On the Bombieri-De Giorgi-Giusti minimal graph and its applications

Juncheng Wei

Department of Mathematics
University of British Columbia

Singapore, December 12, 2014.

The minimal surface equation

$$H_\Gamma := \nabla \cdot \left(\frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = 0 \quad \text{in } \Omega \subset \mathbb{R}^{N-1}.$$

$$\Gamma = \{(x', F(x')) \in \mathbb{R}^{N-1} \times \mathbb{R} \ / \ x' \in \Omega \subset \mathbb{R}^{N-1}\}$$

is a minimal surface (minimal graph) in $\mathbb{R}^N$

Euler-Lagrange equation for the *area functional*

$$A(\Gamma) = \int_\Omega \sqrt{1 + |\nabla F|^2}\, dx'$$

Problem (Bernstein, 1910): Are all (entire) solutions of the minimal surface equation

$$H_\Gamma := \nabla \cdot \left(\frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = 0 \quad \text{in } \mathbb{R}^{N-1}.$$

just linear functions $F(x') = a \cdot x + b$?

Or: Is an entire minimal graph in $\mathbb{R}^N$ necessarily a hyperplane?

True for $N \leq 8$:

- Bernstein (1910), Fleming (1962) $N = 3$
- De Giorgi (1965) $N = 4$
- Almgren (1966), $N = 5$
- Simons (1968), $N = 6, 7, 8$.

False for $N \geq 9$:

• Bombieri-De Giorgi-Giusti found a counterexample (Invent Math 1969).

The BDG minimal graph:

Explicit construction by super and sub-solutions, $N = 9$, of a non-trivial solution of

$$\nabla \cdot \left(\frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = 0 \quad \text{in } \mathbb{R}^8.$$

$$F : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}, \quad (\mathbf{u}, \mathbf{v}) \mapsto F(|\mathbf{u}|, |\mathbf{v}|).$$

In addition, $F(|\mathbf{u}|, |\mathbf{v}|) > 0$ for $|\mathbf{v}| > |\mathbf{u}|$ and

$$F(|\mathbf{u}|, |\mathbf{v}|) = -F(|\mathbf{v}|, |\mathbf{u}|).$$

Polar coordinates:

$$|\mathbf{u}| = r\cos\theta,\ \ |\mathbf{v}| = r\sin\theta, \quad \theta \in (0, \frac{\pi}{2})$$

We have that for large r,

$$F(r,\theta) \approx F_0(r,\theta) = r^3 g(\theta)$$

$$g > 0 \text{ in } (\frac{\pi}{4}, \frac{\pi}{2}], \quad g(\frac{\pi}{2} - \theta) = -g(\theta), \quad g'(\frac{\pi}{2}) = 0.$$

$$g(\theta) \sim \cos(2\theta)$$

and g is such that

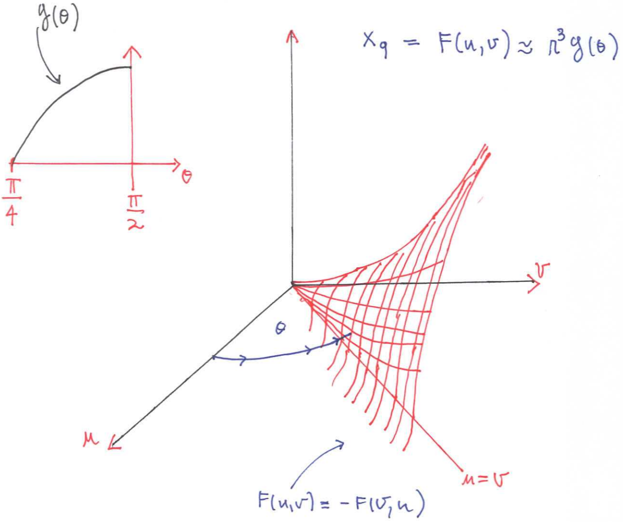
$$\nabla \cdot \left(\frac{\nabla F_0}{|\nabla F_0|} \right) = 0 \quad \text{in } \mathbb{R}^8.$$

Equivalent to an ODE for g

$$\frac{21g\ \sin^3 2\theta}{\sqrt{9g^2+g'^2}} + \left(\frac{g' \sin^3 2\theta}{\sqrt{9g^2+g'^2}}\right)' = 0 \quad \text{in } \left(\frac{\pi}{4}, \frac{\pi}{2}\right),$$

$$g\left(\frac{\pi}{4}\right) = 0 = g'\left(\frac{\pi}{2}\right).$$

This problem has a solution $g > 0$ in $(\frac{\pi}{4}, \frac{\pi}{2}]$.



