# Spherical T-duality

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Workshop on "Positive Curvature and Index Theory", Institute for Mathematical Sciences, National University of Singapore, 18 Nov 2014 Recent work:

P. Bouwknegt, J. Evslin and V. Mathai, Spherical T-duality, [arXiv:1405.5844 [hep-th]].

P. Bouwknegt, J. Evslin and V. Mathai, Spherical T-duality II: An infinity of spherical T-duals for non-principal SU(2)-bundles, [arXiv:1409.1296 [hep-th]].

Review based on:

P. Bouwknegt, J. Evslin and V. Mathai, *T-duality: Topology Change from H-flux*, Comm. Math. Phys. **249** (2004) 383-415, [arXiv:hep-th/0306062].

# **Fourier Transform**

Fourier series for  $f: S^1 \to \mathbb{R}$ 

$$\widehat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx$$
$$f(x) = \sum_{n \in \mathbb{Z}} \widehat{f}(n) e^{inx}$$

Fourier transform for  $f : \mathbb{R} \to \mathbb{R}$ 

$$\widehat{f}(p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ipx} dx$$
$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(p) e^{ipx} dp$$

# Fourier Transform - cont'd

More generally, for G a locally compact, abelian group, we have a Fourier transform  $\mathcal{F}:\mathsf{Fun}(G)\to\mathsf{Fun}(\widehat{G})$ 

$$\widehat{f}(p) = \int_{G} f(x) e^{-ipx} dx = \mathcal{F}(f)(p)$$
$$f(x) = \int_{\widehat{G}} \widehat{f}(p) e^{ipx} dp$$

where

$$\widehat{G} = Hom(G, U(1)) = char(G)$$

is the Pontryagin dual of G. I.e. a character is a U(1) valued function on G, satisfying  $\chi(x + y) = \chi(x)\chi(y)$ .

The characters form a locally compact, abelian group  $\widehat{\mathsf{G}}$  under pointwise multiplication.

# Fourier Transform - cont'd

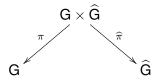
$$\begin{split} \mathbf{G} &= \boldsymbol{\mathcal{S}}^{1} \,, \qquad \widehat{\mathbf{G}} = \mathbb{Z} \,, \qquad \boldsymbol{e}^{\textit{inx}} \\ \mathbf{G} &= \mathbb{R} \,, \qquad \widehat{\mathbf{G}} = \mathbb{R} \,, \qquad \boldsymbol{e}^{\textit{ipx}} \end{split}$$

We can think of  $\chi(x, p) = e^{ipx} \in Fun(G \times \widehat{G})$  as the universal character.

Fourier transform expresses the fact that the characters of G span Fun(G).

# Fourier Transform - cont'd

I.e. we have the following "correspondence"



$$\mathcal{F}f = \widehat{\pi}_*(\pi^*(f) \times \chi(x, p))$$

T-duality is a geometric version of harmonic analysis, i.e. by replacing functions by geometric objects (such as bundles, sheaves, D-modules, ...) or, as an intermediate step, by topological characteristics associated to these objects (cohomology, K-theory, derived categories, ...).

# Fourier-Mukai transform

Consider a manifold  $P = M \times S^1$ . By the Künneth theorem we have

$$H^{\bullet}(P) \cong H^{\bullet}(M) \otimes H^{\bullet}(S^{1})$$

I.e.

$$H^n(P) \cong H^n(M) \oplus H^{n-1}(M)$$

We have a similar decomposition at the level of forms

$$\Omega^n(P)^{\operatorname{inv}} \cong \Omega^n(M) \oplus \Omega^{n-1}(M)$$
.

I.e. invariant degree *n* forms on *P* are of the form  $\omega$  or  $\omega \wedge d\theta$ , where  $\omega$  is an *n*, respectively n - 1, form on *M*.

