Symplectic critical surfaces in Kähler surfaces

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Outline

Let M be a compact Kähler surface, let ω be the Kähler form.

For a compact oriented real surface Σ without boundary which is smoothly immersed in M, one defines, following Chern and Wolfson, the Kähler angle α of Σ in M as

$$\omega|_{\Sigma} = \cos\alpha\, d\mu_{\Sigma},$$

where $d\mu_{\Sigma}$ is the area element of Σ.

As a function on Σ, α is continuous everywhere and is smooth possibly except at the complex or anti-complex points of Σ, i.e. where $\alpha = 0$ or π.

We say that,
Σ is a holomorphic curve if $\cos\alpha \equiv 1$,
Σ is a Lagrangian surface if $\cos\alpha \equiv 0$,
Σ is a symplectic surface if $\cos\alpha > 0$.

Since

$$\cos\alpha d\mu_\Sigma = \omega|_\Sigma,$$

and

$$d\omega = 0,$$

one gets that

$$l := \int_\Sigma \cos\alpha d\mu_\Sigma \text{ is homotopy invariant}.$$

Recall that the area functional is

$$A = \int_\Sigma d\mu_\Sigma.$$

It is clear that

$$\cos\alpha \le 1 \le \frac{1}{\cos\alpha},$$

it follows that

$$\int_\Sigma \cos\alpha d\mu_\Sigma \le A \le L.$$

We have

$$l \le A \le L.$$

We (Han-Li) consider a new functional:

$$L_\beta = \int_\Sigma \frac{1}{\cos^\beta \alpha} d\mu_\Sigma.$$

It is obvious that holomorphic curves minimize the functional if $\beta > 0$.

The first variation formula

Theorem

Let M be a Kähler surface. The first variational formula of the functional L_β is, for any smooth vector field X on Σ,

$$\delta_X L_\beta = -(\beta+1)\int_\Sigma \frac{X\cdot \boldsymbol{H}}{\cos^\beta\alpha}d\mu \quad (2.1)$$

$$+\beta(\beta+1)\int_\Sigma \frac{X\cdot (J(J\nabla\cos\alpha)^\top))^\perp}{\cos^{\beta+3}\alpha}d\mu, \quad (2.2)$$

where $\boldsymbol{H}$ is the mean curvature vector of Σ in M, and $()^\top$ means tangential components of $()$, $()^\perp$ means the normal components of $()$. The Euler-Lagrange equation of the functional L_β is

$$\cos^3\alpha\boldsymbol{H} - \beta(J(J\nabla\cos\alpha)^\top)^\perp = 0. \quad (2.3)$$

Remarks

We call it a β-symplectic critical surface.

$$\cos^3\alpha\mathbf{H}-\beta(J(J\nabla\cos\alpha)^{\top})^{\perp}=0$$

- $\beta=0$, we get minimal surface equation;
- If $\beta\to\infty$, we get $\cos\alpha = constant$.

Proposition

If a β-symplectic critical surface is minimal, then $\cos\alpha\equiv$ Constant.

Let $\{e_1, e_2, v_3, v_4\}$ be a orthonormal frame around $p \in \Sigma$ such that J takes the form

$$J = \begin{pmatrix} 0 & \cos\alpha & \sin\alpha & 0 \\ -\cos\alpha & 0 & 0 & -\sin\alpha \\ -\sin\alpha & 0 & 0 & \cos\alpha \\ 0 & \sin\alpha & -\cos\alpha & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} & (J(J\nabla\cos\alpha)^\top)^\perp \\ = \quad & \cos\alpha\sin^2\alpha\partial_1\alpha v_4 + \cos\alpha\sin^2\alpha\partial_2\alpha v_3 \\ = \quad & \cos\alpha\sin^2\alpha(\partial_1\alpha v_4 + \partial_2\alpha v_3). \end{aligned}$$

Set $\vec{V} = \partial_2\alpha v_3 + \partial_1\alpha v_4$.

Furthermore, we have

$$\begin{aligned}\partial_1 \cos\alpha &= \omega(\bar{\nabla}_{e_1} e_1, e_2) + \omega(e_1, \bar{\nabla}_{e_1} e_2) \\ &= h_{11}^{\alpha} \langle J v_{\alpha}, e_2 \rangle + h_{12}^{\alpha} \langle J e_1, v_{\alpha} \rangle \\ &= (h_{11}^4 + h_{12}^3) \sin\alpha.\end{aligned}$$

Similarly, we can get that,

$$\partial_2 \cos\alpha = (h_{22}^3 + h_{12}^4) \sin\alpha.$$

Note that

$$\partial_i \cos\alpha = -\sin\alpha \partial_i \alpha, \quad \text{for} \quad i = 1, 2.$$

Then

$$\vec{V} = -(h_{22}^3 + h_{12}^4) v_3 - (h_{11}^4 + h_{12}^3) v_4.$$

And consequently the Euler-Lagrange equation of the function L_β is

$$\cos^2\alpha H - \beta \sin^2\alpha V = 0. \tag{2.4}$$

Proposition

For a β-symplectic critical surface with $\beta \geq 0$, the Euler-Lagrange equation is an elliptic system modulo tangential diffeomorphisms of Σ.