$g(\theta)$
$\frac{\pi}{4}$
$\frac{\pi}{2}$
θ
$X_q = F(u,v) \approx r^3 g(\theta)$
v
u
θ
$u=v$
$F(u,v) = -F(v,u)$

Asymptotic behavior of F

Asymptotic behavior of BDG surface $x_9 = F(r, \theta)$, $\theta \in (\frac{\pi}{4}, \frac{\pi}{2})$:
$\sigma \in (0, 1)$

$$F(r, \theta) = r^3 g(\theta) + O(r^{-\sigma}) \quad \text{as } r \to +\infty.$$

(del Pino, Kowalczyk, Wei, 2011)

$$F(r, \theta) = r^3 g(\theta) + O(r^{-1}) \quad \text{as } r \to +\infty.$$

(Daskalopoulous, del Pino, Davila, Wei, 2014)

$$H'(F_0)[\phi] = L(\phi) = \nabla \cdot \left(\frac{\nabla \phi}{\sqrt{1+|\nabla F_0|^2}} - \frac{(\nabla F_0 \cdot \nabla \phi)\nabla F_0}{(1+|\nabla F_0|^2)^{\frac{3}{2}}} \right).$$

Degenerate in the direction of ∇F_0.

Key idea: New Orthogonal Coordinate System

$$t = F_0 = r^3 g(\theta)$$

$$s = \left(\frac{r^7 \sin^3(2\theta) g_\theta}{56\sqrt{9g^2 + g_\theta^2}}\right)^{\frac{1}{7}}$$

These coordinates correspond to "geographical" orthogonal coordinates for the graph of F_0. The coordinate t is simply its height and s measures a weighted length along the level sets.

Variation #1: the Allen-Cahn equation, De Giorgi's conjecture

(AC) $$\Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^n$$

Euler-Lagrange equation for the *energy functional*

$$J(u) = \frac{1}{2}\int |\nabla u|^2 + \frac{1}{4}\int (1 - u^2)^2$$

$u = +1$ and $u = -1$ are *global minimizers* of the energy representing, in the gradient theory of phase transitions, two distinct phases of a material. Problem: to find solutions where the two phases ± 1 coexist.

The case $N = 1$.

The function

$$w(t) := \tanh\left(\frac{t}{\sqrt{2}}\right)$$

connects monotonically -1 and $+1$ and solves

$$w'' + w - w^3 = 0, \quad w(\pm\infty) = \pm 1, \quad w' > 0.$$

For any $p, \nu \in \mathbb{R}^N$, $|\nu| = 1$, the functions

$$u(x) := w(z), \quad z = (x - p) \cdot \nu$$

solve equation (AC). $z =$ normal coordinate to the hyperplane through p, unit normal ν.

De Giorgi's conjecture (1978): *Let u be a bounded solution of equation*

$$\text{(AC)} \qquad \Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^N,$$

which is monotone in one direction, say $\partial_{x_N} u > 0$. Then, **at least** *when $N \le 8$, there exist p, ν such that*

$$u(x) = w(\,(x-p)\cdot \nu).$$

This statement is equivalent to:

At least when $N \le 8$, all level sets of u, $[u = \lambda]$ must be hyperplanes.

De Giorgi's Conjecture: *u bounded solution of (AC), $\partial_{x_N} u > 0$ then level sets $[u = \lambda]$ are hyperplanes.*

- True for $N = 2$. Ghoussoub and Gui (Math Ann 1998).
- True for $N = 3$. Ambrosio and Cabré (JAMS 2000).
- True for $4 \leq N \leq 8$ Savin (Ann of Math 2009), if in addition

$$\lim_{x_N \to \pm\infty} u(x', x_N) = \pm 1 \quad \text{for all} \quad x' \in \mathbb{R}^{N-1}.$$

(A new proof by Kelei Wang, Arxiv 2014)

Connection between solutions of (AC) and minimal surfaces arises (Modica-Mortola, 1977).

