R}^3} u^2 - \frac{\lambda}{p} e^{(p+6)\theta} \int_{\mathbb{R}^3} (u^+)^p - \frac{1}{6} e^{12\theta} \int_{\mathbb{R}^3} (u^+)^6.$$

$I_m \circ \Phi(\theta, v) > 0$ **for all** (θ, v) **with** $|\theta|$**,** $\|v\|_{H^1(\mathbb{R}^3)}$ **small and** $(I_m \circ \Phi)(0, u_0) < 0$**,**
i.e. $I_m \circ \Phi$ **possesses the Mountain-Pass geometry in** $\mathbb{R} \times H^1(\mathbb{R}^3)$**.**

The Mountain-Pass level of $I_m \circ \Phi$:

$$\tilde{c}_m := \inf_{\tilde{\gamma} \in \tilde{\Gamma}_m} \sup_{t \in [0,1]} (I_m \circ \Phi)(\tilde{\gamma}(t)),$$

where

$$\tilde{\Gamma}_m := \left\{ \tilde{\gamma} \in C([0,1], \mathbb{R} \times H^1(\mathbb{R}^3)) : \tilde{\gamma}(0) = (0,0) \text{ and } (I_m \circ \Phi)(\tilde{\gamma}(1)) < 0 \right\}.$$

As $\Gamma_m = \{\Phi \circ \tilde{\gamma} : \tilde{\gamma} \in \tilde{\Gamma}_m\}$**, the Mountain-Pass levels of** I_m **and** $I_m \circ \Phi$ **coincide, i.e.** $c_m = \tilde{c}_m$**.**

Using the General Minimax Principle, $\exists\{(\theta_n, v_n)\}_{n\in\mathbb{N}} \subset \mathbb{R} \times H^1(\mathbb{R}^3)$, **such that**

$$(I_m \circ \Phi)(\theta_n, v_n) \to c_m, \tag{11}$$

$$(I_m \circ \Phi)'(\theta_n, v_n) \to 0 \text{ in } (\mathbb{R} \times H^1(\mathbb{R}^3))^{-1}, \tag{12}$$

$$\theta_n \to 0. \tag{13}$$

2. The limiting problem

Set $\varepsilon = \varepsilon_n := \frac{1}{n^2}$**,** $\delta = \delta_n := \frac{1}{n}$ **in the General Minimax Principle,**

$$(a),(c) \Rightarrow (11),(12)$$

.

For (13), by the definition of c_m**, for** $\varepsilon = \varepsilon_n := \frac{1}{n^2}$**,** $\exists \gamma_n \in \Gamma_m$**, such that**

$$\sup_{t\in[0,1]} I_m(\gamma_n(t)) \le c_m + \frac{1}{n^2}.$$

Set $\tilde{\gamma}_n(t) = (0, \gamma_n(t))$**, then**

$$\sup_{t\in[0,1]} I_m \circ \Phi(\tilde{\gamma}_n(t)) = \sup_{t\in[0,1]} I_m(\gamma_n(t)) \le c_m + \frac{1}{n^2}.$$

By (b) **of the General Minimax Principle, there exists** $(\theta_n, v_n) \in \mathbb{R} \times H^1(\mathbb{R}^3)$ **such that dist**$((\theta_n, v_n), (0, \gamma_n(t))) \le \frac{2}{n}$**, (13) holds.**

$\forall (h,w) \in \mathbb{R} \times H^1(\mathbb{R}^3)$,

$$\left\langle (I_m \circ \Phi)'(\theta_n, v_n), (h,w) \right\rangle = \left\langle {I'}_m(\Phi(\theta_n, v_n)), \Phi(\theta_n, w) \right\rangle + G_m(\Phi(\theta_n, v_n))h. \tag{14}$$

Taking $h = 1$, $w = 0$ **in (14),**

$$G_m(\Phi(\theta_n, v_n)) \to 0 \text{ as } n \to \infty.$$

Set $u_n := \Phi(\theta_n, v_n)$,

$$G_m(u_n) \to 0 \text{ as } n \to \infty.$$

$\forall v \in H^1(\mathbb{R}^3)$, **set** $w(x) = e^{-\theta_n} v(e^{2\theta_n} x)$, $h = 0$ **in (14),**

$$\left\langle I'_m(u_n), v \right\rangle = o(1) \left\| e^{-\theta_n} v(e^{2\theta_n} x) \right\|_{H^1(\mathbb{R}^3)} = o(1) \|v\|_{H^1(\mathbb{R}^3)},$$

i.e. $I'_m(u_n) \to 0$ **in** $(H^1(\mathbb{R}^3))^{-1}$ **as** $n \to \infty$**.**

Similar to P. Rabinowitz (Z. Angew. Math. Phys. 1992 270-291), we can prove

$$c_m = \inf_{u \in H^1(\mathbb{R}^3)\setminus\{0\}} \max_{t>0} I_m(tu(t^{-2}x)) = \inf_{u \in M_m} I_m(u) > 0. \qquad (15)$$

i.e. the Mountain Pass level equals to the ground state level for I_m. Note that (15) holds only if the potential equals to positive constant.

