Results on curvature flow

Pak Tung Ho
Sogang University

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Content

Conformal metric

M = n-dimensional smooth manifold, compact, $\partial M = \emptyset$

g, g_0 = Riemannian metrics on M.

We say that g is conformal to g_0 if $g = e^{2u} g_0$ for some $u \in C^\infty(M)$.

Uniformization Theorem

M = 2-dimensional smooth manifold, compact, $\partial M = \emptyset$

g_0 = a Riemannian metric on M.

Theorem (Uniformization Theorem)

$\exists$ g conformal to g_0 such that the Gaussian curvature of g, $K_g \equiv$constant.

If $g = e^{2u} g_0$, then

$$K_g = e^{-2u}(-\Delta_{g_0} u + K_{g_0}).$$

Q-curvature

M = a compact 4-dimensional manifold with a metric g

Q-curvature: $Q_g = -\frac{1}{6}(\Delta_g R_g - R_g^2 + 3|Ric_g|_g^2)$.

Here, R_g = scalar curvature of g and Ric_g = Ricci curvature tensor of g.

Paneitz operator: $P_g\phi = \Delta_g^2\phi + d_g^*\left[\left(\frac{2}{3}R_g g - 2Ric_g\right)d\phi\right]$, i.e.

$$\int_M \langle P_g\phi_1, \phi_2\rangle_g = \int_M \phi_2\Delta_g^2\phi_1 + \frac{2}{3}\int_M R_g\langle d\phi_1, d\phi_2\rangle_g - 2\int_M Ric_g(d\phi_1, d\phi_2)$$

Q-curvature

If $g = e^{2u}g_0$, then

$$\begin{cases} Q_g = e^{-4u}(P_{g_0}u + Q_{g_0}), \\ P_g = e^{-4u}P_{g_0}. \end{cases}$$

In 2-dim, if $g = e^{2u}g_0$, then

$$\begin{cases} K_g = e^{-2u}(-\Delta_{g_0}u + K_{g_0}), \\ -\Delta_g = -e^{-2u}\Delta_{g_0}. \end{cases}$$

2-dim	4-dim
$-\Delta_g$	P_g
K_g	Q_g

Q-curvature

In 2-dim, by Gauss-Bonnet Theorem,

$$\int_M K_g dV_g = \int_M K_{g_0} dV_{g_0} = 2\pi\chi(M)$$

is a conformal invariant.

In 4-dim, by Chern-Gauss-Bonnet Theorem,

$$\int_M \left(Q_g + \frac{1}{4}|W_g|^2\right) dV_g = 8\pi^2\chi(M).$$

Here W_g = the Weyl tensor of M. Since $|W_g|^2 dV_g = |W_{g_0}|^2 dV_{g_0}$,

$$\int_M Q_g dV_g = \int_M Q_{g_0} dV_{g_0}$$

is a conformal invariant.

Q-curvature

On even-dimensional manifold M, Fefferman-Graham defined Q-curvature Q_g and Paneitz operator P_g such that:

- P_g is self-adjoint with leading term $(-\Delta_g)^{n/2}$.
- If $g = e^{2u}g_0$,
$$Q_g = e^{-nu}(P_{g_0}u + Q_{g_0}).$$
- If $g = e^{2u}g_0$,
$$\int_M Q_g dV_g = \int_M Q_{g_0} dV_{g_0}.$$

If $g_{S^n} =$ standard metric on even-dimensional sphere S^n, then

$$\begin{cases} Q_{g_{S^n}} = (n-1)!, \\ P_{g_{S^n}} = \prod_{k=0}^{(n-2)/2}(-\Delta_{g_{S^n}} + k(n-k-1)). \end{cases}$$

Q-curvature

Questions:

1. (Uniformization Theorem) $\exists$ g conformal to g_0 such that $Q_g \equiv$constant ?
2. (Prescribed Q-curvature problem) Given $f \in C^\infty(M)$, $\exists$ g conformal to g_0 such that $Q_g = f$?