Examples in $\mathbf{C}^2$

We consider the β-symplectic critical surfaces of the following form.

$$F(r,\theta) = (r\cos\theta, r\sin\theta, f(r), 0). \tag{2.5}$$

The equation

$$\cos^3\alpha\mathbf{H} = \beta(J(J\nabla\cos\alpha)^\top)^\perp$$

is equivalent to

$$r(1+\beta(f')^2)f'' + (1+(f')^2)f' = 0. \tag{2.6}$$

Set $h = f'$, then (2.6) can be written as

$$r(1+\beta h^2)h' + (1+h^2)h = 0, \tag{2.7}$$

which in turn implies that

$$(r(1+h^2)^{\frac{\beta-1}{2}}h')' \equiv 0. \tag{2.8}$$

For simplicity, we will consider the special solution of the form

$$rh(1+h^2)^{\frac{\beta-1}{2}} \equiv 1, \tag{2.9}$$

which implies that for $r > 0$

$$h(1+h^2)^{\frac{\beta-1}{2}} = \frac{1}{r} > 0. \tag{2.10}$$

- $\beta = 0$, we get the catenoid, minimal surface.
- $\beta = 1$, we get

$$F(u,v) = (v\cos u, v\sin u, -\ln v, 0),$$

$$u \in [0, 2\pi], \quad v > 0.$$

  It is in fact a surface $z = -\frac{1}{2}\log(x^2+y^2)$ in $\mathbf{R}^3$, we consider it as a surface in $\mathbf{C}^2$.
- $\beta \to \infty$, we get the plane.

In fact, for $\beta \geq 1$,

$$\frac{1}{r} = h(1+h^2)^{\frac{\beta-1}{2}} \geq h\left(1+\frac{\beta-1}{2}h^2\right) \geq \frac{\beta-1}{2}h^3,$$

we see that for each $r > 0$,

$$0 \leq h(r) \leq \sqrt[3]{\frac{2}{(\beta-1)r}}. \tag{2.11}$$

This means that $f' = h$ converges to 0 uniformly on each compact subset of $\mathbf{C}^2$. Therefore, we see that f converges to a constant uniformly on each compact subset of $\mathbf{C}^2$.

We prove a Liouville theorem for β-symplectic critical surfaces in $\mathbf{C}^2$.

Theorem

If Σ is a complete β-symplectic critical surface in $\mathbf{C}^2$ with area quadratic growth, and $\cos^2\alpha > \frac{1}{2}$, then it is a holomorphic curve.

Theorem

If Σ is a closed symplectic surface which is smoothly immersed in M with the Kähler angle α, then α satisfies the following equation ,

$$\begin{aligned}\Delta\cos\alpha &= \cos\alpha(-|h^3_{1k}-h^4_{2k}|^2-|h^4_{1k}+h^3_{2k}|^2)\\ &\quad +\sin\alpha(H^4_{,1}+H^3_{,2})-\frac{\sin^2\alpha}{\cos\alpha}(K_{1212}+K_{1234}). \quad (3.1)\end{aligned}$$

where K is the curvature operator of M and $H^\alpha_{,i}=\langle\bar\nabla^N_{e_i}H,v_\alpha\rangle$.

Theorem

Suppose that M is Kähler surface and Σ is a β-symplectic critical surface in M with Kähler angle α, then $\cos\alpha$ satisfies,

$$\begin{aligned}\Delta\cos\alpha &= \frac{2\beta\sin^2\alpha}{\cos\alpha(\cos^2\alpha+\beta\sin^2\alpha)}|\nabla\alpha|^2-2\cos\alpha|\nabla\alpha|^2\\ &\quad -\frac{\cos^2\alpha\sin^2\alpha}{\cos^2\alpha+\beta\sin^2\alpha}Ric(Je_1,e_2). \qquad (3.2)\end{aligned}$$

Corollary

Assume M is Kahler-Einstein surface with scalar curvature K, then $\cos\alpha$ *satisfies,*

$$\begin{aligned}\Delta\cos\alpha &= \frac{2\beta\sin^2\alpha}{\cos\alpha(\cos^2\alpha+\beta\sin^2\alpha)}|\nabla\alpha|^2 - 2\cos\alpha|\nabla\alpha|^2 \\ &\quad -\frac{K}{4}\frac{\cos^3\alpha\sin^2\alpha}{\cos^2\alpha+\beta\sin^2\alpha}. \end{aligned} \tag{3.3}$$

Corollary

Any β*-symplectic critical surface in a Kähler-Einstein surface with nonnegative scalar curvature is a holomorphic curve for* $\beta \geq 0$.