Denote

$$v_\delta := \psi_\delta / \Big(\int_{B_2(0)} |\psi_\delta|^6 \Big)^{1/6},$$

where $\psi_\delta(x) := \varphi(x) w_\delta(x)$, $\varphi \in C_c^\infty(B_2(0))$ **satisfying** $\varphi \equiv 1$ **on** $B_1(0)$, $0 \le \varphi \le 1$ **on** $B_2(0)$ **and**

$$w_\delta(x) = (3\delta)^{1/4} \frac{1}{(\delta + |x|^2)^{1/2}}$$

is a minimizer for $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$. **We construct a special path** $t v_\delta(t^{-2}x)(t \ge 0)$ **to show that**

$$\max_{t>0} I_m(t v_\delta(t^{-2}x)) < \frac{1}{4} abS^3 + \frac{1}{24} b^3 S^6 + \frac{1}{24} (b^2 S^4 + 4aS)^{\frac{3}{2}}$$

for $\lambda > 0$ **large. Then we get the following Lemma:**

★ **Lemma 3.5**

$$c_m < \frac{1}{4}abS^3 + \frac{1}{24}b^3S^6 + \frac{1}{24}(b^2S^4 + 4aS)^{\frac{3}{2}}$$

for $\lambda > 0$ large, where S is the best Sobolev constant for the embedding $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$.

★ **Lemma 3.6 The sequence got in Proposition 3.4 is bounded in $H^1(\mathbb{R}^3)$.**

$$
\begin{aligned}
& c_m + o(1) \\
&= I_m(u_n) - \frac{1}{p+6} G_m(u_n) \\
&= \frac{p+2}{2(p+6)} a \int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{p-2}{4(p+6)} b \Big(\int_{\mathbb{R}^3} |\nabla u_n|^2 \Big)^2 \\
&\quad + \frac{p-2}{2(p+6)} m \int_{\mathbb{R}^3} |u_n|^2 + \frac{6-p}{6(p+6)} \int_{\mathbb{R}^3} (u_n^+)^6 .
\end{aligned}
$$

By the Vanishing Theorem and Lemma 3.5, we have
Lemma 3.7: There is a sequence $\{x_n\} \subset \mathbb{R}^3$ **and** $R > 0$, $\beta > 0$ **such that**

$$\int_{B_R(x_n)} u_n^2 \geq \beta.$$

Denote $\tilde{u}_n(x) = u_n(x + x_n)$**, by Lemma 3.7,**

$$\tilde{u}_n \rightharpoonup \tilde{u} \text{ in } H^1(\mathbb{R}^3)\backslash\{0\}.$$

$\tilde{u}$ **satisfies**

$$-(a + bA^2)\Delta\tilde{u} + m\tilde{u} = \lambda(\tilde{u}^+)^{p-1} + (\tilde{u}^+)^5, \qquad (16)$$

where $A^2 := \lim_{n\to\infty} \int_{\mathbb{R}^3} |\nabla\tilde{u}_n|^2 \geq \int_{\mathbb{R}^3} |\nabla\tilde{u}|^2$**.**

Following (G. Li, H. Ye J. Differential Equations 2014 566-600), we have

$$\begin{cases} (a+bA^2)\int_{\mathbb{R}^3}|\nabla\tilde{u}|^2+m\int_{\mathbb{R}^3}\tilde{u}^2-\lambda\int_{\mathbb{R}^3}(\tilde{u}^+)^p-\int_{\mathbb{R}^3}(\tilde{u}^+)^6=0, \\ \frac{1}{2}(a+bA^2)\int_{\mathbb{R}^3}|\nabla\tilde{u}|^2+\frac{3}{2}m\int_{\mathbb{R}^3}\tilde{u}^2-\frac{3}{p}\lambda\int_{\mathbb{R}^3}(\tilde{u}^+)^p-\frac{1}{2}\int_{\mathbb{R}^3}(\tilde{u}^+)^6=0. \end{cases}$$

The first one follows by multiplying (16) by $\tilde{u}$ and integrating. The second one is the Pohozaev's identity applying to (16).