Consider  $\widehat{P} = M \times \widehat{S}^1$ . We have an isomorphism

$$\mathcal{F} \; : \; H^{\overline{i}}(P) \; \overset{\cong}{\longrightarrow} \; H^{\overline{i+1}}(\widehat{P})$$

### Fourier-Mukai transform - cont'd

### where

$$H^{\overline{0}}(P) = \bigoplus_{i \ge 0} H^{2i}(P), \quad H^{\overline{1}}(P) = \bigoplus_{i \ge 0} H^{2i+1}(P),$$

#### Explicitly

$$\omega \ \mapsto \ \mathbf{d}\widehat{\theta} \wedge \omega \,, \qquad \mathbf{d}\theta \wedge \omega \ \mapsto \ \omega$$

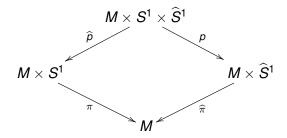
or

$$\mathcal{F}\Omega = \int_{\mathcal{S}^1} (1 + d\theta \wedge d\widehat{\theta}) \,\Omega = \int_{\mathcal{S}^1} e^{d\theta \wedge d\widehat{\theta}} \,\Omega = \int_{\mathcal{S}^1} e^{\mathcal{F}} \,\Omega$$

### Fourier-Mukai transform - cont'd

I.e.  ${\mathcal F}$  is given by a correspondence

$$\mathcal{F}\Omega = p_*\left(\widehat{p}^* \Omega \wedge e^F\right)$$



# Fourier-Mukai transform - cont'd

Once we recognize that  $F = d\theta \wedge d\hat{\theta}$  is the curvature of a canonical linebundle  $\mathcal{P}$  (the Poincaré linebundle) over  $S^1 \times \hat{S}^1$ , in fact  $e^F = ch(\mathcal{P})$ , this immediately suggests a 'geometrization' in terms of vector bundles over P and  $\hat{P}$ 

$$\mathcal{F} E = p_* \left( \widehat{p}^* E \otimes \mathcal{P} \right)$$

This gives rise to the so-called Fourier-Mukai transform

$$\mathcal{F} : K^{i}(P) \xrightarrow{\cong} K^{i+1}(\widehat{P})$$

which has many of the properties of the Fourier transform discussed earlier.

The discussion can be generalized to complexes of vector bundles (complexes of sheaves) and thus gives rise to a Fourier-Mukai correspondence between derived categories D(P) and  $D(\widehat{P})$ .

# T-duality - Closed string on $M \times S^1$

Closed strings on  $M \times S^1$  are described by

$$X : \Sigma \rightarrow M \times S^{2}$$

where  $\Sigma = \{(\sigma, \tau)\}$  is the closed string worldsheet. Upon quantization, we find

- Momentum modes:  $p = \frac{n}{B}$
- Winding modes:  $X(0, \tau) \sim X(1, \tau) + mR$

$${m E}=\left(rac{n}{R}
ight)^2+(mR)^2+{
m osc.}$$
 modes

We have a duality  $R \to 1/R$ , such that ST on  $M \times S^1$  is equivalent to ST on  $M \times \widehat{S}^1$  (or a duality between IIA and IIB ST, for susy ST) Suppose we have a pair (P, H), consisting of a principal circle bundle



and a so-called H-flux H on P, a Čech 3-cocycle.

Topologically, *P* is classified by an element in  $F \in H^2(M, \mathbb{Z})$ while *H* gives a class in  $H^3(P, \mathbb{Z})$ 

# T-duality - Principal S<sup>1</sup>-bundles

The (topological) T-dual of (P, H) is given by the pair  $(\widehat{P}, \widehat{H})$ , where the principal *S*<sup>1</sup>-bundle



and the dual H-flux  $\widehat{H} \in H^3(\widehat{P},\mathbb{Z})$ , satisfy

$$\widehat{F} = \pi_* H$$
,  $F = \widehat{\pi}_* \widehat{H}$ 

where  $\pi_* : H^3(P, \mathbb{Z}) \to H^2(M, \mathbb{Z})$ , is the pushforward map ('integration over the  $S^1$ -fibre').

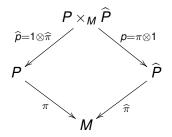
# T-duality - Principal S<sup>1</sup>-bundles

The ambiguity in the choice of  $\hat{H}$  is (almost) removed by requiring that

$$\widehat{p}^*H - p^*\widehat{H} \equiv 0 \quad \in H^3(P imes_M \widehat{P}, \mathbb{Z})$$

where  $P \times_M \widehat{P}$  is the correspondence space

$$P imes_M \widehat{P} = \{(x, \widehat{x}) \in P imes \widehat{P} \mid \pi(x) = \widehat{\pi}(\widehat{x})\}$$