By the equations obtained by Micallef-Wolfson, we see that, on a β-symplectic critical surface we have

$$\frac{\partial \sin \alpha}{\partial \bar{\zeta}} = (\sin \alpha) h,$$

where h is a smooth complex function, ζ is a local complex coordinate on Σ, and consequently, we have

Proposition

A non holomorphic β-symplectic critical surface in a Kähler surface has at most finite complex points.

Theorem

Suppose that Σ is a non holomorphic β-symplectic critical surface in a Kähler surface M. Then

$$\chi(\Sigma)+\chi(v)=-P,$$

and

$$c_1(M)([\Sigma])=-P,$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ, $\chi(v)$ is the Euler characteristic of the normal bundle of Σ in M, $c_1(M)$ is the first Chern class of M, $[\Sigma]\in H_2(M,\mathbf{Z})$ is the homology class of Σ in M, and P is the number of complex tangent points.

Theorem gives a proof of Webster's formula for β-symplectic surfaces:

Corollary

Suppose that Σ is a β-symplectic critical surface in a Kähler surface M. Then

$$\chi(\Sigma) + \chi(\nu) = c_1(M)([\Sigma]).$$

Consider

$$F_{t,\varepsilon} : \Sigma \times (-\delta, \delta) \times (-a, a) \to M$$

with $F_{0,0} = F$, where $F : \Sigma \to M$ is a β-symplectic critical surface. Let

$$\frac{\partial F_{t,0}}{\partial t}\Big|_{t=0} = \mathbf{X}, \quad \frac{\partial F_{0,\varepsilon}}{\partial \varepsilon}\Big|_{\varepsilon=0} = \mathbf{Y}, \quad \text{and} \quad \frac{\partial^2 F_{t,\varepsilon}}{\partial t \partial \varepsilon}\Big|_{t=0,\varepsilon=0} = \mathbf{Z}.$$

Denote

$$\nu_{\beta,t,\varepsilon} = \frac{\det^{(\beta+1)/2}(g_{t,\varepsilon})}{\omega^{\beta}(\partial F_{t,\varepsilon}/\partial x^1, \partial F_{t,\varepsilon}/\partial x^2)}$$

so that

$$L_\beta(\phi_{t,\varepsilon}) = \int_\Sigma \nu_{\beta,t,\varepsilon} dx^1 \wedge dx^2.$$

It is easy to see that

$$\frac{\partial}{\partial t}\Big|_{t=0,\varepsilon=0} g_{ij} = \langle \overline{\nabla}_{e_i}\mathbf{X}, e_j\rangle + \langle e_i, \overline{\nabla}_{e_j}\mathbf{X}\rangle, \tag{4.1}$$

$$\frac{\partial}{\partial \varepsilon}\Big|_{t=0,\varepsilon=0} g_{ij} = \langle \overline{\nabla}_{e_i}\mathbf{Y}, e_j\rangle + \langle e_i, \overline{\nabla}_{e_j}\mathbf{Y}\rangle, \tag{4.2}$$

and

$$\begin{aligned}\frac{\partial^2}{\partial t\partial \varepsilon}\Big|_{t=0,\varepsilon=0} g_{ij} &= \langle \overline{\nabla}_{e_i}\mathbf{Z} + \bar{R}(\mathbf{Y}, e_i)\mathbf{X}, e_j\rangle + \langle e_i, \overline{\nabla}_{e_j}\mathbf{Z} + \bar{R}(\mathbf{Y}, e_j)\mathbf{X}\rangle \\ &\quad + \langle \overline{\nabla}_{e_i}\mathbf{X}, \overline{\nabla}_{e_j}\mathbf{Y}\rangle + \langle \overline{\nabla}_{e_i}\mathbf{Y}, \overline{\nabla}_{e_j}\mathbf{X}\rangle. \end{aligned} \tag{4.3}$$

Here, $\bar{R}$ is the curvature tensor on M.