Hence

$$
\begin{aligned}
& G_m(\tilde{u}) \\
&= 2a\int_{\mathbb{R}^3}|\nabla\tilde{u}|^2 + 4m\int_{\mathbb{R}^3}\tilde{u}^2 + 2b\Big(\int_{\mathbb{R}^3}|\nabla\tilde{u}|^2\Big)^2 \\
&\quad - \frac{p+6}{p}\lambda\int_{\mathbb{R}^3}(\tilde{u}^+)^p - 2\int_{\mathbb{R}^3}(\tilde{u}^+)^6 \\
&\le 2a\int_{\mathbb{R}^3}|\nabla\tilde{u}|^2 + 4m\int_{\mathbb{R}^3}\tilde{u}^2 + 2bA^2\int_{\mathbb{R}^3}|\nabla\tilde{u}|^2 \\
&\quad - \frac{p+6}{p}\lambda\int_{\mathbb{R}^3}(\tilde{u}^+)^p - 2\int_{\mathbb{R}^3}(\tilde{u}^+)^6 \\
&= 0.
\end{aligned}
$$

To prove that $G_m(\tilde{u}) = 0$**, just suppose that** $G_m(\tilde{u}) < 0$**, then** $\tilde{u} \neq 0$ **and there is a unique** $0 < t < 1$ **such that** $G_m(t\tilde{u}(t^{-2}x)) = 0$**. So**

$$
\begin{aligned}
c_m \le I_m((\tilde u)_t) &= I_m((\tilde u)_t) - \frac{1}{p+6} G_m((\tilde u)_t) \\
&= \frac{p+2}{2(p+6)} a t^4 \int_{\mathbb{R}^3} |\nabla \tilde u|^2 + \frac{p-2}{2(p+6)} m t^8 \int_{\mathbb{R}^3} \tilde u^2 \\
&+ \frac{p-2}{4(p+6)} b t^8 \Big(\int_{\mathbb{R}^3} |\nabla \tilde u|^2 \Big)^2 + \frac{6-p}{6(p+6)} t^{12} \int_{\mathbb{R}^3} (\tilde u^+)^6 \\
&< \frac{p+2}{2(p+6)} a \int_{\mathbb{R}^3} |\nabla \tilde u|^2 + \frac{p-2}{2(p+6)} m \int_{\mathbb{R}^3} \tilde u^2 \\
&+ \frac{p-2}{4(p+6)} b \Big(\int_{\mathbb{R}^3} |\nabla \tilde u|^2 \Big)^2 + \frac{6-p}{6(p+6)} \int_{\mathbb{R}^3} (\tilde u^+)^6 \\
&\le \varliminf_{n\to\infty} \Big[\frac{p+2}{2(p+6)} a \int_{\mathbb{R}^3} |\nabla \tilde u_n|^2 + \frac{p-2}{2(p+6)} m \int_{\mathbb{R}^3} \tilde u_n^2 \\
&+ \frac{p-2}{4(p+6)} b \Big(\int_{\mathbb{R}^3} |\nabla \tilde u_n|^2 \Big)^2 + \frac{6-p}{6(p+6)} \int_{\mathbb{R}^3} (\tilde u_n^+)^6 \Big] \\
&= \lim_{n\to\infty} \Big[I_m(\tilde u_n) - \frac{1}{p+6} G_m(\tilde u_n) \Big] = \lim_{n\to\infty} \Big[I_m(u_n) - \frac{1}{p+6} G_m(u_n) \Big] = c_m;
\end{aligned}
$$

a contradiction. Hence $G_m(\tilde{u}) = 0$. Using the above inequality again with $t = 1$, we conclude that $\tilde{u}_n \to \tilde{u}$ in $H^1(\mathbb{R}^3)$, then $I_m(\tilde{u}) = c_m$ and $I'_m(\tilde{u}) = 0$, i.e. $\tilde{u}$ is a ground state solution for the limiting problem.

Let S_m the set of ground state solutions U of (9) satisfying $U(0) = \max_{x\in\mathbb{R}^3} U(x)$.

★ **Proposition 3.7 For each $m > 0$, S_m is compact in $H^1(\mathbb{R}^3)$.**

The proof of Proposition 3.7 involves applying Brezis-Kato type argument, i.e.

