# Rigidity Results for Elliptic PDEs

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# Ennio De Giorgi

Research Area:

- Minimal Surfaces and Bernstein's problem
- Geometric Measure Theory
- Regularity of Solutions of Elliptic Equations (Hilbert's 19th problem with John Nash 56-57)
- Γ-Convergence Theory

Awards:

- Caccioppoli Prize (1960)
- Wolf Prize (1990)

Quote:
If you can't prove your theorem, keep shifting parts of the conclusion to the assumptions, until you can.

Figure : 1928-1996

Allen-Cahn Equation:

$$-\Delta u = u - u^3 \quad \text{in} \quad \mathbb{R}^n.$$

Euler-Lagrange equation for the energy functional:

$$E(u) = \frac{1}{2}\int |\nabla u|^2 + \frac{1}{4}\int (1-u^2)^2$$

$u = 1$ and $u = -1$ are global minimizers of the energy and representing, in the gradient theory of phase transitions, two distinct phases of a material.

$$F(u) = -\frac{1}{4}(1-u^2)^2$$

is called "double-well potential":

$$F(+1) = F(-1) = 0 \text{ and } F(u) \neq 0 \text{ if } u \neq \pm 1$$

Example: In dimension one $w(x) = \tanh\left(\frac{x}{\sqrt{2}}\right)$ solves the equation and $w' > 0$ and $w$ connects -1 to 1 that is $w(\pm\infty) = \pm 1$.

Ennio De Giorgi (1978) expected that the interface between the phases $u = 1$ and $u = -1$ has to approach a minimal surface.
Bernstein's Conjecture: Any minimal surface in $\mathbb{R}^n$ must be a hyperplane.
Equivalent to any entire solution of the form $x_n = F(x_1, \cdots, x_{n-1})$ of

$$\nabla \cdot \left( \frac{\nabla F}{\sqrt{1 + |\nabla F|^2}} \right) = 0 \text{ in } \mathbb{R}^{n-1}$$

must be a linear function that is $F(x_1, \cdots, x_{n-1}) = a \cdot (x_1, \cdots, x_{n-1}) + b$ for some $a \in \mathbb{R}^{n-1}$ and $b \in \mathbb{R}$.
True for $n \leq 8$: Bernstein (1910 Math Z), Fleming (1962 Math Palermo), De Giorgi (1965 Annali Pisa), Almgren (1966 Annals Math), Simons (1968 Annals Math).
False for $n \geq 9$: counterexample by Bombieri-De Giorgi-Giusti (1969 Invent Math).
This led him to state his conjecture.

Allen-Cahn equation
Lane-Emden equation

De Giorgi's Conjecture (1978): Suppose that $u$ is bounded and monotone (in one direction) solution of the Allen-Cahn equation

$$-\Delta u = u - u^3 \quad \text{in} \quad \mathbb{R}^n$$

Then, at least for $n \leq 8$, solutions are one-dimensional, i.e. $u(x) = u^*((x-\nu)\cdot p)$ for some $\nu, p$.
$\Longrightarrow u(x) = \tanh\left(\frac{x.a-b}{\sqrt{2}}\right)$ where $b \in \mathbb{R}$, $|a| = 1$ and $a_n > 0$.

- For $n = 2$ by Ghoussoub-Gui (1997 Math Ann)
- For $n = 3$ by Ambrosio-Cabré (2000 J. AMS)
- For $n = 4, 5$, if $u$ is anti-symmetric, by Ghoussoub-Gui (2003 Annals Math)
- For $4 \leq n \leq 8$, if $u$ satisfies the additional (natural) assumption

  $$\lim_{x_n \to \pm\infty} u(\mathbf{x}', x_n) \to \pm 1. \quad \text{Savin (2003 Annals Math)}$$

  2nd Proof: Wang (2014)
- Counterexample for $n \geq 9$, by del Pino-Kowalczyk-Wei (2008 Annals Math)

Note: In lower dimensions, it is proved for any nonlinearity $-\Delta u = f(u)$. For $n = 2$ the same paper and for $n = 3$ by Alberti-Ambrosio-Cabré (2001 Acta Appl. Math.)