In entire space (AC) is equivalent to

$$\varepsilon^2 \Delta u + (1 - u^2)u = 0 \quad \text{in } \mathbb{R}^N \qquad (AC)_\varepsilon$$

A sequence of solutions to $(AC)_\varepsilon$ in a bounded domain, which are local minimizers and have suitably bounded energy must approach a function that takes only the values ± 1 in two complementary regions separated by a (generalized) minimal surface.

We take the opposite view:

Given an embedded minimal surface Γ in $\mathbb{R}^N$ that splits its complement into two components $\Omega_\pm$ we want to find u_ε such that

$$\varepsilon^2 \Delta u + (1-u^2)u = 0 \quad \text{in } \mathbb{R}^N \qquad (AC)_\varepsilon$$

and $u_\varepsilon \to \pm 1$ in $\Omega_\pm$.

For Γ a BDG graph in $\mathbb{R}^9$ we find:

Theorem (del Pino, Kowalczyk, Wei; Ann. of Math 2011)

Let Γ be a BDG minimal graph in $\mathbb{R}^9$, ν its upward unit normal. For all small $\varepsilon > 0$, there exists a bounded solution u_ε of $(AC)_\varepsilon$, monotone in the x_9-direction, with

$$u_\varepsilon(x) = \tanh\left(\frac{z}{\sqrt{2}\varepsilon}\right) + O(\varepsilon), \quad x = y + z\nu(y), \quad y \in \Gamma, \ |z| < \delta,$$

$$\lim_{x_9 \to \pm\infty} u_\varepsilon(x', x_9) = \pm 1 \quad \textit{for all} \quad x' \in \mathbb{R}^8.$$

u_ε is a "counterexample" to De Giorgi's conjecture in dimension 9 (hence in any dimension higher).

The infinite dimensional gluing method:

Local coordinates near Γ:

$$x = y + z\nu(y), \quad y \in \Gamma, \ |z| < \delta$$

Then

$$\boxed{\Delta_x \ = \ \partial_{zz} \ + \ \Delta_{\Gamma^z} \ - \ H_{\Gamma_z}(y)\,\partial_z}$$

$$\Gamma^z := \{y + z\nu(y) \ / \ y \in \Gamma\}.$$

Δ_{Γ^z} is the Laplace-Beltrami operator on Γ^z (applied to functions of y) and $H_{\Gamma^z}(y)$ its mean curvature. Let $k_1, \ldots, k_8$ be principal curvatures of Γ. Then

$$\boxed{H_{\Gamma^z} \ = \ \sum_{i=1}^{8} \frac{k_i}{1 - zk_i}}$$

Let $f(u) = u - u^3$ the equation

$$S(u) := \varepsilon^2 \Delta u + f(u) = 0 \quad \text{in } \mathbb{R}^9$$

becomes, for

$$u(y,z) := u(x), \quad x = y + z\nu(y), \quad y \in \Gamma_\varepsilon, \ |z| < \delta.$$

$$S(u) = \varepsilon^2(\partial_z^2 u + \Delta_{\Gamma_z} u - H_{\Gamma^z}(y)\,\partial_z u) + f(u) \ = \ 0.$$

For a small function h defined on Γ (to be determined) we set

$$u_0(y,z) = w\left(\frac{z}{\varepsilon} + \varepsilon h(y)\right)$$

where $w(t) = \tanh(t/\sqrt{2})$, so that $w''(t) + f(w(t)) = 0$

$u_0(y,z) = w(t)$, $t = \frac{z}{\varepsilon} - \varepsilon h(y)$. We compute the error

$$S(u_0) = \varepsilon^4 |\nabla_{\Gamma^z} h(y)|^2 w''(t) - w'(t)(\varepsilon^3 \Delta_{\Gamma^z} h(y) + \varepsilon H_{\Gamma^z}(y))$$