★ **Lemma 2.2 (i)Assume that** $\{v_n\}$ **is a sequence of weak solutions to**

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\right)\Delta u+V_n(x)u=f_n(x,u) \text{ in } \mathbb{R}^3$$

satisfying $\|v_n\|_{H^1(\mathbb{R}^3)}\le C$ **where** $V_n(x)\ge\alpha>0$ **and** $\forall\delta>0$, $\exists C_\delta>0$ **such that**

$$|f_n(x,t)|\le\delta|t|+C_\delta|t|^5,\ \forall(x,t)\in\mathbb{R}^3\times\mathbb{R}.$$

If $\{|v_n|^6\}$ **is uniformly integrable in any bounded domain in** $\mathbb{R}^3$**, then for any** $x_0\in\mathbb{R}^3$, $\exists R_0(x_0)>0$ **such that**

$$\|v_n\|_{L^\infty(B_{R_0(x_0)/4}(x_0))}\le C(R_0(x_0)).$$

The proof of Lemma 2.2 (i) mainly comes from Zhu. X. and Yang. J. (System Sci. Math. 1989, 47-52)

(1) can be rewritten as

$$-\Big(a+b\int_{\mathbb{R}^3}|\nabla v|^2\Big)\Delta v+V(\varepsilon x)v=\lambda|v|^{p-2}v+|v|^4v,\ \ v\in H^1(\mathbb{R}^3),\ \ v>0 \tag{17}$$

with the energy functional

$$I_\varepsilon(v)=\frac{a}{2}\int_{\mathbb{R}^3}|\nabla v|^2+\frac{b}{4}\Big(\int_{\mathbb{R}^3}|\nabla v|^2\Big)^2+\frac{1}{2}\int_{\mathbb{R}^3}V(\varepsilon x)v^2$$
$$-\frac{\lambda}{p}\int_{\mathbb{R}^3}(v^+)^p-\frac{1}{6}\int_{\mathbb{R}^3}(v^+)^6,\ \ v\in H^1(\mathbb{R}^3).$$

$$H_\varepsilon := \{v \in H^1(\mathbb{R}^3) | \int_{\mathbb{R}^3} V(\varepsilon x)v^2 < \infty\}$$

endowed with the norm

$$\|v\|_{H_\varepsilon} := \Big(\int_{\mathbb{R}^3} |\nabla v|^2 + \int_{\mathbb{R}^3} V(\varepsilon x)v^2\Big)^{1/2}.$$

Define

$$\chi_\varepsilon(x) = \begin{cases} 0 \text{ if } x \in \Lambda/\varepsilon, \\ \varepsilon^{-1} \text{ if } x \notin \Lambda/\varepsilon \end{cases}$$

and

$$Q_\varepsilon(v) = \Big(\int_{\mathbb{R}^3} \chi_\varepsilon v^2 - 1\Big)_+^2.$$

Set $J_\varepsilon : H_\varepsilon \to \mathbb{R}$ **be given by**

$$J_\varepsilon(v) = I_\varepsilon(v) + Q_\varepsilon(v).$$

Note that this type of penalization was firstly introduced by J. Byeon, Z. Q. Wang (Calc. Var. Partial Differential Equations 2003 207-219).

Let $c_{V_0} = I_{V_0}(w)$ **for** $w \in S_{V_0}$ **and** $10\delta = \textbf{dist}\{\mathcal{M}, \mathbb{R}^3 \backslash \Lambda\}$**, we fix a** $\beta \in (0, \delta)$ **and a cut-off function** $\varphi \in C_c^\infty(\mathbb{R}^3)$ **such that** $0 \le \varphi \le 1$**,** $\varphi(x) = 1$ **for** $|x| \le \beta$**,** $\varphi(x) = 0$ **for** $|x| \ge 2\beta$ **and** $|\nabla \varphi| \le C/\beta$**.**

Denote

$$X_\varepsilon := \left\{ \varphi(\varepsilon x - x') w \Big(x - \frac{x'}{\varepsilon} \Big) : x' \in \mathcal{M}^\beta, w \in S_{V_0} \right\}$$

for sufficiently small $\varepsilon > 0$**, where** $\mathcal{M}^\beta := \{y \in \mathbb{R}^3 : \inf_{z \in \mathcal{M}} |y - z| \le \beta\}$**.**

$\forall U^* \in S_{V_0}$, **define** $W_{\varepsilon,t}(x) := t\varphi(\varepsilon x)U^*(t^{-2}x)$, **by studying the behavior of** $J_\varepsilon(W_{\varepsilon,t})$, **we can check that** J_ε **possesses the Mountain Pass geometry for** $\varepsilon > 0$ **small.**

The Mountain-Pass value of J_ε:

$$c_\varepsilon := \inf_{\gamma\in\Gamma_\varepsilon} \max_{s\in[0,1]} J_\varepsilon(\gamma(s))$$

where $\Gamma_\varepsilon := \{\gamma \in C([0,1], H_\varepsilon) | \gamma(0) = 0,\ \gamma(1) = W_{\varepsilon,t_0}\}$.