Studied by Brendle, Chang-Yang, Malchiodi-Struwe, Wei-Xu, etc.

prescribed Q-curvature flow

Brendle introduced prescribed Q-curvature flow:

$$\frac{\partial}{\partial t} g(t) = (\alpha(t) f - Q_{g(t)}) g(t), \;\; g(0) = g_0$$

where $\alpha(t)$ is defined by

$$\alpha(t) \int_M f \, dV_{g(t)} = \int_M Q_{g(t)} \, dV_{g(t)}.$$

When $\ker P_{g_0} = \{\text{constant}\}$ and

$$\int_M Q_{g_0} dV_{g_0} < \int_{S^n} Q_{g_{S^n}} dV_{g_{S^n}},$$

he proved that the flow exists and converges to g_∞ such that $Q_{g_\infty} = \alpha_\infty f$.

Q-curvature flow on S^n

Theorem (Brendle for $n = 4$, H.___ for general n)

On S^n, the Q-curvature flow:

$$\frac{\partial}{\partial t} g(t) = -(Q_{g(t)} - \overline{Q}_{g(t)}) g(t),$$

where

$$\overline{Q}_{g(t)} = \frac{\int_{S^n} Q_{g(t)} dV_{g(t)}}{\int_{S^n} dV_{g(t)}},$$

exists for all $t \geq 0$ and converges to a metric of constant sectional curvature.

prescribed Q-curvature flow on S^n

Theorem (Malchiodi-Struwe for $n = 4$, Chan-Xu and independently H.___ for general n)

Suppose that $f > 0$. On S^n, under some condition on the Morse index of f, the prescribed Q-curvature flow

$$\frac{\partial}{\partial t} g(t) = (\alpha(t) f - Q_{g(t)}) g(t),$$

where $\alpha(t)$ is defined by

$$\alpha(t) \int_{S^n} f \, dV_{g(t)} = \int_{S^n} Q_{g(t)} \, dV_{g(t)}.$$

exists for all $t \geq 0$ and converges to g_∞ such that $Q_{g_\infty} = \alpha_\infty f$.

Yamabe problem

$M = n$-dimensional smooth manifold, compact, $\partial M = \emptyset$ where $n \geq 3$.

$g_0 =$ a Riemannian metric on M.

Yamabe problem: Find g conformal to g_0 such that the scalar curvature of g, $R_g \equiv$constant.

Yamabe problem

If $g = u^{\frac{4}{n-2}} g_0$ where $0 < u \in C^\infty(M)$, then

$$-\frac{4(n-1)}{(n-2)}\Delta_{g_0} u + R_{g_0} u = R_g u^{\frac{n+2}{n-2}} \tag{1}$$

Here, Δ_{g_0} = Laplacian of g_0,

R_{g_0} = scalar curvature of g_0,

R_g = scalar curvature of g.

Yamabe problem is to solve (1) with $R_g \equiv$constant.

Yamabe problem

The Yamabe constant is defined as

$$Y(M, g_0) = \inf\{E(u) | 0 < u \in C^\infty(M)\}$$

where the energy $E(u)$ is given by

$$E(u) = \frac{\int_M (|\nabla u|^2 + R_{g_0} u^2) dV_{g_0}}{(\int_M u^{\frac{2n}{n-2}} dV_{g_0})^{\frac{n-2}{n}}}.$$

If $E(u) = Y(M, g_0)$, then u satisfies (1).

Yamabe problem

- For $Y(M, g_0) \leq 0$, solved by Trudinger.
- For $Y(M, g_0) > 0$,
  • solved by Aubin when $n \geq 6$ and (M, g_0) is not locally conformally flat.
  • solved by Schoen when $3 \leq n \leq 5$ or (M, g_0) is locally conformally flat using positive mass theorem.
- Bahri obtained same result of Schoen using critical point at infinity.