Since $H_{\Gamma^z} = \sum_{i=1}^{8} \frac{k_i}{1-k_i z}$, $H_\Gamma = 0$, $z = \varepsilon(t+\varepsilon h)$, we get

$$\varepsilon H_{\Gamma_z}(y) = \varepsilon^2 (t + \varepsilon h(y)) \underbrace{\sum_{i=1}^{8} k_i(y)^2}_{|A_\Gamma(y)|^2} + \varepsilon^3 (t + \varepsilon h(y))^2 \sum_{i=1}^{8} k_i^3(y) + \cdots$$

$$S(u_0) = -\varepsilon^3 [(\Delta_\Gamma h + |A_\Gamma|^2 h)\, w' + \sum_{i=1}^{8} k_i^3(y) t^2 w'] - \varepsilon^2 |A_\Gamma|^2 t w' + h.o.t$$

We want ϕ so that

$$0 = S(u_0 + \phi(t,y)) \approx \varepsilon^2 \Delta_\Gamma \phi + \phi_{tt} + f'(w(t))\phi + S(u_0)$$

Writing $\phi = \varepsilon^2 \phi_2(t,y) + \varepsilon^3 \phi_3(t,y) + \cdots$ we formally find

$$\partial_t^2 \phi_2 + f'(w(t))\phi_2 = |A_\Gamma(y)|^2 t w'(t)$$

$$\partial_t^2 \phi_3 + f'(w(t))\phi_3 = (\Delta_\Gamma h + |A_\Gamma|^2 h)\, w' + \sum_{i=1}^{8} k_i^3(y) t^2 w'$$

Need solvability conditions:

$$\psi''(t) + f'(w(t))\psi = p(t) \in L^\infty(\mathbb{R})$$

has a bounded solution if and only if $\int_{\mathbb{R}} p w' dt = 0$. Need a condition in h:

To solve for ϕ_3 need that

$$(\Delta_\Gamma h + |A_\Gamma|^2 h) \int_{-\infty}^{\infty} |w'|^2 dt + \sum_{i=1}^{8} k_i^3 \int_{-\infty}^{\infty} t^2 |w'|^2 dt = 0 \ \forall y \in \Gamma$$

This determines h:

$$\Delta_\Gamma h + |A_\Gamma|^2 h = \sum_{i=1}^{8} k_i^3(y) = g(y)$$

The Jacobi operator of Γ can be accurately analyzed thanks to the precise asymptotics we have for Γ. In particular we have at main order $g(y) = \alpha(\theta)/r^3$ and we can essentially solve by a function of the form $h = \beta(\theta)/r$.

An important example for $N = 3$: finite Morse index solutions.

Theorem (del Pino, Kowalczyk, Wei (JDG 2013))

Let Γ *be a complete, embedded minimal surface in* $\mathbb{R}^3$ *with finite total curvature:* $\int_\Gamma |K| < \infty$*,*K *Gauss curvature.*

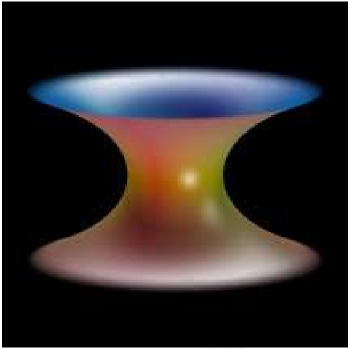
If Γ *is non-degenerate then for small* $\varepsilon > 0$ *there is a solution* u_ε *to* $(AC)_\varepsilon$ *with*

$$u_\varepsilon(x) \approx w(z/\varepsilon), \quad x = y + z\nu(y).$$

In addition $i(u_\varepsilon) = i(\Gamma)$ *where* i *denotes Morse index.*

Examples: nondegeneracy and Morse index are known for the catenoid and Costa-Hoffmann-Meeks surfaces (Nayatani (1990), Morabito, (2008)).

$\Gamma =$ a catenoid: $\quad \exists\ u_\varepsilon(x) = w(z) + O(\varepsilon)$, $x = y + z\nu(y)$.



