Noncommutative Geometry & Conformal Geometry

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Main Results

We use tools of noncommutative geometry to obtain:

- Local index formula in conformal-diffeomorphism invariant geometry.
- A new class of conformal invariants.
- Vafa-Witten inequality in the setting of twisted spectral triples.

References

PW0 Ponge, R.; Hang, W.: *Index map, σ-connections and Connes-Chern character in the setting of twisted spectral triples*. arXiv 1310.6131.

PW1 Ponge, R.; Wang, H.: *Noncommutative geometry and conformal geometry. I. Local index formula and conformal invariants*. arXiv 1411.3701.

PW2 Ponge, R.; Hang, W.: *Noncommutative geometry and conformal geometry. II. Connes-Chern character and the local equivariant index theorem.* arXiv 1411.3703.

PW3 Ponge, R.; Wang, H.: *Noncommutative geometry and conformal geometry. III. Vafa-Witten inequality and Poincaré duality*. arXiv 1310.6138.

Part 1 Background

1. Twisted spectral triples in noncommutative geometry.
2. Index map and Connes-Chern character of a twisted spectral triple.
3. Conformal invariants and local index formulas in conformal geometry.
4. Vafa-Witten inequality for twisted spectral triples.

Manifolds and Spectral triples

Let (M^n, g) be a compact Riemannian spin manifold (n even) with spinor bundle $\$ = \$^+ \oplus \$^-$. Then

$$M \rightsquigarrow C^\infty(M), L^2_g(M, \$), D\!\!\!/_g.$$

- $C^\infty(M)$ acts by multiplication on $L^2_g(M, \$)$.
- $D\!\!\!/_g : C^\infty(M, \$) \to C^\infty(M, \$)$ is the Dirac operator of (M, g).

Conversely,

$$C^\infty(M), L^2_g(M, \$), D\!\!\!/_g \rightsquigarrow \text{ Riemannian geometry of } M.$$

- The algebra $C^\infty(M)$ encodes space information.
- The operator $D\!\!\!/_g$ encodes geometric information.
  - $[D\!\!\!/_g, f] = c(df)$.
  - $d(x, y) = \sup_f \{|f(x) - f(y)| : \|[D\!\!\!/_g, f]\| \leq 1\}$.

Spectral Triples

Definition (Connes)

A *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ consists of

1. A $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$.
2. An involutive algebra $\mathcal{A}$ represented in $\mathcal{H}$.
3. A selfadjoint unbounded operator D on $\mathcal{H}$ such that
   - 3.1 D maps $\mathcal{H}^\pm$ to $\mathcal{H}^\mp$.
   - 3.2 $(D \pm i)^{-1}$ is compact.
   - 3.3 $[D, a]$ is bounded for all $a \in \mathcal{A}$.

Example (Dirac Spectral Triple)

In the previous slide, $\left(C^\infty(M), L^2_g(M, \$), \not{D}_g\right)$ is a spectral triple:

- (M^n, g) compact Riemannian spin manifold (n even) with spinor bundle $\$ = \$^+ \oplus \$^-$.
- $\not{D}_g : C^\infty(M, \$) \to C^\infty(M, \$)$ is the Dirac operator of (M, g).
- $C^\infty(M)$ acts by multiplication on $L^2_g(M, \$)$.

Group Actions on Manifolds

Fact

If G is a group of diffeomorphisms of a manifold M, then M/G need not be Hausdorff.

Solution Provided by NCG

Trade the space M/G for the crossed product algebra,

$$C_c^\infty(M) \rtimes G = \left\{ \sum f_\phi u_\phi; \; f_\phi \in C_c^\infty(M) \right\},$$
$$u_\phi^* = u_\phi^{-1} = u_{\phi^{-1}}, \qquad u_\phi f = (f \circ \phi^{-1}) u_\phi.$$

Proposition (Green)

If G acts freely and properly, then $C^\infty(M/G)$ is Morita equivalent to $C_c^\infty(M) \rtimes G$.

Geometry of Manifolds with Group Actions

Example (Equivariant Dirac Spectral Triple)

- (M^n, g) compact Riemannian spin manifold (n even) with spinor bundle $\not{S} = \not{S}^+ \oplus \not{S}^-$ with the action of a group G of isometries preserving orientation and spin structure.
- $C^\infty(M) \rtimes G$ is the (discrete) crossed product.
- $\not{D}_g : C^\infty(M, \not{S}) \to C^\infty(M, \not{S})$ is the Dirac operator of (M, g).

Then $\left(C^\infty(M) \rtimes G, L^2_g(M, \not{S}), \not{D}_g \right)$ is a spectral triple.