Denote

$$\tilde{c}_\varepsilon := \max_{s\in[0,1]} J_\varepsilon(\gamma_\varepsilon(s)).$$

★ **Lemma 4.1, Lemma 4.2**

$$\lim_{\varepsilon\to 0} c_\varepsilon = \lim_{\varepsilon\to 0} \tilde{c}_\varepsilon = c_{V_0}.$$

Note that Lemma 4.1, Lemma 4.2 will be used in Lemma 4.3 below.

★ **Lemma 4.3**

(i) There exists a $d_0 > 0$ **such that for any** $\{\varepsilon_i\}_{i=1}^{\infty}$, $\{R_{\varepsilon_i}\}$, $\{u_{\varepsilon_i}\}$ **with**

$$\begin{cases} \lim\limits_{i\to\infty} \varepsilon_i = 0,\ \ R_{\varepsilon_i} \geq R_0/\varepsilon_i,\ \ u_{\varepsilon_i} \in X_{\varepsilon_i}^{d_0} \cap H_0^1(B_{R_{\varepsilon_i}}(0)), \\ \lim\limits_{i\to\infty} J_{\varepsilon_i}(u_{\varepsilon_i}) \leq c_{V_0} \text{ and } \lim\limits_{i\to\infty} \left\| J'_{\varepsilon_i}(u_{\varepsilon_i}) \right\|_{*,\varepsilon_i,R_{\varepsilon_i}} = 0, \end{cases} \tag{18}$$

then there exists, up to a subsequence, $\{y_i\}_{i=1}^{\infty} \subset \mathbb{R}^3$, $x_0 \in \mathcal{M}$, $U \in S_{V_0}$ **such that**

$$\lim_{i\to\infty} |\varepsilon_i y_i - x_0| = 0 \text{ and } \lim_{i\to\infty} \|u_{\varepsilon_i} - \varphi(\varepsilon_i x - \varepsilon_i y_i) U(x - y_i)\|_{H_{\varepsilon_i}} = 0.$$

(ii) If we drop $\{R_{\varepsilon_i}\}$ **and replace(18) by**

$$\lim_{i\to\infty} \varepsilon_i = 0,\ \ u_{\varepsilon_i} \in X_{\varepsilon_i}^{d_0},\ \ \lim_{i\to\infty} J_{\varepsilon_i}(u_{\varepsilon_i}) \leqslant c_{V_0}$$
$$\text{and } \lim_{i\to\infty} \left\| J'_{\varepsilon_i}(u_{\varepsilon_i}) \right\|_{(H_{\varepsilon_i})^{-1}} = 0,$$

then the same conclusion holds.

Note that Lemma 4.3 is a key for the proof of Theorem 1.1 and the idea of Lemma 4.3 mainly comes from Byeon and Jeanjean (Arch. Rational Mech. Anal. 2007, 185-200), but for the critical case, the method of Byeon and Jeanjean seems to be hard to be used directly and some more tricks are needed.

★ **Lemma 4.4 Let d_0 be the number given in Lemma 4.3, then for any $d \in (0, d_0)$, there exist $\varepsilon_d > 0$, $\rho_d > 0$ and $\omega_d > 0$ such that**

$$\left\| {J'}_{\varepsilon}(u) \right\|_{*,\varepsilon,R} \geq \omega_d > 0$$

for all $u \in J_{\varepsilon}^{cV_0+\rho_d} \cap (X_{\varepsilon}^{d_0} \backslash X_{\varepsilon}^{d}) \cap H_0^1(B_R(0))$ with $\varepsilon \in (0, \varepsilon_d)$ and $R \geq R_0/\varepsilon$.

★ **Lemma 4.5 There exists $T_0 > 0$ with the following property: for any $\delta > 0$ small, there exist $\alpha_\delta > 0$ and $\varepsilon_\delta > 0$ such that if $J_\varepsilon(\gamma_\varepsilon(s)) \geq c_{V_0} - \alpha_\delta$ and $\varepsilon \in (0, \varepsilon_\delta)$, then $\gamma_\varepsilon(s) \in X_\varepsilon^{T_0\delta}$, where $\gamma_\varepsilon(s) := W_{\varepsilon, st_0}$, $s \in [0, 1]$.**

Inspired by G. M. Figueiredo, N. Ikoma, J. R. Santos Junior(Arch. Rational Mech. Anal. 2014 931-979), using Lemma 4.4, Lemma 4.5 and a version of quantitative deformation lemma due to G. M. Figueiredo, N. Ikoma, J. R. Santos Junior, we can construct a bounded (PS) sequence of the penalized functional J_ε near the compact set S_{V_0}, i.e.

