Spin manifolds and proper group actions

Workshop on Positive Curvature and Index Theory

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References

[HM-14]

Peter Hochs and V. M.,
Spin manifolds and proper group actions,
10 pages, [arXiv:math/1411.0781].

Outline of talk

1. Summary of results.

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4. Outlook and relation to elliptic and Witten type genera.

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Summary of results

Recall the following remarkable result.

Theorem (Atiyah-Hirzebruch, 1970)

Let N be a compact, connected even dimensional manifold and K be a compact connected Lie group acting smoothly and non-trivially on N. Suppose also that N has a K-invariant Spin structure. Then the equivariant index of the Dirac operator on N vanishes in the representation ring of K,

$$\mathrm{Index}_K(\not\partial_N) = 0 \quad \in R(K). \tag{0.1}$$

In particular, $\int_N \hat{A}(N) = 0$.

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In particular, $\displaystyle\int_N \hat{A}(N) = 0$.

Their result then inspired many, especially Witten who studied two-dimensional quantum field theories and the index of the Dirac operator on free loop space LN.

Summary of results

Witten related this index to the rigidity of certain Dirac-type operators on N and the elliptic genus, which was proved in [Bott-Taubes], [Taubes]. Since then, there have been various generalisations to many interesting new variants, some by researchers in the audience.

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Our goal in this note is to extend the theorem to the new non-compact setting.

More precisely, let M be a complete Riemannian manifold, on which a connected Lie group G acts properly and isometrically. Suppose M/G is compact. Suppose M has a G-equivariant Spin-structure. Let

$$\mathrm{Index}_G(\partial\!\!\!/_M) \in K_\bullet(C_r^* G)$$

be the equivariant index of the associated Spin-Dirac operator.

Summary of results

Here $K_\bullet(C_r^* G)$ is the K-theory of the reduced group C^*-algebra of G, and Index_G denotes the analytic assembly map used in the Baum–Connes conjecture [Baum-Connes-Higson], [Kasparov]. If G is compact, then $K_\bullet(C_r^* G) = R(G)$, and the analytic assembly map is the usual equivariant index.

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Atiyah and Hirzebruch's result generalises as follows.

Theorem (Hochs-VM)

If there is a point in M whose stabiliser in G is not a maximal compact subgroup of G, then

$$\mathrm{Index}_G(\partial\!\!\!/_M) = 0,$$

if G has a property (*).

Summary of results

From this theorem, we will deduce vanishing of characteristic classes related to the $\hat{A}$ class, as well as an application of it. In another Corollary, we give an equivalent statement of Theorem (Hochs-VM) that does not use $C_r^* G$ or the analytic assembly map.

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There are many group actions that satisfy the hypotheses of Theorem (Hochs-VM). Indeed, let $K < G$ be a maximal compact subgroup, and suppose that G has the property mentioned in Theorem (Hochs-VM). Then if K acts on a compact Spin-manifold N as in the Atiyah-Hirzebruch Theorem, then Theorem (Hochs-VM) applies to the action by G on the fibred product $G \times_K N$, as we will see.

Summary of results

If $K = S^1$, then it is proved in the theorem in Section 2.3 in [Atiyah-Hirzebruch] that any compact oriented manifold X with $\int_X \hat{A}(X) = 0$ has the property that mX (for some $m \in \mathrm{N}$) is oriented cobordant to a compact Spin manifold N which has a non-trivial S^1-action on each of its components. Then the action by K on N satisfies the hypotheses of the Atiyah-Hirzebruch Theorem, so that the to the action by G on $G \times_K N$ satisfies the conditions of Theorem (Hochs-VM).

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Note that if N is a compact Spin-manifold with the trivial K-action, the action by G on $G/K \times N$ does ***not*** satisfy the hypotheses of Theorem (Hochs-VM): all stabilisers are conjugate to K.

Outline of proof

Let M be a smooth manifold, and let G be a connected Lie group acting properly and co-compactly on M. Then roughly speaking, the equivariant K-homology $K_\bullet^G(M)$ consists of equivalence classes of G-invariant elliptic operators, for example the Dirac operator associated to a G-invariant Riemannian metric which is G-spin.

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Kasparov's induction to the crossed product is a certain canonical morphism,

$$j_G : K^G_\bullet(M) \to KK_\bullet(C_0(M) \rtimes G, C^*_r(G)).$$

where $KK_\bullet(A, \mathbb{C})$ is the K-homology of A and $KK_\bullet(\mathbb{C}, A)$ is the K-theory of A.