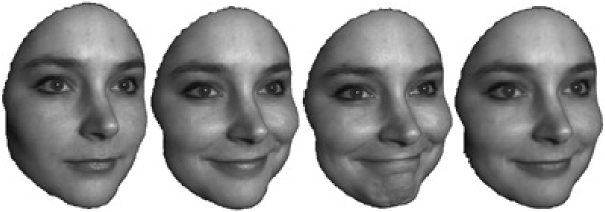
Remark

When the group G acts by conformal diffeomorphisms on M, commutators

$$[\not{D}_g, a] \qquad a \in C^\infty(M) \rtimes G$$

are not bounded.

Conformal Geometry



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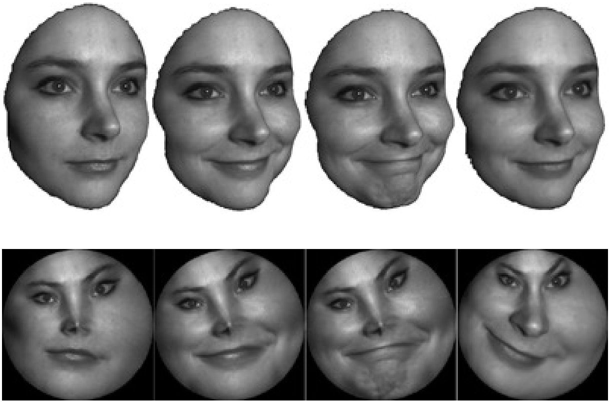

Twisted Spectral Triples

Twisted Spectral Triples

Definition (Connes-Moscovici)

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A twisted *spectral triple* $(\mathcal{A}, \mathcal{H}, D)_\sigma$ consists of

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   3.2 $(D \pm i)^{-1}$ is compact.
   3.3 $[D, a]_\sigma := Da - \sigma(a)D$ is bounded for all $a \in \mathcal{A}$.

Conformal Deformations of Spectral Triples

Example (Connes-Moscovici)

- An *ordinary* spectral triple $(\mathcal{A}, \mathcal{H}, D)$.
- A positive element $k \in \mathcal{A}$ withinner automorphism $\sigma(a) = k^2 a k^{-2}$, $a \in \mathcal{A}$.

Then $(\mathcal{A}, \mathcal{H}, kDk)_\sigma$ is a *twisted* spectral triple.

Conformal Change of Metric

Example

- (M^n, g) compact Riemannian spin manifold (n even) with spinor bundle $\not{S} = \not{S}^+ \oplus \not{S}^-$.
- $\not{D}_g : C^\infty(M, \not{S}) \to C^\infty(M, \not{S})$ is the Dirac operator of (M, g).
- $C^\infty(M)$ acts by multiplication on $L^2_g(M, \not{S})$.

Consider a conformal change of metric,

$$\hat{g} = k^{-2} g, \qquad k \in C^\infty(M),\ k > 0.$$

Then the Dirac spectral triple $\left(C^\infty(M), L^2_{\hat{g}}(M, \not{S}), \not{D}_{\hat{g}}\right)$ is unitarily equivalent to $\left(C^\infty(M), L^2_g(M, \not{S}), \sqrt{k}\not{D}_g\sqrt{k}\right)$ (i.e., the spectral triples are intertwined by a unitary operator).

Further examples

1. (Conformal Dirac spectral triple, Connes-Moscovici)
   - G is a group of conformal diffeomorphisms of (M, g).
   - $\not{D}_g$ is the Dirac operator.
   - σ is an automorphism of $C^\infty(M) \rtimes G$ given by

   $$\sigma_g(fv_\phi) = e^{2h_\phi} fv_\phi, \qquad f \in C^\infty(M), \phi \in G.$$

   Then

   $$\left(C^\infty(M) \rtimes G, L^2_g(M, \not{S}), \not{D}_g\right)_{\sigma_g}$$

   is a twisted spectral triple.
2. Twisted spectral triples over NC tori associated to conformal weights (Connes-Tretkoff).
3. Twisted spectral triples associated to some quantum statistical systems (e.g., Connes-Bost systems).

Overview of Noncommutative Geometry

Classical	NCG
Manifold M	Spectral Triple $(\mathcal{A}, \mathcal{H}, D)$
Vector Bundle E over M	Projective Module $\mathcal{E}$ over $\mathcal{A}$ $\mathcal{E} = e\mathcal{A}^q, \; e \in M_q(\mathcal{A}), \; e^2 = e$
K-Theory $K^0(M)$	K-Theory $K_0(A)$
de Rham Cohomology $H^{ev}(M)$	Cyclic Homology $HP_0(\mathcal{A})$
de Rham Homology $H_{ev}(M)$	Cyclic Cohomology $HP^0(\mathcal{A})$
Atiyah-Singer Index Formula $\mathrm{ind} \not{D}_E = \int \hat{A}(R^M) \wedge \mathrm{Ch}(E)$	Connes-Chern Character $\mathrm{Ch}(D)$ $\mathrm{ind}\, D_{\mathcal{E}} = \langle \mathrm{Ch}(D), \mathrm{Ch}(\mathcal{E}) \rangle$

Part 2 Index Theory of Twisted Spectral Triples

1. Twisted spectral triples in noncommutative geometry.
2. Index map and Connes-Chern character of a twisted spectral triple.
3. Conformal invariants and local index formulas in conformal geometry.
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Connections over a Spectral Triple

- $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple.
- $\mathcal{E}$ is a finitely generated projective (right) module over $\mathcal{A}$.
- Differential 1 forms:

$$\Omega^1_D(\mathcal{A}) = \mathrm{Span}\{ad(b);\ a, b \in \mathcal{A}\} \subset \mathcal{L}(\mathcal{H}),$$

where $d(a) = [D, a] = Da - aD$.