Catenoid

u_ε axially symmetric: $u_\varepsilon(x) = u_\varepsilon(\sqrt{x_1^2 + x_2^2}\,, x_3)$, x_3 rotation axis coordinate. $i(u_\varepsilon) = 1$

$\Gamma =$ CHM surface genus $\ell \geq 1$:

$\exists\ u_\varepsilon(x) = w(z/\varepsilon) + O(\varepsilon),\ x = y + z\nu(y).\ \ i(u_\varepsilon) = 2\ell + 3.$

The above gluing procedure suggests the following correspondence

$$\boxed{\begin{array}{c}\text{Study of Entire Solutions of} \\ \text{Allen-Cahn Equation} \\ \Delta u + u - u^3 = 0 \\ \text{in } \mathbb{R}^N\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{Theory of} \\ \text{Minimal Surfaces}\end{array}}$$

It is attempting to think about

$$\Delta u + u = u^3 = 0 \quad \sim\!=\!=\!=\!=\!=\!\sim \quad \text{minimal surfaces}$$

but this is not true: Let $N \geq 4$ and consider the N-dimensional catenoid:

$$F = F(r),\ \nabla\left(\frac{\nabla F}{\sqrt{1+|\nabla F|^2}}\right) = 0,$$

$$F \sim 1 + O(r^{2-N})$$

It is known that the $N-$dimensional catenoid has Morse index one.

Agudelo, del Pino, Wei, (J. Math. Pures Appl 2015): For every $N \geq 4$ and any sufficiently small $\epsilon > 0$ there exist a solution u_ϵ to Allen-Cahn equation with $\{u_\epsilon = 0\}$ being the largely dilated catenoid. For $\epsilon > 0$ small and for dimensions $4 \leq N \leq 10$ the Morse index of $u_\epsilon = \infty$.

Variation # 2: Overdetermined semilinear equation in an epigraph

A classical application of the method of moving planes was by Serrin in 1971. He considered the following overdetermined problem: Let Ω be a bounded domain and u be a solution of

$$(S) \qquad \begin{cases} \Delta u + f(u) = 0 \text{ in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \\ \frac{\partial u}{\partial \nu} = \text{Constant on } \partial\Omega \end{cases}$$

Then Ω must be a ball.

Serrin's Overdetermined Problem

We reformulate Serrin's problem: Find a domain Ω such that there exists a solution u to

$$(S) \quad \begin{cases} \Delta u + f(u) = 0 \text{ in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \\ \frac{\partial u}{\partial \nu} = \text{Constant on } \partial\Omega \end{cases}$$

Serrin's result: The only bounded domain is ball.

Serrin's proof was based on the Alexandrov reflection principle, introduced in 1956 by Alexandrov to prove the following famous result:

A compact, connected, embedded hypersurface in RN whose mean curvature is constant, must necessarily be an Euclidean sphere.

The reflection maximum principle based procedure was used in 1979 by Gidas-Ni-Nirenberg to derive radial symmetry results for positive solutions of semilinear equations. The reflection principle, named after Gidas-Ni-Nirenberg as the moving plane method, has become a standard and powerful tool for the analysis of symmetries of solutions of nonlinear elliptic equations.

Serrin's Problem in Unbounded Domains

In this talk, we consider Serrin's problem when Ω is unbounded.
When $f(u) = 0$,

$$\begin{cases} \Delta u = 0 \text{ in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \\ \frac{\partial u}{\partial \nu} = \text{Constant on } \partial\Omega \end{cases}$$

this kind of domains are called exceptional domains and the function u is called root function.
Helen, Hauswirth, Pacard (PJM-2010) derived Weierstrass representation for such domains in $\mathbb{R}^3$.

In this talk, we consider $f \neq 0$ and Ω is unbounded.
A natural situation to consider is epigraphs:

$$\Omega = \{x_N > \varphi(x^{'})\}$$

where $x^{'} = (x_1, ..., x_{N-1})$.

With this notation, Serrin's problem becomes

$$(S) \quad \begin{cases} \Delta u + f(u) = 0 \text{ in } \Omega = \{x_N > \varphi(x')\} \\ u > 0 \quad \text{in } \{x_N > \varphi(x')\}, \\ u = 0 \quad \text{on } \partial\Omega = \{x_N = \varphi(x')\}, \\ \frac{\partial u}{\partial \nu} = C \quad \text{on } \partial\Omega = \{x_N = \varphi(x')\} \end{cases}$$

An Obvious Solution: $\varphi = 0, u = u(x_N)$ which satisfies an ODE

$$u^{''} + f(u) = 0, u(0) = 0, u^{'}(0) = C$$

The question is then: is this the only solution?