Assume that $\mathbf{X} = \mathbf{Y}$ is a normal vector field,

$$
\begin{aligned}
= \quad & \frac{\beta+1}{\cos^\beta\alpha} J_0(\mathbf{X}) - \frac{\beta(\beta+1)}{\cos^{\beta+1}\alpha}\overline{\nabla}^\perp_{\nabla\cos\alpha}\mathbf{X} \\
& - \frac{\beta^2(\beta+1)^2}{\cos^{\beta+6}\alpha}\langle \mathbf{X}, (J(J\nabla\cos\alpha)^\top)^\perp\rangle (J(J\nabla\cos\alpha)^\top)^\perp \\
& -2\frac{\beta^2(\beta+1)}{\cos^{\beta+4}\alpha}[\omega(\overline{\nabla}_{e_1}\mathbf{X}, e_2) + \omega(e_1, \overline{\nabla}_{e_2}\mathbf{X})](J(J\nabla\cos\alpha)^\top)^\perp \\
& - \frac{\beta(\beta+1)}{\cos^{\beta+2}\alpha}[\nabla_{e_1}\cos\alpha(J\overline{\nabla}_{e_2}\mathbf{X})^\perp - \nabla_{e_2}\cos\alpha(J\overline{\nabla}_{e_1}\mathbf{X})^\perp] \\
& -\beta(\beta+1)\nabla_{e_1}\frac{\bar{\omega}(\overline{\nabla}_{e_1}\mathbf{X}, e_2) + \bar{\omega}(e_1, \overline{\nabla}_{e_2}\mathbf{X})}{\cos^{\beta+2}\alpha}(Je_2)^\perp \\
& +\beta(\beta+1)\nabla_{e_2}\frac{\bar{\omega}(\overline{\nabla}_{e_1}\mathbf{X}, e_2) + \bar{\omega}(e_1, \overline{\nabla}_{e_2}\mathbf{X})}{\cos^{\beta+2}\alpha}(Je_1)^\perp \\
:= \quad & -\int_\Sigma \langle J_\beta \mathbf{X}, \mathbf{X}\rangle d\mu,
\end{aligned}
$$

Theorem

If we choose $\boldsymbol{X} = x_3e_3 + x_4e_4$ *and* $\boldsymbol{Y} = -J_\nu \boldsymbol{X} = x_4e_3 - x_3e_4$*, then the second variation formula is*

$$
\begin{aligned}
& II_\beta(\boldsymbol{X}) + II_\beta(\boldsymbol{Y}) \\
= \; & -2(\beta+1)\int_\Sigma \frac{R|\boldsymbol{X}|^2\sin^2\alpha}{\cos^\beta\alpha}d\mu \\
& +(\beta+1)\int_\Sigma \frac{|\bar\partial \boldsymbol{X}|^2(2\cos^2\alpha+\beta\sin^2\alpha)}{\cos^{\beta+2}\alpha}d\mu \\
& -(\beta+1)\int_\Sigma \frac{(2\cos^2\alpha+\beta\sin^2\alpha)(\cos^2\alpha+\beta\sin^2\alpha)}{\cos^{\beta+4}\alpha}|\boldsymbol{X}|^2|\nabla\alpha|^2 d\mu.
\end{aligned}
$$

As applications of the stability inequality above, we can obtain some rigidity results for stable β-symplectic critical surfaces.

Corollary

Let M be a Kähler surface with positive scalar curvature R. If Σ is a stable β-symplectic critical surface in M with $\beta \geq 0$, whose normal bundle admits a nontrivial section $\boldsymbol{X}$ with

$$\frac{|\bar{\partial}\boldsymbol{X}|^2}{|\boldsymbol{X}|^2} \leq \frac{\cos^2\alpha + \beta\sin^2\alpha}{\cos^2\alpha}|\nabla\alpha|^2,$$

then Σ is a holomorphic curve.

Corollary

Let M be a Kähler surface with positive scalar curvature R. If Σ is a stable β-symplectic critical surface in M with $\beta \geq 0$ and $\chi(\nu) \geq g$, where $\chi(\nu)$ is the Euler characteristic of the normal bundle ν of Σ in M and g is the genus of Σ, then Σ is a holomorphic curve.

we define the set

$$S: \; = \; \{\beta \in [0,\infty) \mid \exists \; strictly \; stable \; \beta - symplecitc \; critical \; surface \; \Sigma \; with \int_\Sigma |A|^2 d\mu \leq C(s)\}$$

where A is the second fundamental form of Σ in M, and $C(s)$ is a positive continuous function.

Theorem

The set S is open and closed in $[0,\infty)$. In other words, $S = [0,\infty)$.

Convergence?

Conjecture *Let M be a Kähler surface. There is a holomorphic curve in the homotopy class of a symplectic stable minimal surface in M.*

There does exist symplectic stable minimal surfaces which are not holomorphic (Claudi).

Thanks for your attention