★ **Lemma 4.6** $\exists \bar{\varepsilon} > 0$ **such that for each** $\varepsilon \in (0, \bar{\varepsilon}]$ **and** $R > R_0/\varepsilon$, **there exists a sequence** $\{v_{n,\varepsilon}^R\}_{n=1}^{\infty} \subset J_\varepsilon^{\tilde{c}_\varepsilon + \varepsilon} \cap X_\varepsilon^{d_0} \cap H_0^1(B_R(0))$ **such that** $J'_\varepsilon(v_{n,\varepsilon}^R) \to 0$ **in** $(H_0^1(B_R(0)))^{-1}$ **as** $n \to \infty$**.**

3. Proof of the main result

Using standard argument, we can find a solution $v_\varepsilon \in H^1(\mathbb{R}^3)$ to the penalized equation

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2\right)\Delta u + V(\varepsilon x)u + 4\left(\int_{\mathbb{R}^3} \chi_\varepsilon u^2 dx - 1\right)_+ \chi_\varepsilon u = \lambda u^{p-1} + u^5$$

To show that, v_ε is, in fact, a solution to (17), we need the following Brezis-Kato type argument:

★ **Lemma 2.2 (ii) Assume that $\{v_n\}$ is a sequence of weak solutions to**

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\right)\Delta u+V_n(x)u=f_n(x,u) \textbf{ in } \mathbb{R}^3$$

satisfying $\|v_n\|_{H^1(\mathbb{R}^3)}\le C$ where $V_n(x)\ge\alpha>0$ and $\forall\delta>0$, $\exists C_\delta>0$ such that

$$|f_n(x,t)|\le\delta|t|+C_\delta|t|^5,\ \forall(x,t)\in\mathbb{R}^3\times\mathbb{R}.$$

If $\{|v_n|^6\}$ is uniformly integrable near ∞, i.e. $\forall\varepsilon>0$, $\exists R>0$, for any $r>R$, $\int_{\mathbb{R}^3\setminus B_r(0)}|v_n|^6<\varepsilon$, then

$$\lim_{|x|\to\infty}v_n(x)=0 \textbf{ uniformly for } n.$$

The proof mainly comes from G. Li(Ann. Acad. Sci. Fenn. A I Math. 1990 27-36)

For any sequence $\{\varepsilon_j\}$ **with** $\varepsilon_j \to 0$, **by Lemma 4.3(ii),** $\exists\{y_j\}_{i=j}^{\infty} \subset \mathbb{R}^3$, $x_0 \in \mathcal{M}$, $U \in S_{V_0}$ **such that**

$$\lim_{j\to\infty} |\varepsilon_j y_j - x_0| = 0 \text{ and } \lim_{j\to\infty} \left\| v_{\varepsilon_j}(x) - \varphi(\varepsilon_j x - \varepsilon_j y_j) U(x - y_j) \right\|_{H_{\varepsilon_j}} = 0 \tag{20}$$

$$\Rightarrow$$

$$w_{\varepsilon_j}(x) := v_{\varepsilon_j}(x + y_j) \to U(x) \text{ in } L^6(\mathbb{R}^3).$$

By Lemma 2.2 (ii), we get

$$\lim_{|x|\to\infty} w_{\varepsilon_j}(x) = 0 \text{ uniformly for all } \varepsilon_j. \tag{21}$$

Proceeding as G. Li and S. Yan (Commun. Partial Differential Equations 1989 1291-1314), we get the uniform exponential decay

$$w_{\varepsilon_j}(x) \leq C_1 e^{-C_2|x|},\ x \in \mathbb{R}^3.$$

Thus

$$\varepsilon_j^{-1} \int_{\mathbb{R}^3 \setminus (\Lambda/\varepsilon_j)} v_{\varepsilon_j}^2(x) = \varepsilon_j^{-1} \int_{\mathbb{R}^3 \setminus (\Lambda/\varepsilon_j - y_j)} w_{\varepsilon_j}^2(x)$$
$$\leqslant \varepsilon_j^{-1} \int_{\mathbb{R}^3 \setminus B_{\beta/\varepsilon_j}(0)} (C_1)^2 e^{-2C_2|x|} \to 0,\ \textbf{as } j \to \infty,$$

i.e. $Q_{\varepsilon_j}(v_{\varepsilon_j}) = 0$ **for** ε_j **small. Therefore** v_{ε_j} **is a solution of (17). Set** $u_\varepsilon(x) = v_\varepsilon(\frac{x}{\varepsilon})$, u_{ε_j} **is a solution of (1).**