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where $KK_\bullet(A, \mathbb{C})$ is the K-homology of A and $KK_\bullet(\mathbb{C}, A)$ is the K-theory of A. Moreover there is an intersection product,

$$KK_\bullet(\mathbb{C}, A) \times KK_\bullet(A, B) \to KK_\bullet(\mathbb{C}, B)$$

Outline of proof

There is a dempotent in $C_0(M) \rtimes G$ defined as,

$$e(g,x) = \sqrt{c(x)c(g^{-1}.x)}, \qquad x \in M, g \in G,$$

where c is a cutoff function, that is a non-negative function satisfying for all $x \in M$,

$$\int_G c(g^{-1}x)\mathrm{d}g = 1$$

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Then $[e] \in KK_0(\mathbb{C}, C_0(M) \rtimes G)$ is independent of the choice of c by convexity.

Finally by composing induction to the crossed product and the intersection product with $[e]$, we get the equivariant index

$$\mathrm{Index}_G(\xi) = j_G(\xi) \otimes_{C_0(M)\rtimes G} [e] \in K_\bullet(C_r^*(G)).$$

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Theorem (Abels slice theorem for proper actions)

There is a smooth, K-invariant submanifold $N \subset M$, such that the map $[g, n] \mapsto gn$ is a G-equivariant diffeomorphism

$$G \times_K N \cong M \tag{0.2}$$

Here the left hand side is the quotient of $G \times N$ by the action by K given by

$$k \cdot (g, n) = (gk^{-1}, kn),$$

for $k \in K$, $g \in G$ and $n \in N$.

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for $k \in K$, $g \in G$ and $n \in N$.

We call (0.2) an associated Abels fibration of M, as it is a fibre bundle over G/K with fibre N. From now on, fix a choice of N as in Abels'Theorem.

Outline of proof

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Lemma

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$$M_{(K)} = G \cdot N^K \cong G/K \times N^K,$$

where $M_{(K)}$ = set of points in M with stabilisers conjugate to K.

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Lemma

One has

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where $M_{(K)}$ = set of points in M with stabilisers conjugate to K.

Proof.

Let $m \in M_{(K)}$, and write $m = [g, n]$ for $g \in G$ and $n \in N$, under the correspondence (0.2). Then $G_m = gK_ng^{-1}$. So G_m is conjugate to K if and only if K_n is. Since $K_n < K$, it is conjugate to K precisely if it equals K. $\square$

Outline of proof

Now fix a K-invariant inner product on the Lie algebra $\mathfrak{g}$ of G, and let $\mathfrak{p} \subset \mathfrak{g}$ be the orthogonal complement to the Lie algebra $\mathfrak{k}$ of K. Suppose $\mathrm{Ad} : K \to \mathrm{SO}(\mathfrak{p})$ lifts to

$$\widetilde{\mathrm{Ad}} : K \to \mathrm{Spin}(\mathfrak{p}). \tag{0.3}$$

This is always possible if one replaces G by a double cover.

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This is always possible if one replaces G by a double cover. Indeed, consider the diagram

$$\begin{tikzcd} \widetilde{K} \arrow[r, "\widetilde{\mathrm{Ad}}"] \arrow[d, "\pi_K"'] & \mathrm{Spin}(\mathfrak{p}) \arrow[d, "\pi"', "2:1"] \\ K \arrow[r, "\mathrm{Ad}"] & \mathrm{SO}(\mathfrak{p}), \end{tikzcd}$$

where

$$\widetilde{K} := \{(k, a) \in K \times \mathrm{Spin}(\mathfrak{p}); \mathrm{Ad}(k) = \pi(a)\};$$

$$\pi_K(k, a) := k; \qquad \widetilde{\mathrm{Ad}}(k, a) := a,$$

for $k \in K$ and $a \in \mathrm{Spin}(\mathfrak{p})$.

Outline of proof

Then for all $k \in K$,

$$\pi_K^{-1}(k) \cong \pi^{-1}(\mathrm{Ad}(k)) \cong \mathbb{Z}_2,$$

so π_K is a double covering map. Since G/K is contractible, $\widetilde{K}$ is the maximal compact subgroup of a double cover of G.

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This explains the relevant assumption in Theorem (Hochs-VM).