# Observations to prove the De Giorgi's conjecture

Consider PDE:

$$-\Delta u = f(u) \qquad x \in \mathbb{R}^n.$$

1. Monotonicity $\Longrightarrow$ Pointwise Stability $\Longleftrightarrow$ Stability.
   - Pointwise Stability: $\exists \phi > 0$ that
     $$-\Delta \phi = f'(u)\phi \quad \text{in} \quad \mathbb{R}^n.$$
   - Stability (or Stability Inequality): if the second variation of the energy is non-negative:
     $$\int f'(u)\zeta^2 \leq \int |\nabla \zeta|^2 \qquad \forall \zeta \in C_c^2(\mathbb{R}^n)$$
2. Set $\phi := \partial_{x_n} u$ and $\psi := \partial_{x_i} u$, then the quotient $\sigma = \frac{\psi}{\phi}$ satisfies a linear equation $\operatorname{div}(\phi^2 \nabla \sigma) = 0$.
   It is shown by Berestycki-Caffarelli-Nirenberg, Ambrosio-Cabre and Ghoussoub-Gui in 97-98 that if $\phi > 0$ and
   $$\int_{B_R} \phi^2 \sigma^2 < R^2, \qquad \forall R > 1$$
   then $\sigma = 0$.

Note: Is this optimal? Consider $R^{a_n}$ then

- $a_n < n$ where $n \geq 3$. Barlow (1998 Can J Math)
- $a_n < 2 + 2\sqrt{n-1}$ when $n \geq 7$. Ghoussoub-Gui (1998 Math Ann)
- IF $a_n \geq n-1$ then conjecture would establish in $n$-D.

- Modica's estimate:

$$|\nabla u|^2 \leq 2F(u) \text{ for bounded solutions of } \Delta u = f(u) \text{ in } \mathbb{R}^n$$

when $F' = f$ and $F \geq 0$ by Modica (1980 CPAM).
Ex.: $|\nabla u|^2 \leq \frac{1}{2}(u^2 - 1)^2$ for the Allen-Cahn equation.
Ex.: $|\nabla u|^2 \leq 2(1 - \cos u)$ for bd solutions of $\Delta u = \sin u$

- Monotonicity Formula:

$$\Gamma_R = \frac{1}{R^{n-1}} \int_{B_R} \frac{1}{2}|\nabla u|^2 + F(u)$$

is nondecreasing in $R$.

- 2nd proof in $n = 2$ by Farina-Sciunzi-Valdinoci (2008 Ann. Pisa) via

$$\int_{\mathbb{R}^n \cap \{|\nabla u| \neq 0\}} \left( |\nabla u|^2 \mathcal{A}^2 + |\nabla_T |\nabla u||^2 \right) \eta^2 \leq \int_{\mathbb{R}^n} |\nabla u|^2 |\nabla \eta|^2$$

For any $\eta \in C^1_c(\mathbb{R}^n)$. How?

  - Test stability on $|\nabla u|\eta$.
  - Apply a geometric identity by Sternberg-Zumbrun (1998 ARMA): For any $w \in C^2$ and where $|\nabla w| > 0$;

$$\sum_{k=1}^{n} |\nabla \partial_{x_k} w|^2 - |\nabla |\nabla w||^2 = |\nabla w|^2 (\sum_{l=1}^{n-1} \kappa_l^2) + |\nabla_T |\nabla w||^2$$

$\kappa_l$ are the principal curvatures of the level set of $w$.

- If the limit

$$\lim_{x_n \to \pm\infty} u(\mathbf{x}', x_n) \to \pm 1 \quad \text{in} \quad \mathbf{R}^{n-1}$$

is uniform $\Longrightarrow$ called Gibbon's conjecture and proved (1999) in all dimensions by Farina (Mat e Appli), Barlow-Bass-Gui (CPAM), Berestycki-Hamel-Monneau (Duke Math)

- Stability Conjecture: Let $u$ be a bounded stable solution of Allen-Cahn equation. Then the level sets $u = \lambda$ are all hyperplanes.
  - True in $n = 2$ by Ambrosio-Cabre and Ghoussoub-Gui.
  - False in $n = 8$ by Pacard-Wei (2013 JFA)
  - Classification is open in other dimensions.
- Fractional Laplacian case: $(-\Delta)^s u = f(u)$ and $s \in (0, 1)$
  - Existence when $n = 1$ by Cabre-Sire (2009 Annales Poincare).
  - For any $s$ when $n = 2$ by Sire-Valdinoci (2009 J FA).
  - For any $s \in [1/2, 1)$ when $n = 3$ by Cabre-Cinti (2012 DCDS).
  - Open for other cases.