Yamabe flow

Hamilton introduced Yamabe flow:

$$\frac{\partial}{\partial t}g(t) = -(R_{g(t)} - \overline{R}_{g(t)})g(t) \;\; t \geq 0.$$

Here, $R_{g(t)}$ = scalar curvature of $g(t)$,

and $\overline{R}_{g(t)}$ is defined by

$$\overline{R}_{g(t)} = \frac{\int_M R_{g(t)} dV_{g(t)}}{\int_M dV_{g(t)}}.$$

Yamabe flow

- $$0=\frac{\partial}{\partial t}g(t)=-(R_{g(t)}-\overline{R}_{g(t)})g(t) \quad\Leftrightarrow\quad R_{g(t)}=\overline{R}_{g(t)}.$$
- If we write $g(t)=u(t)^{\frac{4}{n-2}}g_0$, then
$$\begin{aligned}&\frac{\partial}{\partial t}\left(u(t)^{\frac{n+2}{n-2}}\right)\\&=\frac{n+2}{4}\left(\frac{4(n-1)}{n-2}\Delta_{g_0}u(t)-R_{g_0}u(t)+\overline{R}_{g(t)}u(t)^{\frac{n+2}{n-2}}\right).\end{aligned}$$

Yamabe flow

Along the Yamabe flow, we have

$$\frac{d}{dt}E(u(t)) = -\frac{n}{2}\frac{\int_M (R_{g(t)} - \overline{R}_{g(t)})^2 dV_{g(t)}}{\text{Vol}(M, g_0)^{\frac{n-2}{n}}} \leq 0.$$

$$\Longrightarrow \int_0^\infty \int_M (R_{g(t)} - \overline{R}_{g(t)})^2 dV_{g(t)} dt < \infty$$

$$\Longrightarrow \liminf_{t\to\infty} \int_M (R_{g(t)} - \overline{R}_{g(t)})^2 dV_{g(t)} = 0$$

Yamabe flow

- Hamilton proved (i) long time existence of the Yamabe flow
  (ii) convergence for $Y(M, g_0) \leq 0$.
- For $Y(M, g_0) > 0$, convergence was studied by Chow, Ye, and Schwetlick-Struwe.
- Brendle proved convergence using positive mass theorem.

CR Yamabe problem

(M,θ_0)=compact, strictly pseudoconvex CR manifold of real dimension $2n+1$

CR manifold: $\exists$ subbundle $T^{1,0}\subset\mathbb{C}\otimes TM$ such that

- $T^{1,0}\cap\overline{T^{1,0}}=\{0\}$,
- $\dim_{\mathbb{C}} T^{1,0}=n$,
- $[T^{1,0},T^{1,0}]\subset T^{1,0}$.

strictly pseudoconvex: θ_0 is called contact 1-form

Levi form $-\sqrt{-1}d\theta_0>0$ on $T^{1,0}\times\overline{T^{1,0}}$.

CR Yamabe problem

Example: odd-dimensional sphere $S^{2n+1} \subset \mathbb{C}^n$ with contact form $\theta_{S^{2n+1}} = \sqrt{-1}\sum_{j=1}^{n+1}(z_j d\overline{z}_j - \overline{z}_j dz_j)$ is a strictly pseudoconvex CR manifold.

On a strictly pseudoconvex CR manifold (M, θ_0), one can define the Webster scalar curvature R_{θ_0}.

For $(S^{2n+1}, \theta_{S^{2n+1}})$, we have $R_{\theta_{S^{2n+1}}} \equiv n(n+1)/2$.

CR Yamabe problem

CR Yamabe problem: Given a strictly pseudoconvex CR manifold (M, θ_0), find a contact form θ conformal to θ_0 such that its Webster curvature $R_\theta \equiv$constant.

If $\theta = u^{\frac{2}{n}}\theta_0$ where $0 < u \in C^\infty(M)$, then

$$-(2 + \frac{2}{n})\Delta_{\theta_0} u + R_{\theta_0} u = R_\theta u^{1+\frac{2}{n}}$$

Here, Δ_{θ_0} = sub-Laplacian of θ_0,

R_{θ_0} = Webster scalar curvature of θ_0,

R_θ = Webster scalar curvature of θ.