Outline of proof

Suppose M has a G-equivariant Spin-structure $P_M \to M$. In [Hochs-VM], an induction procedure of equivariant Spin^c-structures from N to M is described, and which can be adapted to the Spin setting, which we will denote by Ind_N^M.

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We will use the fact that any G-equivariant Spin-structure on M can be obtained via this induction procedure.

Lemma

There is an induced K-equivariant Spin*-structure $P_N \to N$ such that*

$$P_M = \mathrm{Ind}_N^M(P_N).$$

Outline of proof

Suppose M/G is compact.
The ***quantisation commutes with induction*** techniques of [Hochs][Hochs-VM], suitably adapted to the Spin-setting, allow us to deduce our main result from Atiyah and Hirzebruch's Theorem.

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The ***quantisation commutes with induction*** techniques of [Hochs][Hochs-VM], suitably adapted to the Spin-setting, allow us to deduce our main result from Atiyah and Hirzebruch's Theorem. This involves the

Theorem (Chabert-Echterhoff-Nest)

The Dirac induction map

$$\text{D-Ind}_K^G : R(K) \longrightarrow K_\bullet(C_r^* G),$$

is an isomorphism for almost connected Lie groups

We will use the fact that it relates the equivariant indices of the Spin-Dirac operators $\partial\!\!\!/_N$ on N and $\partial\!\!\!/_M$ on M, associated to the Spin-structures P_N and P_M, respectively, to each other.

Outline of proof

Theorem (Spin Quantisation commutes with Induction)

The following diagram commutes:

$$\begin{tikzcd} K^G_\bullet(M) \arrow[r, "\mathrm{Index}_G"] & K_\bullet(C^*G) \\ K^K_\bullet(N) \arrow[u, "\text{K-Ind}_K^G"] \arrow[r, "\mathrm{Index}_K"] & R(K). \arrow[u, "\text{D-Ind}_K^G"'] \end{tikzcd}$$

That is,

$$\text{D-Ind}_K^G\big(\mathrm{Index}_K(\partial\!\!\!/_N)\big) = \mathrm{Index}_G(\partial\!\!\!/_M) \quad \in K_\bullet(C^*_rG).$$

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That is,

$$\text{D-Ind}_K^G\big(\mathrm{Index}_K(\partial\!\!\!/_N)\big) = \mathrm{Index}_G(\partial\!\!\!/_M) \quad \in K_\bullet(C_r^*G).$$

Here $K_\bullet^K(N)$ and $K_\bullet^G(M)$ are the equivariant K-homology groups of N and M, respectively. Then there is a map

$$\text{K-Ind}_K^G : K_\bullet^K(N) \to K_\bullet^G(M), \quad \text{with} \quad \text{K-Ind}_K^G[\partial\!\!\!/_N] = [\partial\!\!\!/_M].$$

Outline of proof

Proof of Theorem (Hochs-VM). In the setting of Theorem (Hochs-VM)., let $N \subset M$ be as in Abels' Theorem. Consider the induced K-equivariant Spin-structure on N. By Proposition (Spin Quantisation commutes with Induction) we have

$$\mathrm{Index}_G(\partial\!\!\!/_M) = \text{D-Ind}_K^G\big(\mathrm{Index}_K(\partial\!\!\!/_N)\big).$$

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Proof of Theorem (Hochs-VM). In the setting of Theorem (Hochs-VM)., let $N \subset M$ be as in Abels' Theorem. Consider the induced K-equivariant Spin-structure on N. By Proposition (Spin Quantisation commutes with Induction) we have

$$\mathrm{Index}_G(\partial\!\!\!/_M) = \text{D-Ind}_K^G\big(\mathrm{Index}_K(\partial\!\!\!/_N)\big).$$

The stabiliser of a point $m \in M$ is a maximal compact subgroup of G if and only if $m \in M_{(K)}$. Hence, by Lemma 4, the condition on the stabilisers of the action by G on M is equivalent to the action by K on N being nontrivial. So the Atiyah-Hirzebruch Theorem implies that

$$\mathrm{Index}_K(\partial\!\!\!/_N) = 0,$$

and the result follows. $\square$

Consequences

Let $c \in C_c^\infty(M)$ be a cutoff function, that is a non-negative function satisfying

$$\int_G c(g^{-1}m)\mathrm{d}g = 1$$

for all $m \in M$.