  The proof strongly relies on the extension function given by Caffarelli-Silvestre (2007 CPDE), i.e.

$$\begin{cases} \operatorname{div}(y^{1-2s} \nabla u_e) &= 0 \ \text{ in } \ \mathbf{R}^{n+1}_+ = \{x \in \mathbf{R}^n, y > 0\}, \\ -\lim_{y \to 0} y^{1-2s} \partial_y u_e &= k_s f(u_e) \ \text{ in } \ \partial \mathbf{R}^{n+1}_+, \end{cases}$$

# What about systems?

To extend the De Giorgi's conjecture to systems, what is the right system?
Consider the gradient system:

$$\Delta u = \nabla H(u) \text{ in } \mathbb{R}^n,$$

where $u : \mathbb{R}^n \to \mathbb{R}^k$ is bounded and $H \in C^2(\mathbb{R}^k)$.
Euler-Lagrange equation for the energy functional:

$$E(u) = \frac{1}{2}\int \sum_{i=1}^{k} |\nabla u_i|^2 + \int (H(u) - \inf_u H(u))$$

Phase Transitions: Minimum points of $H$ are global minimizers of the energy and representing distinct phases of k materials.
Example: For $k = 2$ and $H(u, v) = u^2 v^2$ the global minimizers are $u = 0$ and $v = 0$.
We need Monotonicity and Stability concepts for systems.

Allen-Cahn equation
Lane-Emden equation

- *H*-Monotone:
  1. For every $i \in \{1, \cdots, k\}$, $u_i$ is strictly monotone in the $x_n$-variable (i.e., $\partial_{x_n} u_i \neq 0$).
  2. For $i < j$, we have
  $$H_{u_i u_j} \partial_{x_n} u_i(x) \partial_{x_n} u_j(x) \leq 0 \text{ for all } x \in \mathbb{R}^n.$$
  This condition implies a combinatorial assumption on $H_{u_i u_j}$ and we call such a system orientable.
- Pointwise Stability: $\exists (\phi_i)_{i=1}^k$ non sign changing
$$\Delta \phi_i = \sum_j H_{u_i u_j} \phi_j$$
and $H_{u_i u_j} \phi_j \phi_i \leq 0$ for $1 \leq i < j \leq k$.
- Stability (or Stability Inequality):
$$\sum_i \int_{\mathbb{R}^n} |\nabla \zeta_i|^2 + \sum_{i,j} \int_{\mathbb{R}^n} H_{u_i u_j} \zeta_i \zeta_j \geq 0,$$
for every $\zeta_i \in C_c^1(\mathbb{R}^n), i = 1, \cdots, k$.

Orientable systems: $H$-Monotonicity $\Longrightarrow$ Pointwise Stability $\Longleftrightarrow$ Stability.

Allen-Cahn equation

Lane-Emden equation

## Conjecture

*Suppose $u = (u_i)_{i=1}^k$ is a bounded $H$-monotone solution, then at least in lower dimensions each component $u_i$ must be one-dimensional.*

## Theorem (Fazly-Ghoussoub, Calc PDE 2013)

*Positive answer to this conjecture for $n \le 3$. Moreover, $\nabla u_i = C_{i,j} \nabla u_j$ where $C_{i,j}$ is a constant with opposite sign of $H_{u_i u_j}$.*

Gradients are parallel via geometric Poincaré inequality:

$$\sum_i \int_{\mathbb{R}^n} |\nabla u_i|^2 |\nabla \eta_i|^2 \geq \sum_i \int_{\mathbb{R}^n \cap \{|\nabla u_i| \neq 0\}} \left( |\nabla u_i|^2 \mathcal{A}_i^2 + |\nabla_T |\nabla u_i||^2 \right) \eta_i^2 + \sum_{i \neq j} \int_{\mathbb{R}^n} \left( \nabla u_i \cdot \nabla u_j \eta_i^2 - |\nabla u_i||\nabla u_j| \eta_i \eta_j \right) H_{u_i u_j},$$

How? Test stability on $|\nabla u_i| \eta_i$.