3. Proof of the main result (concentration)

Let P_j **be a maximum point of** w_{ε_j}**, we can check that** $\exists C_0 > 0$ **such that** $w_{\varepsilon_j}(P_j) > C_0$**, then by (21),** $\{P_j\}$ **must be bounded. Since** $u_{\varepsilon_j}(x) = w_{\varepsilon_j}(\frac{x}{\varepsilon_j} - y_j)$**,** $x_j := \varepsilon_j P_j + \varepsilon_j y_j$ **is a maximum point of** u_{ε_j}**. From (20),** $x_j \to x_0 \in \mathcal{M}$ **as** $j \to \infty$**.**

A. Arosio, S. Panizzi, On the well-posedness of the Kirchhoff string, Trans. Amer. Math. Soc. 348 (1996) 305-330.

A. Ambrosetti, P. Rabinowitz, Dual variational methods in critical points theory and applications, J. Funct. Anal. 14 (1973) 349-381.

P. D'Ancona, S. Spagnolo, Global solvability for the degenerate Kirchhoff equation with real analytic data, Invent. Math. 108 (1992) 247-262.

V. Benci and G. Cerami, Existence of positive solutions of the equation $-\Delta u + a(x)u = u^{\frac{N+2}{N-2}}$ in $\mathbb{R}^N$, J. Funct. Anal. 88 (1990) 90-117.

H. Berestycki, P. L. Lions, Nonlinear scalar field equations, I existence of a ground state, Arch. Rational Mech. Anal. 82 (1983) 313-345.

H. Berestycki, P. L. Lions, Nonlinear scalar field equations, II existence of infinitely many solutions, Arch. Rational Mech. Anal. 82 (1983) 347-375.

J. Byeon, L. Jeanjean, Standing waves for nonlinear Schrödinger equations with a general nonlinearity, Arch. Rational Mech. Anal. 185 (2007) 185-200.

J. Byeon, Z. Q. Wang, Standing waves with a critical frequency for nonlinear Schrödinger equations II. Calc. Var. Partial Differential Equations 18 (2003) 207-219.

K. C. Chang, Infinite dimensional morse theory and multiple solution problems, Birkhäuser, Boston, 1993.

C. Y. Chen, Y. C. Kuo, T. F. Wu, The Nehari manifold for a Kirchhoff type problem involving sign-changing weight functions, J. Differential Equations 250 (2011) 1876-1908.

- S. Cingolani, N. Lazzo, Multiple semiclassical standing waves for a class of nonlinear Schrödinger equations, Topol. Methods Nonlinear Anal. 10 (1997) 1-13.
- G. M. Figueiredo, N. Ikoma, J. R. Santos Junior, Existence and concentration result for the Kirchhoff type equations with general nonlinearities, Arch. Rational Mech. Anal. 213 (2014) 931-979.
- A. Floer, A. Weinstein, Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential, J. Funct. Anal. 69 (1986) 397-408.
- C. Gui, Existence of multi-bump solutions for nonlinear Schrödinger equations via variational method, Commun. Partial Differential Equations 21 (1996) 787-820.
- D. Gilbarg, N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, 2nd ed., Grundlehren Math. Wiss., vol. 224, Springer, Berlin, 1983.

- J. Hirata, N. Ikoma, K. Tanaka, Nonlinear scalar field equations in $\mathbb{R}^N$: mountain pass and symmetric mountain pass approaches, Topol. Methods Nonlinear Anal. 35 (2010) 253-276.
- Y. He, G. Li, The existence and concentration of weak solutions to a class of p-Laplacian type problems in unbounded domains, Sci. China Math. 57 (2014) 1927-1952.
- Y. He, G. Li, S. Peng, Concentrating Bound States for Kirchhoff type problems in $\mathbb{R}^3$ involving critical Sobolev exponents, Adv. Nonlinear Stud. 14 (2014) 441-468.
- X. He, W. Zou, Infinitely many positive solutions for Kirchhoff-type problems, Nonlinear Anal. 70 (2009) 1407-1414.
- X. He, W. Zou, Existence and concentration behavior of positive solutions for a kirchhoff equation in $\mathbb{R}^3$, J. Differential Equations 2 (2012) 1813-1834.