Definition

A *connection* on $\mathcal{E}$ is a linear map $\nabla^{\mathcal{E}} : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A})$ such that

$$\nabla^{\mathcal{E}}(\xi a) = \xi \otimes d(a) + \left(\nabla^{\mathcal{E}} \xi\right) a \qquad \forall a \in \mathcal{A}\ \forall \xi \in \mathcal{E}.$$

Example

Any connection ∇^E on a vector bundle E over M defines a connection $\nabla^{\mathcal{E}}$ on $\mathcal{E} = C^\infty(M, E)$.

σ-Connections, Set up

Setup/Notation

- $(\mathcal{A}, \mathcal{H}, D)_\sigma$ twisted spectral triple.
- $\mathcal{E}$ finitely generated projective (right) module over $\mathcal{A}$.
- Twisted differential forms:

$$\Omega^1_{D,\sigma}(\mathcal{A}) = \mathrm{Span}\{a d_\sigma b;\ a, b \in \mathcal{A}\} \subset \mathcal{L}(\mathcal{H}),$$

where $d_\sigma(a) = [D, a]_\sigma = Da - \sigma(a)D$.

Definition

A *σ-translate* of $\mathcal{E}$ is a finitely generated projective module $\mathcal{E}^\sigma$ together with a linear isomorphism $\sigma^{\mathcal{E}} : \mathcal{E} \to \mathcal{E}^\sigma$ such that

$$\sigma^{\mathcal{E}}(\xi a) = \sigma^{\mathcal{E}}(\xi)\sigma(a) \qquad \forall \xi \in \mathcal{E}\ \forall a \in \mathcal{A}.$$

σ-Connections

σ-Connections

Definition (Connes-Moscovici)

A connection on a finitely generated projective module $\mathcal{E}$ is a linear map $\nabla^{\mathcal{E}} : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_{D,\sigma}(\mathcal{A})$ such that

$$\nabla^{\mathcal{E}}(\xi a) = \xi \otimes da + \left(\nabla^{\mathcal{E}} \xi\right) a \qquad \forall a \in \mathcal{A} \; \forall \xi \in \mathcal{E},$$

where $da = [D, a]$.

σ-Connections

Definition (PW0)

A σ-connection on a finitely generated projective module $\mathcal{E}$ is a linear map $\nabla^{\mathcal{E}} : \mathcal{E} \to \mathcal{E}^{\sigma} \otimes_{\mathcal{A}} \Omega^1_{D,\sigma}(\mathcal{A})$ such that

$$\nabla^{\mathcal{E}}(\xi a) = \sigma^{\mathcal{E}}(\xi) \otimes d_{\sigma} a + \left(\nabla^{\mathcal{E}} \xi\right) a \qquad \forall a \in \mathcal{A} \; \forall \xi \in \mathcal{E},$$

where $d_{\sigma} a = [D, a]_{\sigma} = Da - \sigma(a) D$.

σ-Connections

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$$\nabla^{\mathcal{E}}(\xi a) = \sigma^{\mathcal{E}}(\xi) \otimes d_{\sigma} a + \left(\nabla^{\mathcal{E}} \xi\right) a \qquad \forall a \in \mathcal{A} \ \forall \xi \in \mathcal{E},$$

where $d_{\sigma} a = [D, a]_{\sigma} = Da - \sigma(a) D$.

Example

If $\mathcal{E} = e\mathcal{A}^q$ with $e = e^2 \in M_q(\mathcal{A})$, then

1. $\mathcal{E}^{\sigma} = \sigma(e)\mathcal{A}^q$ is a σ-translate.
2. We have the Grassmanian σ-connection,

$$\nabla_0^{\mathcal{E}} \xi := \sigma(e) d_{\sigma} \xi, \qquad \xi \in \mathcal{E}.$$