- Alama, Bronsard, Gui (1997 Calc PDE) constructed 2D solutions $u : \mathbb{R}^2 \to \mathbb{R}^2$ that are not $H$-monotone. $H$-monotonicity is a crucial assumption!.
- Brendan Pass (2011 PhD Thesis) observed a similar concept called “compatible cost” in multi-marginal optimal transport. Equivalent: Ghoussoub-Pass (2014 CPDE)
- Modica's estimates does not hold in general, by Farina (2004 J FA)

$$\sum_{i=1}^{k} |\nabla u_i|^2 \le 2H(u) \quad \textit{Nope}!$$

However a Hamiltonian identity given by Gui (2008 J FA)

$$\int_{R^{n-1}} \left[ \sum_{i=1}^{k} \left( |\nabla_{x'} u_i|^2 - |\partial_{x_n} u_i|^2 \right) - 2H(u(x', x_n)) \right] dx' = C \quad \text{for} \quad x_n \in \mathbb{R}$$

- When is $\Gamma_R = \frac{E_R(u)}{R^{n-1}}$ increasing? Not known when $k > 1$. $\tilde{\Gamma}_R = \frac{E_R(u)}{R^{n-2}}$ is nondecreasing.

# More Comments:

- Fractional system $-(-\Delta)^s u = \nabla H(u)$ when $n = 2$ and $s \in (0,1)$ and $n = 3$ and $1/2 \leq s < 1$ by Fazly-Sire (2014 CPDE).
  - $\tilde{\Gamma}_R = \frac{E_R(u)}{R^{n-2s}}$ is increasing where
  $$E_R(u) = \frac{1}{2}\int_{B_R \cap \mathbb{R}^{n+1}_+} \sum_{i=1}^{k} y^{1-2s}|\nabla u_i|^2 d\mathbf{x}dy + \int_{B_R \cap \partial\mathbb{R}^{n+1}_+} H(u)d\mathbf{x}$$
  (Idea: Pohozaev Identity)
  - If $u = u(|\mathbf{x}|, y)$ then $I_r(u)$ is nondecreasing in $r$ where
  $$I_r(u) = \sum_{i=1}^{k}\int_0^\infty y^{1-2s}\left[(\partial_r u_i)^2 - (\partial_y u_i)^2\right] dy + 2H(u(r,0))$$
  - Let $n = 1$ and $\lim_{x\to\infty} u = \alpha$ then for $x \in \mathbb{R}$
  $$\sum_{i=1}^{k}\int_0^\infty y^{1-2s}\left[(\partial_x u_i)^2 - (\partial_y u_i)^2\right] dy + 2H(u(x,0)) = 2H(\alpha).$$
- For the case $k = 2$ and $H(u,v) = \frac{1}{2}u^2v^2$ and $\Delta u = \nabla H(u)$ then
  - there exists 1-D solutions of the form $u(x - x_0) = v(x_0 - x)$. Berestycki-Lin-Wei-Zhao (2013 ARMA)
  - 1-D solution is unique. Berestycki-Terracini-Wang-Wei (2013 Adv Math)
- For the case $k = 2$ and $H(u,v) = \frac{1}{2}u^2v^2$ and $-(-\Delta)^s u = \nabla H(u)$ then
  - there exists a unique 1-D solution. Wang-Wei (2014)

# Why three dimensions?

Set $\phi_i := \partial_{x_n} u_i$ and $\psi_i := \nabla u_i \cdot \eta$ for $\eta = (\eta', 0) \in \mathbb{R}^{n-1} \times \{0\}$ then $\sigma_i := \frac{\psi_i}{\phi_i}$ satisfies a linear equation

$$\operatorname{div}(\phi_i^2 \nabla \sigma_i) + \sum_{j=1}^{k} h_{i,j}(x)(\sigma_i - \sigma_j) = 0 \ \text{ in } \ \mathbb{R}^n$$

where $h_{i,j}(x) = H_{u_i u_j} \phi_i \phi_j$.