Berestycki-Cafferalli-Nirenberg Conjecture (1997):

$\boxed{\text{The epigraph } \{x_N > \varphi(x^{'})\} \text{ is a half plane and } u \text{ depends on } x_N \text{ only.}}$

Previous Results

Yes, if φ satisfies

$$\lim_{|x'|\to+\infty}[\varphi(x'+\tau)-\varphi(x')]=0, \forall\tau\in\mathbb{R}^{N-1}$$

Berestycki, Cafferalli, Nirenberg (CPAM 1997)

Yes, if φ is globally Lipschitz, and $N=2,3$

Farina, Valdinoci (ARMA-2010)

Theorem (del Pino, Pacard, Wei, Arxiv 2014, to appear in Duke Math Journal)

In Dimension $N \geq 9$ there exists a solution to Problem (S) with $f(u) = u - u^3$, in an entire epigraph Ω which is not a half-space.

As before we consider the problem introducing a scaling parameter,

$$\varepsilon^2\Delta u + f(u) = 0, \;\; u > 0 \quad \text{in } \Omega, \;\; u \in L^\infty(\Omega) \qquad (S)_\varepsilon$$

$$u = 0, \quad \partial_\nu u = constant \quad \text{on } \partial\Omega$$

The proof consists of finding the region Ω whose boundary is

$$\partial\Omega = \{y + \varepsilon^2 h(y)\nu(y) \;/\; y \in \Gamma\}.$$

for h a small decaying function on Γ, with Γ a BDG graph.
The construction carries over for more general surfaces Γ

Let us set

$$u_0(y,z) = w\left(\frac{z}{\varepsilon} - \varepsilon h(y)\right) = w(t), \quad x = y + z\nu(y) \quad \Omega = \{t > 0\}.$$

We look for a solution in $t > 0$ with $u(t,y) = w(t) + \phi(t,y)$. Then at main order we should have

$$\partial_{tt}\phi + \varepsilon^2 \Delta_\Gamma \phi + f'(w(t))\phi = E$$

$$\phi(0,y) = 0, \phi_t(0,y) = \alpha \quad \forall y \in \Gamma$$

for a certain constant α.

$$E = \Delta u_0 + f(u_0) \ =$$

$$\varepsilon^4 |\nabla_{\Gamma^z} h(y)|^2 w''(t) - [\varepsilon^3 \Delta_{\Gamma^z} h(y) \ + \varepsilon H_{\Gamma^z}(y)]\, w'(t),$$

$$E = \varepsilon H_\Gamma(y)\, w'(t) + O(\varepsilon^2)$$

Integrating the equation for ϕ against w' we find

$$-w'(0)\phi_t(0,y) = \int_0^\infty E(y,t) w'(t) dt = -\varepsilon H_\Gamma(y) \int_0^\infty w'(t)^2 dt + O(\varepsilon^2)$$

We need

$$H_\Gamma \equiv H = constant$$

Namely Γ should be a **constant mean curvature surface.** Then we solve imposing $\alpha = \varepsilon(H/w'(0)) \int_0^\infty w'(t)^2 dt$.

Let us assume that that Γ is a smooth surface such that

$$H_\Gamma \equiv H = constant$$

The approximation can then be improved as follows:

For $x = y + \varepsilon(t + \varepsilon h(y))$, we look now for a solution for $t > 0$ with

$$u(t,y) = w(t) + \phi(t,y), \quad \phi(0,y) = 0.$$

Imposing $\alpha = (H/w'(0)) \int_0^\infty w'(t)^2 dt$. we can solve

$$\psi'' + f'(w(t))\psi = Hw'(t), \quad t > 0, \quad \psi(0) = 0, \psi'(0) = \alpha$$

which is solvable with ψ bounded. Then the approximation $u_1(x) = w(t) + \varepsilon\psi(t)$ produces a new error of order ε^2. And the equation for $\phi = \varepsilon\psi(t) + \phi_1$ now becomes

$$\partial_{tt}\phi_1 + \Delta_{\Gamma_\varepsilon}\phi_1 + f'(w(t))\phi_1 = E_1 = O(\varepsilon^2)$$

$$\phi_1(0, y) = 0, \phi_{1,t}(0, y) = 0$$

Then we proceed to the adjustment of h.