Operators Coupling with σ-connections

Operators Coupling with σ-connections

Definition

Let $\nabla^{\mathcal{E}}$ be a connection on $\mathcal{E}$. Then the coupled operator $D_{\nabla^{\mathcal{E}}} : \mathcal{E} \otimes_{\mathcal{A}} \operatorname{dom} D \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$ is given by

$$D_{\nabla^{\mathcal{E}}}(\xi \otimes \zeta) = \xi \otimes D\zeta + c(\nabla^{\mathcal{E}})(\xi \otimes \zeta),$$

where $c(\nabla^{\mathcal{E}})$ is the composition,

$$\mathcal{E} \otimes \mathcal{H} \xrightarrow{\nabla^{\mathcal{E}} \otimes \mathrm{id}_{\mathcal{H}}} \mathcal{E} \otimes \Omega^1_D(\mathcal{A}) \otimes \mathcal{H} \xrightarrow{\mathrm{id}_{\mathcal{E}} \otimes c} \mathcal{E} \otimes \mathcal{H},$$

where $c(\omega \otimes \zeta) = \omega(\zeta)$ (recall that $\Omega^1_D(\mathcal{A}) \subset \mathcal{L}(\mathcal{H})$).

Operators Coupling with σ-connections

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$$D_{\nabla^{\mathcal{E}}}(\xi \otimes \zeta) = \sigma^{\mathcal{E}}(\xi) \otimes D\zeta + c(\nabla^{\mathcal{E}})(\xi \otimes \zeta),$$

where $c(\nabla^{\mathcal{E}})$ is the composition,

$$\mathcal{E} \otimes \mathcal{H} \xrightarrow{\nabla^{\mathcal{E}} \otimes \mathrm{id}_{\mathcal{H}}} \mathcal{E}^{\sigma} \otimes \Omega^1_{D,\sigma}(\mathcal{A}) \otimes \mathcal{H} \xrightarrow{\mathrm{id}_{\mathcal{E}^{\sigma}} \otimes c} \mathcal{E}^{\sigma} \otimes \mathcal{H},$$

where $c(\omega \otimes \zeta) = \omega(\zeta)$.

Operators Coupling with σ-connections

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where $c(\omega \otimes \zeta) = \omega(\zeta)$.

Example

For a Dirac spectral triple $(C^{\infty}(M), L^2_g(M, \$), \not{D}_g)$ and the projective module $\mathcal{E} = C^{\infty}(M, E)$,

$$D_{\nabla^{\mathcal{E}}} = D_{\nabla^{E}}.$$

Index Map

Proposition (PW0)

The coupling of D with any σ-connection $\nabla^{\mathcal{E}}$ gives rise to a Fredholm operator $D_{\nabla^{\mathcal{E}}}$ with the form

$$D_{\nabla^{\mathcal{E}}} = \begin{pmatrix} 0 & D^-_{\nabla^{\mathcal{E}}} \\ D^+_{\nabla^{\mathcal{E}}} & 0 \end{pmatrix}, \quad D^{\pm}_{\nabla^{\mathcal{E}}} : \mathcal{E} \otimes \operatorname{dom} D^{\pm} \to \mathcal{E}^{\sigma} \otimes \mathcal{H}^{\mp}.$$

Define the index of $D_{\nabla^{\mathcal{E}}}$ to be $\operatorname{ind} D_{\nabla^{\mathcal{E}}} = \frac{1}{2}\left(\operatorname{ind} D^+_{\nabla^{\mathcal{E}}} - \operatorname{ind} D^-_{\nabla^{\mathcal{E}}}\right)$.

Proposition (Connes-Moscovici, PW0)

The Fredholm indices,

$$\operatorname{ind} D^{\pm}_{\nabla^{\mathcal{E}}} := \dim\ker D^{\pm}_{\nabla^{\mathcal{E}}} - \dim\ker \left(D^{\pm}_{\nabla^{\mathcal{E}}}\right)^*,$$

depend only on the K-theory class of $\mathcal{E}$. There is a additive map

$$\operatorname{ind}_{D,\sigma} : K_0(\mathcal{A}) \to \frac{1}{2}\mathbb{Z} \quad \operatorname{ind}_{D,\sigma}[\mathcal{E}] = \operatorname{ind} D_{\nabla^{\mathcal{E}}} \quad \forall (\mathcal{E}, \nabla^{\mathcal{E}}).$$

Connes-Chern Character

Theorem (Connes-Moscovici, PW0)

Assume $(\mathcal{A}, \mathcal{H}, D)_\sigma$ *is p-summable, i.e.,* $\operatorname{Tr}|D|^{-p} < \infty$ *for some* $p \geq 1$. *Then there is an even periodic cyclic cohomology class* $\operatorname{Ch}(D)_\sigma \in \operatorname{HP}^0(\mathcal{A})$*, called the Connes-Chern character, such that*

$$\operatorname{ind} D_{\nabla^{\mathcal{E}}} = \langle \operatorname{Ch}(D)_\sigma, \operatorname{Ch}(\mathcal{E}) \rangle \quad \forall (\mathcal{E}, \nabla^{\mathcal{E}}),$$

where $\operatorname{Ch}(\mathcal{E})$ *is the Chern character in the periodic cyclic homology* $\operatorname{HP}_0(\mathcal{A})$.