- Linear Liouville Theorem: If $\sigma_i$ satisfies the above, $\phi_i > 0$, $h_{i,j} = h_{j,i} \leq 0$ and

$$\sum_{i=1}^{k} \int_{B_{2R} \setminus B_R} \phi_i^2 \sigma_i^2 < CR^2, \qquad \forall R > 1$$

$\implies$ then each $\sigma_i$ is constant. [Little imp. by Fazly (2014 PAMS)]

- Energy estimates:

$$\text{Bounded stable} \implies \sum_{i=1}^{k} \int_{B_R} |\nabla u_i|^2 \leq E_R(u) \leq CR^{n-1}.$$

Optimality not known when $k > 1$.

Allen-Cahn equation
Lane-Emden equation

Nonnegative solutions and $p > 1$:

$$-\Delta u = u^p \quad \text{in} \quad \mathbb{R}^n$$

**Theorem (Gidas and Spruck, 1980)**

*Let $n \geq 3$ and $p$ be under the Sobolev exponent, $1 < p < \frac{n+2}{n-2} =: p^*(n)$. Then $u = 0$.*

Critical case $p = \frac{n+2}{n-2}$:

- Gidas-Ni-Nirenberg (1981 MAA) proved that all solutions with $u(x) = O(|x|^{2-n})$ are radially symmetric about some $x_0 \in \mathbb{R}^n$ and of the form

$$u(x) = C_n \left( \frac{\lambda}{1 + \lambda^2 |x - x_0|^2} \right)^{\frac{n-2}{2}}$$

  where $C_n = (n(n-2))^{\frac{2-n}{4}}$, $\lambda > 0$ and some $x_0 \in \mathbb{R}^n$.
- Caffarelli-Gidas-Spruck (1989 CPAM) removed the condition.
- Chen and Li (1991 Duke Math) via moving plane methods.

Note:

- Fourth order case: Wei-Xu (1999 Math Ann). Here $p^*(n) := \frac{n+4}{n-4}$.
- Fractional case: YanYan Li (2004 JEMS) and Chen-Li-Ou (2006 CPAM). Here $p_s^*(n) := \frac{n+2s}{n-2s}$.

Stable solutions and $p > 1$:

$$(-\Delta)^s u = |u|^{p-1} u \quad \text{in} \quad \mathbb{R}^n$$

there exists $p_s^{**}(n)$, called Joseph-Lundgren exponent, such that for $1 < p < p_s^{**}(n)$, $u = 0$.

- For $s = 1$ Farina (2007 J Math Pure Appl) where

$$p_1^{**}(n) = \begin{cases} \infty & \text{if } n \leq 10, \\ \frac{(n-2)^2 - 4n + 8\sqrt{n-1}}{(n-2)(n-10)} & \text{if } n \geq 11, \end{cases}$$

- For $s = 2$ Davila-Dupagine-Wang-Wei (2014 Adv Math) where

$$p_2^{**}(n) = \begin{cases} \infty & \text{if } n \leq 12, \\ \frac{n+2-\sqrt{n^2+4-n\sqrt{n^2-8n+32}}}{n-6-\sqrt{n^2+4-n\sqrt{n^2-8n+32}}} & \text{if } n \geq 13, \end{cases}$$

- For $0 < s < 1$ Davila-Dupagine-Wei (2014)
- For $1 < s < 2$ Fazly-Wei (2014) where $p_s^{**}(n)$ can be found from

$$p \frac{\Gamma(\frac{n}{2} - \frac{s}{p-1})\Gamma(s + \frac{s}{p-1})}{\Gamma(\frac{s}{p-1})\Gamma(\frac{n-2s}{2} - \frac{s}{p-1})} > \frac{\Gamma(\frac{n+2s}{4})^2}{\Gamma(\frac{n-2s}{4})^2}$$

Optimal. For $p \geq p_s^{**}(n)$ there is a stable solution that is radially symmetric w.r.t. some point.