CR Yamabe problem

The CR Yamabe problem was solved by

- Jerison-Lee when $n \geq 2$ and M is not locally CR equivalent to S^{2n+1},
- Gamara-Yacoub when $n = 1$ or M is locally CR equivalent to S^{2n+1} using critical point at infinity.

CR Yamabe flow

Consider the CR Yamabe flow:

$$\frac{\partial}{\partial t}\theta(t) = -(R_{\theta(t)} - \overline{R}_{\theta(t)})\theta(t) \quad t \geq 0.$$

Here, $R_{\theta(t)}$ = Webster scalar curvature of $\theta(t)$, and $\overline{R}_{\theta(t)}$ is defined by

$$\overline{R}_{\theta(t)} = \frac{\int_M R_{\theta(t)} dV_{\theta(t)}}{\int_M dV_{\theta(t)}}.$$

If we write $\theta(t) = u(t)^{\frac{2}{n}}\theta_0$, then

$$\begin{aligned}&\frac{\partial}{\partial t}\left(u(t)^{\frac{2+n}{n}}\right)\\ &= \frac{n+2}{2}\left(\left(2+\frac{2}{n}\right)\Delta_{\theta_0}u(t) - R_{\theta_0}u(t) + \overline{R}_{\theta(t)}u(t)^{1+\frac{2}{n}}\right).\end{aligned}$$

CR Yamabe flow

- S. C. Chang and J. H. Cheng proved the short time existence and obtained some Harnack inequality.
- When $Y(M, \theta_0) < 0$, Y. B. Zhang proved the long time existence and convergence.
- When $Y(M, \theta_0) > 0$, S. C. Chang, H. L. Chiu, and C. T. Wu proved the long time existence and convergence when $n = 1$ and torsion is zero.

CR Yamabe flow

Theorem (H.___)

When $Y(M, \theta_0) > 0$, the CR Yamabe flow exists for all time $t \geq 0$.

Theorem (H.___)

On S^{2n+1}, the CR Yamabe flow $\theta(t) \to \theta_{S^{2n+1}}$ as $t \to \infty$.

Recently, J. H. Cheng, H. L. Chiu, A. Malchodi, and P. Yang proved the CR positive mass theorem when $n = 1$ or when M is locally CR equivalent to S^{2n+1}.

Theorem (H.___)

The CR Yamabe flow converges when $n = 1$ or when M is locally CR equivalent to S^{2n+1}.

Eigenvalue along the Yamabe flow

Theorem (X. Cao)

The first eigenvalue of $-\Delta_{g(t)} + \frac{1}{2}R_{g(t)}$ *is nondecreasing along the Ricci flow*

$$\frac{\partial}{\partial t}g(t) = -2Ric_{g(t)}$$

on a Riemannian manifold with nonnegative curvature operator.

Theorem (X. Cao et. al.)

For all $a > 0$*, the first eigenvalue of* $-\Delta_{g(t)} + aR_{g(t)}$ *is nondecreasing along the Ricci flow on a surface with* $R_{g(t)} \geq 0$*.*

Eigenvalue along the Yamabe flow

Consider Yamabe flow, because:

- When dim $= 2$, Ricci flow becomes the unnormalized Yamabe flow:
$$\frac{\partial}{\partial t}g(t) = -2Ric_{g(t)} = -R_{g(t)}g(t).$$
- The condition $R_{g(t)} \geq 0$ is preserved along the unnormalized Yamabe flow

Eigenvalue along the Yamabe flow

Theorem (H.___)

Along the unnormalized Yamabe flow, the first eigenvalue of $-\Delta_{g(t)} + aR_{g(t)}$ is nondecreasing

(i) if $0 \leq a < \dfrac{n-2}{4(n-1)}$ *and* $\min R_{g(t)} \geq \dfrac{n-2}{n} \min R_{g(t)} \geq 0$,

(ii) if $a \geq \dfrac{n-2}{4(n-1)}$ *and* $\min R_{g(t)} \geq 0$.

Similar results hold for p-Laplacian and for manifolds with boundary.