Part 3 An Application to Conformal Geometry

1. Twisted spectral triples in noncommutative geometry.
2. Index map and Connes-Chern character of a twisted spectral triple.
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Local Index Formula in NCG

Theorem (Connes-Moscovici)

Let $(\mathcal{A}, \mathcal{H}, D)$ be an ordinary spectral triple. Under suitable conditions, the Connes-Chern character $\mathrm{Ch}(D)$ may be represented by a cocycle $\varphi^{\mathrm{CM}} = (\varphi^{\mathrm{CM}}_{2q})$ whose components are given by "heat-kernel techniques". This cocycle is called the CM cocycle.

Proposition (Connes-Moscovici, Ponge)

For a Dirac spectral triple $(C^\infty(M), L^2_g(M, \$), \not{D}_g)$, we have

$$\varphi^{\mathrm{CM}}_{2q}(f^0, \ldots, f^{2q}) = \frac{(2i\pi)^{-n}}{(2q)!} \int_M f^0 df^1 \wedge \cdots \wedge df^{2q} \wedge \hat{A}(R^M),$$

where

$$\hat{A}\left(R^M\right) := \det{}^{\frac{1}{2}} \left[\frac{R^{TM}/2}{\sinh\left(R^{TM}/2\right)} \right].$$

CM Cocycle and Twisted Spectral Triples

Open Question

Construct a version of the CM cocycle for twisted spectral triples.

Remark

Moscovici derived an Ansatz for such a cocycle, but the Ansatz has been verified only for a narrow class of examples.

Conformal Dirac Spectral Triple

Setup

1. M^n is a compact spin oriented manifold (n even).
2. $\mathcal{C}$ is a conformal structure on M.
3. G is a group of conformal diffeomorphisms preserving $\mathcal{C}$. Thus, given any metric $g \in \mathcal{C}$ and $\phi \in G$,

$$\phi_* g = k_\phi^{-2} g \text{ with } k_\phi \in C^\infty(M),\ k_\phi > 0.$$

4. $C_c^\infty(M) \rtimes G$ is the crossed-product algebra, i.e.,

$$C^\infty(M) \rtimes G = \left\{ \sum f_\phi u_\phi;\ f_\phi \in C^\infty(M) \right\},$$
$$u_\phi^* = u_\phi^{-1} = u_{\phi^{-1}}, \qquad u_\phi f = (f \circ \phi^{-1}) u_\phi.$$

Conformal Dirac Spectral Triple

Lemma (Connes-Moscovici)

For $\phi \in G$ define $U_\phi : L^2_g(M, \$) \to L^2_g(M, \$)$ by

$$U_\phi \xi = k_\phi^{-\frac{n}{2}} \phi_* \xi \quad \forall \xi \in L^2_g(M, \$).$$

Then U_ϕ is a unitary operator, and

$$U_\phi D\!\!\!/_g U_\phi^* = \sqrt{k_\phi} D\!\!\!/_g \sqrt{k_\phi}.$$

Theorem (Connes-Moscovici)

The datum of any metric $g \in \mathcal{C}$ defines a twisted spectral triple $\left(C^\infty(M) \rtimes G, L^2_g(M, \$), D\!\!\!/_g\right)_{\sigma_g}$ given by

1. *The Dirac operator $D\!\!\!/_g$ associated to g.*
2. *The representation $fu_\phi \to fU_\phi$ of $C^\infty(M) \rtimes G$ in $L^2_g(M, \$)$.*
3. *The automorphism $\sigma_g(fu_\phi) := k_\phi^{-1} fu_\phi$.*

Conformal Connes-Chern Character

Main Theorem (PW1)

1. The Connes-Chern character $\mathrm{Ch}(\not{D}_g)_{\sigma_g} \in \mathrm{HP}^0(C^\infty(M) \rtimes G)$ is an invariant of the conformal class $\mathcal{C}$.
2. For any even cyclic homology class $\eta \in \mathrm{HP}_0(C^\infty(M) \rtimes G)$, the pairing,
$$\langle \mathrm{Ch}(\not{D}_g)_{\sigma_g}, \eta \rangle,$$
is a scalar conformal invariant.

Definition

The conformal Connes-Chern character $\mathrm{Ch}(\mathcal{C}) \in \mathrm{HP}^0(C^\infty(M) \rtimes G)$ is the Connes-Chern character $\mathrm{Ch}(\not{D}_g)_{\sigma_g}$ for any metric $g \in \mathcal{C}$.

Computation of Ch($\mathcal{C}$)

Theorem (Ferrand, Obata)

If the conformal structure $\mathcal{C}$ is non-flat, then G is a compact Lie group, and so $\mathcal{C}$ contains a G-invariant metric.

Fact

If $g \in \mathcal{C}$ be G-invariant, then $\left(C^\infty(M) \rtimes G, L^2_g(M, \$), \not{D}_g\right)_{\sigma_g}$ is an ordinary spectral triple (equivariant Dirac spectral triple, $\sigma_g = 1$).