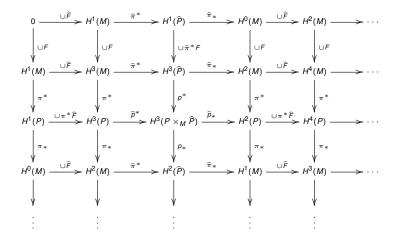


### Gysin sequences

$$\cdots \longrightarrow H^{3}(M) \xrightarrow{\pi^{*}} H^{3}(P) \xrightarrow{\pi_{*}} H^{2}(M) \xrightarrow{\cup F} H^{4}(M) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^{3}(M) \xrightarrow{\widehat{\pi}^{*}} H^{3}(\widehat{P}) \xrightarrow{\widehat{\pi}_{*}} H^{2}(M) \xrightarrow{\bigcup \widehat{F}} H^{4}(M) \longrightarrow \cdots$$

# T-duality - Principal S<sup>1</sup>-bundles



# T-duality - Examples

Consider principal  $S^1$ -bundles P over  $M = S^2$ , then

$$H^2(M,\mathbb{Z})\cong\mathbb{Z}\,,\qquad H^3(P,\mathbb{Z})\cong\mathbb{Z}$$

and we have, for example,

$$(\boldsymbol{S}^2 imes \boldsymbol{S}^1, \boldsymbol{0}) \longrightarrow (\boldsymbol{S}^2 imes \boldsymbol{S}^1, \boldsymbol{0})$$

$$(S^2 \times S^1, 1) \longrightarrow (S^3, 0)$$

or more generally

$$(L_{\rho},k) \longrightarrow (L_{k},\rho)$$

where  $L_{p} = S^{3}/\mathbb{Z}_{p}$  is the lens space.

# T-duality - Twisted cohomology

Using 
$$\Omega^{k}(P)^{inv} \cong \Omega^{k}(M) \oplus \Omega^{k-1}(M)$$
  
 $F = dA, \qquad H = H_{(3)} + A \wedge H_{(2)}$ 

we find

$$\widehat{F} = H_{(2)} = d\widehat{A}, \qquad \widehat{H} = H_{(3)} + \widehat{A} \wedge F$$

such that

$$\widehat{H} - H = \widehat{A} \wedge F - A \wedge \widehat{F} = d(A \wedge \widehat{A}).$$

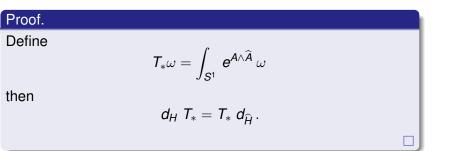
#### Theorem

We have an isomorphism of ( $\mathbb{Z}_2$ -graded) differential complexes

$$T_*: \ (\Omega(P)^{inv}, d_H) \longrightarrow (\Omega(\widehat{P})^{inv}, d_{\widehat{H}})$$

where  $d_H = d + H \wedge$ .

# T-duality - Twisted cohomology



and consequently, we have isomorphisms

$$T_* : H^{\overline{i}}(P, H) \xrightarrow{\cong} H^{\overline{i+1}}(\widehat{P}, \widehat{H})$$

as well as

$$T_* : K^i(P, H) \xrightarrow{\cong} K^{i+1}(\widehat{P}, \widehat{H})$$

For example,

$$\mathcal{K}^{i}(L_{p},k)\cong egin{cases} \mathbb{Z}_{k} & i=0\ \mathbb{Z}_{p} & i=1 \end{cases}$$

# Spherical T-duality - Principal SU(2)-bundles

Much of the above can be generalized to principal SU(2)-bundles: Gysin sequence for principal SU(2)-bundles  $\pi : P \to M$ 

$$\cdots \longrightarrow H^7(M) \xrightarrow{\pi^*} H^7(P) \xrightarrow{\pi_*} H^4(M) \xrightarrow{\cup c_2(P)} H^8(M) \longrightarrow \cdots$$

where

$$c_2(P)=rac{1}{8\pi^2}\operatorname{Tr}(F\wedge F)\in H^4(M)$$

is (a de Rham representative of) the 2nd Chern class of *P*. However, in this case,

$$[M, BSU(2)] \longrightarrow H^4(M, \mathbb{Z})$$

is, in general, neither surjective nor injective.