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for all $m \in M$.
Let $\tau : C^*G \to \mathbb{C}$ be the **von Neumann trace** determined by

$$\tau\big(R(f)^*R(f)\big) = \int_G |f(g)|^2 \mathrm{d}g,$$

for $f \in L^1(G) \cap L^2(G)$, where R denotes the right regular representation. This induces a morphism $\tau_* : K_\bullet(C^*G) \to \mathbb{R}$.

Consequences

The following fact follows immediately from Theorem (Hochs-VM) and Theorem 6.12 in [Hang Wang].

Corollary

Under the hypotheses of Theorem (Hochs-VM), one has

$$0 = \tau_*(\mathrm{Index}_G(\partial\!\!\!/_M)) = \int_M c \cdot \hat{A}(M). \tag{0.4}$$

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Note that the right hand side of (0.4) is independent of the choice of cutoff function c, cf. [Hang Wang].

Consequences

As an application of Corollary 8, one has the following generalisation of the second theorem in Section 2.2 of [Atiyah-Hirzebruch].

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Corollary

Let M be a complete, connected, oriented Riemannian manifold with $w_2(M) = 0$ and suppose that $\int_M c \cdot \hat{A}(M) \neq 0$. Then any closed subgroup G (in the compact–open topology) of the group of all orientation preserving isometries of M is a discrete group, if there is a point in M whose stabiliser in G is not a maximal compact subgroup of G.

Consequences

Proof.

In this setting, the Myer–Steenrod theorem implies that G is a Lie group. The action on M by the identity component G_0 of G satisfies the conditions of Theorem (Hochs-VM). So if G_0 is nontrivial, then $\int_M c \cdot \hat{A}(M) = 0$ by an earlier Corollary. $\square$

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Proof.

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Let G be a connected Lie group acting properly on a manifold M. Then by Abels' Theorem, there is a proper equivariant projection map $p : M \to G/K$, where K is a maximal compact subgroup of G. The map p_* induced on K-homology relates the equivariant indices on M and G/K by the diagram

$$\begin{tikzcd}
K_\bullet^G(M) \arrow[r, "\mathrm{Index}_G"] \arrow[d, "p_*"'] & K_\bullet(C_r^* G). \\
K_\bullet^G(G/K) \arrow[ur, "\cong", "\mathrm{Index}_G"'] &
\end{tikzcd}$$

Consequences

It was shown in [Chabert-Echterhoff-Nest], Theorem 1.1, that the equivariant index on G/K defines an isomorphism $K^G_\bullet(G/K) \cong K_\bullet(C^*_r G)$. Using this, we deduce an equivalent statement of Theorem (Hochs-VM) that does not use $C^*_r G$ or the equivariant index.

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Corollary

Under the hypotheses of Theorem (Hochs-VM), one has

$$p_*[\partial\!\!\!/_M] = 0 \quad \in K^G_\bullet(G/K).$$

Define $\hat{A}_c(M;E) = \int_M c \cdot \hat{A}(M) Ch(E)$ where c is a cutoff function. Then it can be shown that $\hat{A}_c(M;E)$ is independent of the choice of c.

Outlook and relation to elliptic and Witten genera

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Define the corresponding Witten type genus as

$$\varphi_W^c(M) = \hat{A}_c(M; \bigotimes_{n=1}^{\infty} S_{q^n}(TM))\lambda(g) \in \mathbb{R}[[q]]$$

where $\lambda(g) = \prod_{n=1}^{\infty}(1-q^n)^{4k}$ and $S_t(E) = \sum_j S^j(E)t^j$ denote the symmetric powers of E.

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That is,

$$\varphi_W^c(M) = \left(\hat{A}_c(M) + \hat{A}_c(M;TM)q + \cdots\right)\lambda(g) \in \mathbb{R}[[q]]$$

and more generally,

$$\varphi^c_{W,G}(M) = \mathrm{Index}_G(\not{\partial}_M \bigotimes_{n=1}^{\infty} S_{q^n}(TM))\lambda(g) \in K_\bullet(C^*_r(G))[[q]]$$

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Then $\tau_*\left(\varphi^c_{W,G}(M)\right) = \varphi^c_W(M)$, where $\tau : C^*_r(G) \to \mathbb{C}$ denotes the von Neumann trace.

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Conjecture

Under the hypotheses listed earlier, together with the assumption that M is a G-string manifold, one has $\varphi^c_{W,G}(M) = 0 = \varphi^c_W(M)$.