Consequence

When $\mathcal{C}$ is non-flat, we are reduced to the computation of the Connes-Chern character of $\left(C^\infty(M) \rtimes G, L^2_g(M, \$), \not{D}_g\right)$ where G is a group of isometries.

Local Index Formula in Conformal Geometry

Setup

- $\mathcal{C}$ is a nonflat conformal structure on M.
- g is a G-invariant metric in $\mathcal{C}$.

Notation

Let $\phi \in G$. Then

- M^ϕ is the fixed-point set of ϕ; this is a disconnected sums of submanifolds.
  $M^\phi = \bigsqcup M_a^\phi, \quad \dim M_a^\phi = a.$
- $\mathcal{N}^\phi = (TM^\phi)^\perp$ is the normal bundle (vector bundle over M^ϕ).

Local Index Formula in Conformal Geometry

Main Theorem (PW2)

Let g be any G-invariant metric in $\mathcal{C}$,

1. The Connes-Chern character $\mathrm{Ch}(\not{D}_g)_{\sigma_g}$ is represented by the CM cocycle $\varphi^{\mathrm{CM}} = (\varphi^{\mathrm{CM}}_{2q})$.
2. We have

$$\varphi^{\mathrm{CM}}_{2q}(f^0 u_{\phi_0}, \cdots, f^{2q} u_{\phi_{2q}}) =$$
$$\frac{(-i)^{\frac{n}{2}}}{(2q)!} \sum_a (2\pi)^{-\frac{a}{2}} \int_{M^\phi_a} \hat{A}(R^{TM^\phi}) \wedge \nu_\phi\left(R^{\mathcal{N}^\phi}\right) \wedge f^0 d\tilde{f}^1 \wedge \cdots \wedge d\tilde{f}^{2q},$$

where $\phi := \phi_0 \circ \cdots \circ \phi_{2q}$, and $\tilde{f}^j := f^j \circ \phi_0^{-1} \circ \cdots \circ \phi_{j-1}^{-1}$, and

$$\nu_\phi\left(R^{\mathcal{N}^\phi}\right) := \det{}^{-\frac{1}{2}}\left[1 - \phi'_{|\mathcal{N}^\phi} e^{-R^{\mathcal{N}^\phi}}\right].$$

Remark

The n-th degree component is given by

$$\varphi_n(f^0 U_{\phi_0}, \cdots, f^n U_{\phi_n}) = \begin{cases} \int_M f^0 d\tilde{f}^1 \wedge \cdots \wedge d\tilde{f}^n & \text{if } \phi_0 \circ \cdots \circ \phi_n = 1, \\ 0 & \text{if } \phi_0 \circ \cdots \circ \phi_n \neq 1. \end{cases}$$

This represents Connes' transverse fundamental class of M/G.

Equivariant CM cocycles

Remark

- When G is a group of isometries, the Connes-Chern character of $\left(C^\infty(M) \rtimes G, L^2_g(M, \not{S}), \not{D}_g\right)$ is computed by using CM or JLO representatives and a differential version of the local equivariant index theorem (Azmi, Chern-Hu).
- We produce a new approach to equivariant heat kernel asymptotics that proves the local equivariant index theorem and computes the JLO cocycle in the same shot. The approach combines
  - Getzler's rescaling.
  - Greiner-Hadamard's approach to the heat kernel asymptotics.

Cyclic Homology of $C^\infty(M) \rtimes G$

Theorem (Brylinski-Nistor, Crainic)

Along the conjugacy classes of G,

$$HP_0(C^\infty(M) \rtimes G) \simeq \bigoplus_{\langle\phi\rangle} \bigoplus_a H^{ev}_{G^\phi}(M^\phi_a),$$

where G^ϕ is the centralizer of ϕ and $H^{ev}_{G^\phi}(M^\phi_a)$ is the G^ϕ-invariant even de Rham cohomology of M^ϕ_a.

Lemma

Any closed form $\omega \in \Omega^*_{G^\phi}$ defines a cyclic cycle η_ω on $C^\infty(M) \rtimes G$ via the transformation,

$$f^0 df^1 \wedge \cdots \wedge df^k \to U_\phi \tilde{f}^0 \otimes \tilde{f}^1 \otimes \cdots \otimes \tilde{f}^k, \qquad f^j \in C^\infty(M^\phi_a)^{G^\phi},$$

where $\tilde{f}^j$ is a G^ϕ-invariant smooth extension of f^j to M.