Eigenvalue along the CR Yamabe flow

Theorem (H.___)

Along the unnormalized CR Yamabe flow

$$\frac{\partial}{\partial t}\theta(t) = -R_{\theta(t)}\theta(t),$$

the first eigenvalue of $-\Delta_{\theta(t)} + aR_{\theta(t)}$ *is nondecreasing*

(i) if $0 \leq a < \dfrac{n}{2n+2}$ *and* $\min R_{\theta(t)} \geq \dfrac{n}{n+1} \min R_{\theta(t)} \geq 0$,

(ii) if $a \geq \dfrac{n}{2n+2}$ *and* $\min R_{\theta(t)} \geq 0$.

Nirenberg's problem

As a generalization of Yamabe problem, we want to ask:

Given $f \in C^\infty(M)$, $\exists$ g conformal to g_0 such that its scalar curvature $R_g = f$?

If $(M, g_0) = (S^n, g_{S^n})$ the standard sphere, the problem is called Nirenberg's problem.

Studied by Kazdan-Wanrer, Chang-Yang, Struwe, etc.

Nirenberg's problem

Kazdan-Warner obtained a necessary condition, the so-called Kazdan-Warner identity:

If there exists $g = u^{\frac{4}{n-2}} g_{S^n}$ such that $R_g = f$, we must have

$$\int_{S^n} \langle \nabla f, \nabla x_i \rangle_{g_{S^n}} u^{\frac{2n}{n-2}} dV_{g_{S^n}} = 0 \quad \text{for } i = 1, 2, ..., n+1$$

where x_i is the coordinate function of $\mathbb{R}^{n+1}$ restricted to S^n.

Nirenberg's problem

Theorem (Chang-Yang)

If $f > 0$ is a Morse function on S^n such that

$$\sum_{\nabla_{g_{S^n}} f(x)=0, \Delta_{g_{S^n}} f(x)<0} (-1)^{ind(f,x)} \neq -1$$

and $\|f - n(n-1)\|_{C^0}$ is sufficiently small, then $\exists$ g conformal to g_{S^n} such that $R_g = f$.

Nirenberg's problem

Using prescribed scalar curvature flow, X. Chen and X. Xu proved the following:

Theorem (Chen-Xu)

If $f > 0$ is a Morse function on S^n such that

$$\sum_{\nabla_{g_{S^n}} f(x)=0, \Delta_{g_{S^n}} f(x)<0} (-1)^{ind(f,x)} \neq -1$$

and $\|f - n(n-1)\|_{C^0} < \delta_n$ where $\delta_n = 2^{\frac{2}{n-2}}$, then $\exists$ g conformal to g_{S^n} such that $R_g = f$.

Nirenberg's problem

Q: What about the CR case? That is, given f on the CR sphere S^{2n+1}, find θ conformal to $\theta_{S^{2n+1}}$ such that $R_\theta = f$.

J. H. Cheng obtained the necessary condition corresponding to the Kazdan-Warner identity in the Riemannian case.

Nirenberg's problem

Theorem (Malchiodi-Uguzzoni)

If $f > 0$ is a Morse function on S^{2n+1} such that

$$\sum_{\nabla_{g_{S^n}} f(x)=0, \Delta_{\theta_{S^n}} f(x)<0} (-1)^{ind(f,x)} \neq -1$$

and $\|f - n(n+1)/2\|_{C^0}$ is sufficiently small, then $\exists\ \theta$ conformal to θ_{S^n} such that $R_\theta = f$.

Nirenberg's problem

Using prescribed Webster scalar curvature flow, one can obtain

Theorem (H.___)

If $f > 0$ is a Morse function on S^{2n+1} such that

$$\sum_{\nabla_{g_{S^n}} f(x)=0, \Delta_{\theta_{S^n}} f(x)<0} (-1)^{ind(f,x)} \neq -1$$

and $\|f - n(n+1)/2\|_{C^0} < \delta_n$ where $\delta_n = 2^{\frac{1}{n}}$, then $\exists\ \theta$ conformal to θ_{S^n} such that $R_\theta = f$.

Thank you very much for your attention!