Let us take the function h to have the following form:

$$h(y) = h_0 + \epsilon h_1 + \epsilon^2 h_2 + \epsilon^3 h_3$$

Then we find, by successive approximation

$$\Delta_\Gamma h_0 + |A_\Gamma|^2 h_0 = \sum_{i=1}^{8} k_i^2, \quad h_0 = c_0$$

$$\Delta_\Gamma h_1 + |A_\Gamma|^2 h_1 = \sum_{i=1}^{8} k_i^3$$

$$\Delta_\Gamma h_2 + |A_\Gamma|^2 h_2 = \sum_{i=1}^{8} k_i^4$$

$$\Delta_\Gamma h_2 + |A_\Gamma|^2 h_3 = |A_\Gamma|^4$$

where $\mathcal{J}[h] = \Delta_\Gamma h + |A_\Gamma|^2$ is the Jacobi operator.

At ∞, $\Delta_\Gamma \sim \Delta, |A_\Gamma|^2 \sim r^{-2}$. Thus we have a Hardy Type operator

$$\mathcal{J} \sim \Delta + \frac{a(\theta)}{r^2}$$

We will show that $\mathcal{J}$ has indicial roots r^{-2} and r^{-3}. On the other hand,

$$k_i = O(\frac{1}{r})$$

$$\sum_i k_i^3 = O(\frac{1}{r^3})$$

$$\sum_i k_i^4 = O(\frac{1}{r^4})$$

$$|A_\Gamma|^4 = O(\frac{1}{r^4})$$

A new linear problem: Neumann to Dirichlet Map

Let us write $\mathbb{R}^9_+ := \mathbb{R}^8 \times (0, \infty)$. We consider the problem of finding, for given functions $g(y,t)$, $\beta(y)$, a solution (α, ϕ) to the problem

$$\begin{aligned} \Delta\phi + f'(w(t))\phi &= \alpha(y)\, w'(t) + g(y,t) \quad \text{in } \mathbb{R}^9_+, \\ \phi(y,0) &= 0 \qquad \text{for all} \quad y \in \mathbb{R}^8, \\ \partial_t \phi(y,0) &= \beta(y) \quad \text{for all} \quad y \in \mathbb{R}^8. \end{aligned}$$

The principle behind Theorem 2 applies, more generally, to domains enclosed by a large dilation of an embedded CMC surface, provided that sufficient information about the surface (such as nondegeneracy) is available.

Theorem (del Pino, Pacard, Wei 2014)

Assume that $\Omega_0 \subset M$ is a smooth bounded domain whose boundary $\partial\Omega_0$ is a non degenerate hypersurface whose mean curvature is constant. Then, Serrin's overdetermined problem is solvable in $\epsilon^{-1}\Omega_O$

Delaunay surfaces, etc.

Variation #3: Translating solutions to the mean curvature flow. $\Sigma(t)$ in $\mathbb{R}^{N+1}$ orientable, embedded evolves by mean curvature if it is parametrized by a family of diffeormorphisms of $\Sigma(0)$ $Y(\cdot, t)$ where

$$\frac{\partial Y}{\partial t} = H_{\Sigma(t)}(Y)\nu(Y) \qquad (MCF)$$

An *eternal* solution is one defined at all times $t \in (-\infty, \infty)$. MCF typically develops singularities in finite time. An eternal solution usually arises as a limit after suitable scalings, blowing-up of the solution near a singularity.

Simplest eternal solutions: *translating solutions*,
A self-translating solution of mean curvature flow with speed $c \in \mathbb{R}$ and direction $\mathrm{e} \in S^{N-1}$ is a solution to MCF of the form

$$\Sigma(t) = ct\mathrm{e} + \Sigma(0) .$$

Graphical self-translating solution $e = e_{N+1}$

$$F(x, t) = ct + F(x)$$

$$\nabla \cdot \left(\frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = \frac{c}{\sqrt{1 + |\nabla F|^2}} \quad \text{in } \mathbb{R}^N. \qquad (MC)_c$$

A Bernstein problem for $(MC)_c$:

(B) Question by X.J. Wang (2009) *Is it true that entire solutions of $(MC)_c$ for $c \geq 0$ need to be convex?*

This statement for $c = 0$ reduces to Bernstein's problem: If F solving $(MC)_c$ was necessarily convex, then so would be $-F$. Hence F would be a linear affine function.