# SU(2) and quaternions

Recall that

$$\mathrm{SU}(2)=\left\{U(a,b)=\left(egin{array}{cc} a & -ar{b} \\ b & ar{a} \end{array}
ight):\ a,b\in\mathbb{C}, |a|^2+|b|^2=1
ight\}$$

can be identified with the unit sphere  $\mathcal{S}(\mathbb{H}) = \operatorname{Sp}(1) = \mathcal{S}^3$  in the quaternions

$$\mathbb{H} = \{ \alpha + \beta \mathbf{i} + \gamma \mathbf{j} + \delta \mathbf{k} : \mathbf{i} \mathbf{j} = \mathbf{k} = -\mathbf{j} \mathbf{i}, \text{ cyclic} \}$$

The isomorphism is given explicitly as

$$SU(2) 
i U(a,b) \mapsto a+jb \in Sp(1) = S^3$$

The relationship of principal SU(2)-bundles to quaternionic line bundles is analogous to the relationship of principal U(1)-bundles to complex line bundles. Recall that a **quaternionic line bundle** over a manifold *M* is a complex rank 2 vector bundle  $V \rightarrow M$  together with a reduction of structure group to  $\mathbb{H} \setminus \{0\}$ . Note that the unit sphere bundle  $S(V) \rightarrow M$  is an *S*<sup>3</sup>-bundle together with the inherited group structure, i.e. a principal SU(2)-bundle.

Conversely, given a principal SU(2)-bundle  $P \rightarrow M$ , then the associated vector bundle

$$V = P imes_{\mathsf{SU}(2)} \mathbb{H} o M$$

is a quaternionic line bundle.

Principal SU(2)-bundles on  $S^4$  are described by smooth maps  $g : SU(2) \rightarrow SU(2)$ . Let  $g(z) = z, z \in SU(2)$ , which is a degree 1 map. Then  $g(z) = z^r, r \in \mathbb{Z}$  is a degree *r* map. Let  $P(r) \rightarrow S^4$  be the corresponding principal SU(2)-bundle on  $S^4$ . Then  $c_2(P(r)) = r \in \mathbb{Z} \cong H^4(S^4, \mathbb{Z})$ .

The principal SU(2)-bundle  $S^7 = P(1) \rightarrow S^4$  is known as the **Hopf bundle**.

Let *M* be a compact, connected, oriented 4-dimensional manifold. Then one can show fairly easily that isomorphism classes of principal SU(2)-bundles *P* on *M* is canonically identified with homotopy classes  $[M, S^4] \cong H^4(M; \mathbb{Z})$  given by  $c_2(P)$ .

More precisely, given a degree 1 map  $h: M \to S^4$ , then  $h^*(P(r)) \to M$  is a principal SU(2)-bundle on M with  $c_2(h^*(P(r))) = r \in \mathbb{Z} \cong H^4(M, \mathbb{Z}).$ 

Recall the Gysin sequence for principal SU(2)-bundles  $\pi: P \to M$ 

$$\cdots \longrightarrow H^{7}(M) \xrightarrow{\pi^{*}} H^{7}(P) \xrightarrow{\pi_{*}} H^{4}(M) \xrightarrow{\cup c_{2}(P)} H^{8}(M) \longrightarrow \cdots$$

We consider pairs of the form (P, H) consisting of a principal SU(2)-bundle  $P \rightarrow M$  and a 7-cocycle H on P.

The Gysin sequence implies that  $\pi_*$  is a canonical isomorphism  $H^7(P, \mathbb{Z}) \cong H^4(M, \mathbb{Z}) \cong \mathbb{Z}$ , and intuitively spherical T-duality exchanges H with the second Chern class  $c_2$ 

More precisely, the **spherical T-dual** bundle  $\hat{\pi} : \hat{P} \to M$  is defined by  $c_2(\hat{P}) = \pi_* H$  while the dual 7-cocycle  $\hat{H} \in H^7(\hat{P})$  is related to  $c_2(P)$  by the isomorphism  $\hat{\pi}_*$ , via a similar Gysin sequence for  $\hat{P} \to M$ .

Let *M* be a connected compact, oriented, 4 dimensional manifold, and consider the principal SU(2)-bundle P(r) over *M* with  $c_2(P(r)) = r \in \mathbb{Z} \cong H^4(M, \mathbb{Z})$ , together with the 7-cocycle H = s vol on P(r).