Conformal Invariants

Main Theorem (PW1)

Assume that the conformal structure $\mathcal{C}$ is non-flat. Then

1. For any closed even form $\omega \in \Omega^{ev}_{G^\phi}(M^\phi_a)$, the pairing $\langle \mathrm{Ch}(\mathcal{C}), \eta_\omega \rangle$ is a conformal invariant.
2. For any G-invariant metric $g \in \mathcal{C}$, we have

$$\langle \mathrm{Ch}(\mathcal{C}), \eta_\omega \rangle = \int_{M^\phi_a} \hat{A}(R^{TM^\phi}) \wedge \nu_\phi \left(R^{\mathcal{N}^\phi} \right) \wedge \omega.$$

Remark

Branson-Orsted proved that for $\omega = 1$ the above integral is independent of the choice of any metric $g \in \mathcal{C}$ preserved by ϕ.

Part 4 Another Application to Conformal Geometry

1. Twisted spectral triples in noncommutative geometry.
2. Index map and Connes-Chern character of a twisted spectral triple.
3. Conformal invariants and local index formulas in conformal geometry.
4. Vafa-Witten inequality for twisted spectral triples.

Vafa-Witten Inequality

Theorem (Vafa-Witten)

Let (M^n, g) be a compact spin Riemannian manifold. Then there exists a constant $C > 0$ such that, for any Hermitian vector bundle E over M and Hermitian connection ∇^E on E, we have

$$|\lambda_1(\not{D}_{\nabla^E})| \leq C,$$

where $\lambda_1(\not{D}_{\nabla^E})$ is the smallest eigenvalue of the coupled Dirac operator $\not{D}_{\nabla^E}$.

Vafa-Witten Inequality: Sketch of Proof

- Pick (F, ∇^F) and $(F', \nabla^{F'})$ so that $F \oplus F' \simeq F^0$ is trivial.
- Two connections on F^0: $\nabla_0 = d$ and $\nabla_1 \simeq \nabla^F \oplus \nabla^{F'}$.
- $T_F := D_{\nabla^E \otimes \nabla_0} - D_{\nabla^E \otimes \nabla_1}$ is bounded, and so by the max-min principle,

$$\left|\lambda_1(\not{D}_{\nabla^E \otimes \nabla_0})\right| \leq \lambda_1(\left|\not{D}_{\nabla^E \otimes \nabla_1}\right|) + \| T_F \|.$$

  Moreover $\| T_F \|$ does not depend on (E, ∇^E).
- We have $|\lambda_1(\not{D}_{\nabla^E})| = \left|\lambda_1(\not{D}_{\nabla^E \otimes \nabla_0})\right|$.
- If $\operatorname{ind} \not{D}_{\nabla^E \otimes \nabla^F} = \dim \ker \not{D}^+_{\nabla^E \otimes \nabla^F} - \dim \ker \not{D}^-_{\nabla^E \otimes \nabla^F} \neq 0$, then

$$\{0\} \subsetneq \ker \not{D}_{\nabla^E \otimes \nabla^F} \subset \ker \not{D}_{\nabla^E \otimes \nabla_1} \quad \text{and} \quad \lambda_1(\left|\not{D}_{\nabla^E \otimes \nabla_1}\right|) = 0.$$

- Thus, $\operatorname{ind} \not{D}_{\nabla^E \otimes \nabla^F} \neq 0 \Longrightarrow |\lambda_1(\not{D}_{\nabla^E})| \leq \| T_F \|$.

Vafa-Witten Inequality: Sketch of Proof

- From
$$\mathrm{ind}\,\not{D}_{\nabla^E \otimes \nabla^F} \neq 0 \Longrightarrow |\lambda_1(\not{D}_{\nabla^E})| \leq \|T\|,$$
the proof is completed by constructing a finite family $(F_1, \nabla^{F_1}), \ldots, (F_N, \nabla^{F_N})$ such that
$$\forall (E, \nabla^E) \quad \exists (F_i, \nabla^{F_i}) \quad \text{such that } \mathrm{ind}\,\not{D}_{\nabla^E \otimes \nabla^{F_i}} \neq 0.$$

- This last step is carried out by using Poincaré duality: There is a natural bilinear pairing $K^0(M) \times K^0(M) \to \mathbb{Z}$,
$$([E],[F]) \longrightarrow \mathrm{ind}\, D_{\nabla^{E \otimes F}}.$$
$\dim K^0(M) \otimes \mathbb{Q} < \infty$ and the pairing is nondegenerate over $\mathbb{Q}$.

Poincaré Duality

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Definition (Connes-Moscovici)

Two ordinary spectral triples $(\mathcal{A}_1, \mathcal{H}, D)$ and $(\mathcal{A}_2, \mathcal{H}, D)$ are in *Poincaré duality* when

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- $[a_1, a_2] = [[D, a_1], a_2] = 0$ for all $a_j \in \mathcal{A}_j$.
- The following bilinear form $(\cdot, \cdot)_D : K_0(\mathcal{A}_1) \times K_0(\mathcal{A}_2) \to \mathbb{Z}$ is *nondegenerate*,

$$(\mathcal{E}_1, \mathcal{E}_2)_D := \operatorname{ind} D_{\nabla^{\mathcal{E}_1 \otimes \mathcal{E}_2}}.$$

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Example

A Dirac spectral triple $\left(C^\infty(M), L^2(M, \$), \not{D}_g\right)$ is in Poincaré duality with itself.