Connected to B. White's result (JAMS 2004): if $N < 7$ blowing up of a mean convex flow around a singularity leads to a convex surface.

True for $N = 2$ (X.-J. Wang Ann. of Math 2011). Solutions are radial.

Examples of self-translating graphs:

- A unique radially symmetric solution (for $c = 1$, $N \geq 2$)

$$F(|x|) = \frac{|x|^2}{2(N-1)} - \log|x| + O(|x|^{-1}) \quad \text{as } |x| \to \infty.$$

- X.-J. Wang Ann. of Math 2011: Examples for $N \geq 3$ of convex, non-radial solutions.

The answer to **(B)** is **negative** for $c > 0$ and $N \geq 8$, in analogy to the result of Bombieri, De Giorgi and Giusti:

Theorem (Daskalopoulos, Dávila, del Pino, Wei (2014))

Assume that $N \geq 8$. Then there exists a non-convex entire solution to the equation

$$\nabla \cdot \left(\frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = \frac{1}{\sqrt{1 + |\nabla F|^2}} \quad \textit{in } \mathbb{R}^N.$$

$$F(r, \theta) = r^3 g(\theta) + r^2 \beta(\theta) + O(r) \quad \text{as } r \to \infty.$$

Replacing $F_\varepsilon(x)$ with $\varepsilon^{-1}F_\varepsilon(\varepsilon x)$ we are reduced to finding a non-convex solution F_ε of the equation

$$\nabla \cdot \left(\frac{\nabla F}{\sqrt{1+|\nabla F|^2}} \right) = \frac{\varepsilon}{\sqrt{1+|\nabla F|^2}} \quad \text{in } \mathbb{R}^N. \qquad (MCG)_\varepsilon$$

When $\varepsilon = 0$ this is the equation of minimal graph:

$$\nabla \cdot \left(\frac{\nabla F}{\sqrt{1+|\nabla F|^2}} \right) = 0 \quad \text{in } \mathbb{R}^{N-1}$$

The method: construction of ordered sub and super solutions for the equation

$$M[F] := \nabla \cdot \left(\frac{\nabla F}{\sqrt{1 + |\nabla G|^2}} \right) - \frac{\varepsilon}{\sqrt{1 + |\nabla F|^2}} \ = \ 0 \quad \text{in } \mathbb{R}^8,$$

The equation $M[\bar{F}+\varphi]=0$ is at main order, for r large,

$$L_{\bar{F}}[\varphi]=\frac{\varepsilon}{\sqrt{1+|\nabla F_0|^2}}\approx\frac{\varepsilon p_1(\theta)}{r^2}$$

We can solve by barriers equations of the form

$$L_{\bar{F}}[\varphi]=\mathrm{g}=O(r^{-4-\sigma}).$$

where $\sigma>0$. The barrier procedure however does not work for decays $O(r^{-4})$ or slower, and the main error term only has decay $O(r^{-2})$.

To overcome this difficulty, we need to improve the approximation:

There is a smooth function $\varphi_*(r,\theta) = O(\varepsilon r^2)$ as $r \to \infty$ such that for some $\sigma > 0$

$$M[F + \varphi_*] \; = \; O(r^{-4-\sigma}).$$

The function $\varphi_*(r,\theta)$ is found by setting first

$$\varphi_*(r,\theta) = \varepsilon\varphi_1(r,\theta) + \varepsilon^2\varphi_2(r,\theta) + \varepsilon^3\varphi_2(r,\theta) + \cdots$$

and solving (explicitly, up to fast decaying terms) the linear equations for the first 3 coefficients (which at main order separate variables).

Neck-connection

Near the Simons cone (the "neck part"), we use entire solutions to the heat equation to connect: we need to find an entire solution to the heat equation

$$h_t - h_{xx} = 0; x > 1; -\infty < t < +\infty$$

such that

$$h(x;t) \sim t^{\frac{2}{3}} \quad \text{as } t \to +\infty$$

$$h(x;t) \sim -t^{\frac{2}{3}} \quad \text{as } t \to -\infty$$

This and a refinement of the asymptotic behavior of $\bar{F} - F_0$ yields the result.

Thanks for your attention!