Since  $H \cup H = 0$  for dimension reasons, we can define integer-valued H-twisted cohomology as

 $H^{\bullet}(P(r), H; \mathbb{Z}) \equiv H^{\bullet}(H^{\bullet}(P(r); \mathbb{Z}), H\cup).$ 

### Isomorphism of 7-twisted cohomology

Use the Gysin sequence to calculate the cohomology groups  $H^{even/odd}(F(p); \mathbb{Z})$ , and obtain for  $p \neq 0$ 

$$\begin{aligned} H^{j}(P(r);\mathbb{Z}) &= H^{4-j}(M;\mathbb{Z}), \, j = 0, 1, 2, 3\\ H^{4}(P(r);\mathbb{Z}) &= \mathbb{Z}_{r} \oplus H^{1}(M;\mathbb{Z})\\ H^{7-j}(P(r);\mathbb{Z}) &= H^{4-j}(M;\mathbb{Z}), \, j = 0, 1, 2, 3 \end{aligned}$$

Therefore there is an isomorphism of 7-twisted cohomology groups over the integers with a parity change,

#### Theorem

$$\begin{aligned} & H^{even}(P(r), s; \mathbb{Z}) \cong H^{odd}(P(s), r; \mathbb{Z}) \,, \\ & H^{odd}(P(r), s; \mathbb{Z}) \cong H^{even}(P(s), r; \mathbb{Z}) \,. \end{aligned}$$

There is a similar isomorphism of 7-twisted K-theories.

Beyond dimension 4 the situation becomes more complicated as not all integral 4-cocycles of *M* are realized as  $c_2$  of a principal SU(2)-bundle  $\pi : P \to M$  and moreover multiple bundles can have the same  $c_2(P)$ .

More precisely, principal SU(2)-bundles are classified upto isomorphism by homotopy classes of maps into the classifying space  $M \rightarrow BSU(2)$ . However, the complete homotopy type of  $S^3 = SU(2)$  is still unknown, and therefore also for BSU(2).

However Serre's theorem tells us that  $\pi_j(BSU(2)) \otimes \mathbb{Q} \cong \pi_j(K(\mathbb{Z}, 4)) \otimes \mathbb{Q}$ , i.e. the homotopy groups of degree higher than 4 are all torsion.

For example, recall that principal SU(2)-bundles over  $S^5$  are classified by  $\pi_4(SU(2)) \cong \mathbb{Z}_2$ , while  $H^4(S^5, \mathbb{Z}) = 0$ .

By a theorem of Granja, there is a natural number N(d) where  $d = \dim(M)$ , such that if  $\alpha \in N(d) \times H^4(M, \mathbb{Z})$ , then it is the 2nd Chern class of a principal SU(2)-bundle over M. Therefore a pair (P, H) is spherical T-dualizable if  $\pi_*(H) \in N(d) \times H^4(M; \mathbb{Z})$ . Then  $\pi_*(H) = c_2(\widehat{P})$  where  $\widehat{P}$  is a principal SU(2)-bundle over M. However, this does not necessarily uniquely specify  $\widehat{P}$ . But at most, there are finitely many choices.

We will simply assert that a spherical T-dual  $\hat{\pi} : \hat{P} \to M$  be any SU(2)-bundle with  $c_2(\hat{P}) = \pi_* H$ , with  $\hat{H}$  defined such that  $\hat{\pi}_* \hat{H} = c_2(P)$  with  $\hat{p}^* H = p^* \hat{H}$  on the correspondence space  $P \times_M \hat{P}$ .

T-duality induces an isomorphism on twisted cohomologies with real or rational coefficients.

Theorem

$$\begin{aligned} & H^{even}(P,H;\mathbb{Q}) \cong H^{odd}(\widehat{P},\widehat{H};\mathbb{Q}), \\ & H^{odd}(P,H;\mathbb{Q}) \cong H^{even}(\widehat{P},\widehat{H};\mathbb{Q}). \end{aligned}$$

There is a similar isomorphism of 7-twisted K-theories with parity shift, upto  $\mathbb{Z}_2$ -extensions.

### Comments and open questions

- T-duality for non-principal SU(2)-bundles (non-uniqueness, even for S<sup>4</sup>)
- A generalised geometry counterpart of spherical T-duality?
- What is the physics behind spherical T-duality?
- What are useful geometric realisations of integral 7-cocycles?
- Is there a useful geometric description of 7-twisted K-theory?
- When dim*M* ≥ 4, then it is known that not every spherical pair (*P*, *H*) has a spherical T-dual. Can the missing spherical T-duals be obtained some other way?
- Is there a C\*-algebra version of spherical T-duality?

# THANK YOU

Peter Bouwknegt Spherical T-duality