Poincaré Duality

Definition (PW3)

Two twisted spectral triples $(\mathcal{A}_1, \mathcal{H}, D)_{\sigma_1}$ and $(\mathcal{A}_2, \mathcal{H}, D)_{\sigma_2}$ are in *Poincaré duality* when

- $[a_1, a_2] = [[D, a_1]_{\sigma_1}, a_2]_{\sigma_2} = 0$ for all $a_j \in \mathcal{A}_j$.
- The following bilinear form $(\cdot, \cdot)_{D,\sigma} : K_0(\mathcal{A}_1) \times K_0(\mathcal{A}_2) \to \mathbb{Z}$ is *nondegenerate*,

$$(\mathcal{E}_1, \mathcal{E}_2)_{D,\sigma} := \text{ind } D_{\nabla^{\mathcal{E}_1 \otimes \mathcal{E}_2}}.$$

Poincaré Duality. Conformal Deformations

Example (PW3)

Consider the following data:

- $(\mathcal{A}_1, \mathcal{H}, D)$ and $(\mathcal{A}_2, \mathcal{H}, D)$ ordinary spectral triples in Poincaré duality.
- Positive invertible elements $k_j \in \mathcal{A}_j$, $j = 1, 2$.
- Inner automorphisms $\sigma_j(a) = k_j^2 a k_j^{-2}$, $a \in \mathcal{A}_j$.
- $k = k_1 k_2$.

Then the conformal deformations $(\mathcal{A}_1, \mathcal{H}, kDk)_{\sigma_1}$ and $(\mathcal{A}_2, \mathcal{H}, kDk)_{\sigma_2}$ are in Poincaré duality.

Remark

In the special case $k_1 = 1$, the *ordinary* spectral triple $(\mathcal{A}_1, \mathcal{H}, kDk)$ has for Poincaré dual the *twisted* spectral triple $(\mathcal{A}_2, \mathcal{H}, kDk)_{\sigma_2}$.

Poincaré Duality. Further Examples

- Duals of discrete subgroups of Lie groups (Connes).
- Ordinary and twisted spectral triples over noncommutative tori (Connes, PW3).
- Spectral triples describing the Standard Model of particle physics (Chamseddine, Connes, Marcolli).
- Quantum projective line (D'Andrea-Landi).
- Quantum Podleś spheres (Dąbrowski-Sitarz, Wagner).
- Conformal deformations of the above.

Ordinary Spectral Triples

Theorem (PW3)

Let $(\mathcal{A}_1, \mathcal{H}, D)$ *be an ordinary spectral triple such that*

1. $(\mathcal{A}_1, \mathcal{H}, D)$ *has a twisted Poincaré dual* $(\mathcal{A}_2, \mathcal{H}, D)_{\sigma_2}$*.*
2. $\dim K_0(\mathcal{A}) \otimes \mathbb{Q} < \infty$.

Then there is a constant $C > 0$ *such that, for any Hermitian finitely generated projective module* $\mathcal{E}$ *over* $\mathcal{A}_1$ *and any Hermitian connection* $\nabla^{\mathcal{E}}$ *on* $\mathcal{E}$*, we have*

$$|\lambda_1(D_{\nabla^{\mathcal{E}}})| \leq C,$$

where $\lambda_1(D_{\nabla^{\mathcal{E}}})$ *is the smallest eigenvalue of* $D_{\nabla^{\mathcal{E}}}$*.*

Remark

This extends Moscovici's Vafa-Witten inequality for ordinary spectral triples to the case where the Poincaré dual is a *twisted* spectral triple.

Vafa-Witten Inequality in Conformal Geometry

Theorem (PW3)

Let (M, g) be an even dimensional compact Riemannian spin manifold. Then there is a constant $C > 0$ such that, for any conformal factor $k \in C^\infty(M)$, $k > 0$, and any Hermitian vector bundle E equipped with a Hermitian connection ∇^E, we have

$$\left|\lambda_1(D_{\hat{g},\nabla^E})\right| \leq C\|k\|_\infty, \qquad \hat{g} := k^{-2}g,$$

where $\|k\|_\infty$ is the maximum value of k.

Further Results

Remark (PW3)

The main theorem of this paper [PW3] is a more general one where $(\mathcal{A}_1, \mathcal{H}, D)_{\sigma_1}$ is a twisted spectral triple.
As its corollaries, we also obtain versions of Vafa-Witten inequality for

1. Conformal deformations of Connes' spectral triples for duals for cocompact discrete subgroups of semisimple Lie groups.
2. Connes-Tretkoff's twisted spectral triples over noncommutative tori associated to conformal weights (with uniform control on the conformal weights).

Thank You!