- Case $s=1$: Evans (91 ARMA) and Pacard (93 Manuscripta Math).

$$E(x_0,r) := r^{-n+2\frac{p+1}{p-1}} \int_{B_r(x_0)} \left(\frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}|u|^{p+1}\right) + \frac{r^{-1-n+2\frac{p+1}{p-1}}}{p-1} \int_{\partial B_r(x_0)} |u|^2$$

- Case $1<s<2$, $E(x_0,r)$ is the following

$$\begin{aligned}
& r^{2s\frac{p+1}{p-1}-n} \left( \int_{\mathbb{R}^{n+1}_+ \cap B_r(x_0)} \frac{1}{2} y^{3-2s} |\Delta_b u_e|^2 - \frac{1}{p+1} \int_{\partial\mathbb{R}^{n+1}_+ \cap B_r(x_0)} u_e^{p+1} \right) \\
& - C r^{-3+2s+\frac{4s}{p-1}-n} \int_{\mathbb{R}^{n+1}_+ \cap \partial B_r(x_0)} y^{3-2s} u_e^2 \\
& - C \partial_r \left[ r^{\frac{4s}{p-1}+2s-2-n} \int_{\mathbb{R}^{n+1}_+ \cap \partial B_r(x_0)} y^{3-2s} u_e^2 \right] \\
& + \frac{1}{2} r^3 \partial_r \left[ r^{\frac{4s}{p-1}+2s-3-n} \int_{\mathbb{R}^{n+1}_+ \cap \partial B_r(x_0)} y^{3-2s} \left( \frac{2s}{p-1} r^{-1} u + \partial_r u_e \right)^2 \right] \\
& + \frac{1}{2} \partial_r \left[ r^{2s\frac{p+1}{p-1}-n} \int_{\mathbb{R}^{n+1}_+ \cap \partial B_r(x_0)} y^{3-2s} \left( |\nabla u_e|^2 - |\partial_r u_e|^2 \right) \right] \\
& + \frac{1}{2} r^{2s\frac{p+1}{p-1}-n-1} \int_{\mathbb{R}^{n+1}_+ \cap \partial B_r(x_0)} y^{3-2s} \left( |\nabla u_e|^2 - |\partial_r u_e|^2 \right)
\end{aligned}$$

where $\Delta_b u_e := y^{-3+2s} \operatorname{div}(y^{3-2s} \nabla u_e)$. Extension function: Ray Yang (2013).

## Major ideas: Handling Homogenous Solutions

- Monotoniciy Formula implies $u = r^{-\frac{2s}{p-1}}\psi(\theta)$ that is called Homogenous Solution.

Goal: $\psi \equiv 0$ where $1 < p < p_s^{**}(n)$. How?

- Step 1. From PDE:

$$A_{n,s}\int_{\mathbb{S}^{n-1}}\psi^2 + \int_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} K_{\frac{2s}{p-1}}(<\theta,\sigma>)(\psi(\theta)-\psi(\sigma))^2 = \int_{\mathbb{S}^{n-1}}\psi^{p+1}$$

  where $A_{n,s}$ is explicitly known and $K_\alpha(<\theta,\sigma>)$ is decreasing in $\alpha$ for $p > p_s^*(n)$.

- Step 2. From Stability: Test on $r^{-\frac{n-2s}{2}}\psi(\theta)\eta_\epsilon(r)$ for appropriate $\eta_\epsilon(r)$ to get

$$\Lambda_{n,s}\int_{\mathbb{S}^{n-1}}\psi^2 + \int_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} K_{\frac{n-2s}{2}}(<\theta,\sigma>)(\psi(\theta)-\psi(\sigma))^2 \geq p\int_{\mathbb{S}^{n-1}}\psi^{p+1}$$

  where $\Lambda_{n,s}$ is the Hardy constant.

- Note that $K_{\frac{n-2s}{2}} < K_{\frac{2s}{p-1}}$ for $p > p_s^*(n)$. If $\Lambda_{n,s} < pA_{n,s}$ then $\psi = 0$.

Wei's Conjecture: If $p_1^{**}(n) \leq p < p_1^{**}(n-1)$, all stable solutions are radially symmetric?

Note: For $\frac{n+1}{n-3} < p < p_1^{**}(n-1)$ there are unstable nonradial solutions.
Dancer-Guo-Wei (2012 Indiana Math)

Thank you for your attention.