- L. Jeanjean, On the existence of bounded Palais-Smale sequences and application to a Landsman-Lazer-type problem set on $\mathbb{R}^N$, Proc. Roy. Soc. Edingburgh Sect. A Math. 129 (1999) 787-809.

- G. Kirchhoff, Mechanik, Teubner, Leipzig, 1883.

- G. Li, Some properties of weak solutions of nonlinear scalar field equations, Ann. Acad. Sci. Fenn. A I Math. 15 (1990) 27-36.

- P. L. Lions, The concentration-compactness principle in the calculus of variations, The locally compact case, part 2, Ann. Inst. H. Poincaré Anal. Non. Linéaire 2 (1984) 223-283.

- P. L. Lions, The concentration-compactness principle in the calculus of variations, The limit case, part 1, Rev. Mat. H. Iberoamericano 1.1 (1985) 145-201.

- J. L. Lions, On some questions in boundary value problems of mathematical physics, in: Contemporary Developments in

Continuum Mechanics and Partial Differential Equations, Proceedings of International Symposium, Inst. Mat., Univ. Fed. Rio de Janeiro, Rio de Janeiro, 1977, in: North-Holland Math. Stud. vol. 30, North-Holland, Amsterdam, 1978, pp. 284-346.

W. Liu, X. He, Multiplicity of high energy solutions for superlinear Kirchhoff equations, J. Appl. Math. Comput. 39 (2012) 473-487.

G. Li, H. Ye, Existence of positive ground state solutions for the nonlinear Kirchhoff type equations in $\mathbb{R}^3$, J. Differential Equations 257 (2014) 566-600.

G. Li, H. Ye, Existence of positive solutions for nonlinear Kirchhoff type equations in $\mathbb{R}^3$ with critical Sobolev exponent, Math. Meth. Appl. Sci. 37 (2014) 2570-2584.

T. F. Ma, J. E. Munoz Rivera, Positive solutions for a nonlinear nonlocal elliptic transmission problem, Appl. Math. Lett. 16 (2003) 243-248.

- Y. G. Oh, Existence of semi-classical bound states of nonlinear Schrödinger equations with potential on the class $(V)_a$, Commun. Partial Differential Equations 13 (1988) 1499-1519.
- Y. G. Oh, Corrections to existence of semi-classical bound states of nonlinear Schrödinger equations with potential on the class $(V)_a$, Commun. Partial Differential Equations 14 (1989) 833-834.
- Y. G. Oh, On positive multi-lump bound states of nonlinear Schrödinger equations under multiple well potential, Commun. Math. Phys. 131 (1990) 223-253.
- M. del Pino, P. L. Felmer, Local mountain pass for semilinear elliptic problems in unbounded domains, Calc. Var. Partial Differential Equations 4 (1996) 121-137.
- P. Pucci, J. Serrin, A general variational identity, Indiana Univ. Math. J. 35 (1986) 681-703 .

- K. Perera, Z. Zhang, Nontrivial solutions of Kirchhoff-type problems via the Yang index, J. Differential Equations 221 (2006) 246-255.
- P. Rabinowitz, On a class of nonlinear Schrödinger equations, Z. Angew. Math. Phys. 43 (1992) 270-291.
- M. Ramos, Z. Q. Wang, M. Willem, Positive solutions for elliptic equations with critical growth in unbounded domains, Calculus of Variations and Differential Equations, Chapman & Hall/CRC Press, Boca Raton, 2000, pp.192-199.
- P. Tolksdorf, Regularity for a more general class of quasilinear elliptic equations, J. Differential Equations 51 (1984) 126-150.
- X. Wang, On concentration of positive bound states of nonlinear Schrödinger equations, Commun. Math. Phys. 153 (1993) 229-244.

M. Willem, Minimax theorems, Progress in Nonlinear Differential Equations and their Applications, 24. Birkhäuser Boston, Inc., Boston, MA, 1996.

J. Wang, L. Tian, J. Xu, F. Zhang, Multiplicity and concentration of positive solutions for a Kirchhoff type problem with critical growth, J. Differential Equations 253 (2012) 2314-2351.

J. Zhang, Z. Chen, W. Zou, Standing waves for nonlinear Schrödinger equations involving critical growth, arXiv:1209.3074v1.

X. Zhu, J. Yang, Regularity for quasilinear elliptic equations in involving critical Sobolev exponent, System Sci. Math. 9 (1989) 47-52.

Thank You Very Much for Your Attention!