Spherical T-duality

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Fourier Transform

Fourier series for $f : S^1 \to \mathbb{R}$

$$\widehat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, e^{-inx}\, dx$$

$$f(x) = \sum_{n \in \mathbb{Z}} \widehat{f}(n)\, e^{inx}$$

Fourier transform for $f : \mathbb{R} \to \mathbb{R}$

$$\widehat{f}(p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\, e^{-ipx}\, dx$$

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(p)\, e^{ipx}\, dp$$

Fourier Transform - cont'd

More generally, for G a locally compact, abelian group, we have a Fourier transform $\mathcal{F} : \mathrm{Fun}(\mathrm{G}) \to \mathrm{Fun}(\widehat{\mathrm{G}})$

$$\widehat{f}(p) = \int_{\mathrm{G}} f(x)\, e^{-ipx}\ dx = \mathcal{F}(f)(p)$$

$$f(x) = \int_{\widehat{\mathrm{G}}} \widehat{f}(p)\, e^{ipx}\ dp$$

where

$$\widehat{\mathrm{G}} = \mathrm{Hom}(\mathrm{G}, \mathrm{U}(1)) = \mathrm{char}(\mathrm{G})$$

is the Pontryagin dual of G. I.e. a character is a U(1) valued function on G, satisfying $\chi(x+y) = \chi(x)\chi(y)$.
The characters form a locally compact, abelian group $\widehat{\mathrm{G}}$ under pointwise multiplication.

Fourier Transform - cont'd

$$\mathrm{G} = S^1\,, \qquad \widehat{\mathrm{G}} = \mathbb{Z}\,, \qquad e^{inx}$$
$$\mathrm{G} = \mathbb{R}\,, \qquad \widehat{\mathrm{G}} = \mathbb{R}\,, \qquad e^{ipx}$$

We can think of $\chi(x,p) = e^{ipx} \in \mathrm{Fun}(\mathrm{G} \times \widehat{\mathrm{G}})$ as the universal character.
Fourier transform expresses the fact that the characters of G span Fun(G).

Fourier Transform - cont'd

I.e. we have the following "correspondence"

$$\begin{tikzcd} & G \times \widehat{G} \arrow[dl, "\pi"'] \arrow[dr, "\widehat{\pi}"] & \\ G & & \widehat{G} \end{tikzcd}$$

$$\mathcal{F}f = \widehat{\pi}_*(\pi^*(f) \times \chi(x,p))$$

Fourier Transform - Geometric generalisations

T-duality is a geometric version of harmonic analysis, i.e. by replacing functions by geometric objects (such as bundles, sheaves, D-modules, ...) or, as an intermediate step, by topological characteristics associated to these objects (cohomology, K-theory, derived categories, ...).

Fourier-Mukai transform

Consider a manifold $P = M \times S^1$. By the Künneth theorem we have

$$H^\bullet(P) \cong H^\bullet(M) \otimes H^\bullet(S^1)$$

I.e.

$$H^n(P) \cong H^n(M) \oplus H^{n-1}(M)$$

We have a similar decomposition at the level of forms

$$\Omega^n(P)^{\mathrm{inv}} \cong \Omega^n(M) \oplus \Omega^{n-1}(M)\,.$$

I.e. invariant degree n forms on P are of the form ω or $\omega \wedge d\theta$, where ω is an n, respectively $n-1$, form on M.

Consider $\widehat{P} = M \times \widehat{S}^1$. We have an isomorphism

$$\mathcal{F} \;:\; H^{\bar{i}}(P) \xrightarrow{\;\cong\;} H^{\overline{i+1}}(\widehat{P})$$

Fourier-Mukai transform - cont'd

where

$$H^{\bar{0}}(P) = \bigoplus_{i\geq 0} H^{2i}(P)\,, \quad H^{\bar{1}}(P) = \bigoplus_{i\geq 0} H^{2i+1}(P)\,,$$

Explicitly

$$\omega \ \mapsto \ d\widehat{\theta} \wedge \omega\,, \qquad d\theta \wedge \omega \ \mapsto \ \omega$$

or

$$\mathcal{F}\Omega = \int_{S^1} (1 + d\theta \wedge d\widehat{\theta})\,\Omega = \int_{S^1} e^{d\theta \wedge d\widehat{\theta}}\,\Omega = \int_{S^1} e^F\,\Omega$$

Fourier-Mukai transform - cont'd

I.e. $\mathcal{F}$ is given by a correspondence

$$\mathcal{F}\Omega = p_* \left(\widehat{p}^* \, \Omega \wedge e^F \right)$$

$$\begin{tikzcd}
 & M \times S^1 \times \widehat{S}^1 \arrow[dl, "\widehat{p}"'] \arrow[dr, "p"] & \\
M \times S^1 \arrow[dr, "\pi"'] & & M \times \widehat{S}^1 \arrow[dl, "\widehat{\pi}"] \\
 & M &
\end{tikzcd}$$

Fourier-Mukai transform - cont'd

Once we recognize that $F = d\theta \wedge d\widehat{\theta}$ is the curvature of a canonical linebundle $\mathcal{P}$ (the Poincaré linebundle) over $S^1 \times \widehat{S}^1$, in fact $e^F = \mathrm{ch}(\mathcal{P})$, this immediately suggests a 'geometrization' in terms of vector bundles over P and $\widehat{P}$

$$\mathcal{F}E = p_*\left(\widehat{p}^*\, E \otimes \mathcal{P}\right)$$

This gives rise to the so-called Fourier-Mukai transform

$$\mathcal{F} \ : \ K^i(P) \xrightarrow{\ \cong\ } K^{i+1}(\widehat{P})$$

which has many of the properties of the Fourier transform discussed earlier.

The discussion can be generalized to complexes of vector bundles (complexes of sheaves) and thus gives rise to a Fourier-Mukai correspondence between derived categories $D(P)$ and $D(\widehat{P})$.

T-duality - Closed string on $M \times S^1$

Closed strings on $M \times S^1$ are described by

$$X \ : \ \Sigma \ \to \ M \times S^1$$

where $\Sigma = \{(\sigma, \tau)\}$ is the closed string worldsheet.
Upon quantization, we find

- Momentum modes: $p = \frac{n}{R}$
- Winding modes: $X(0, \tau) \sim X(1, \tau) + mR$

$$E = \left(\frac{n}{R}\right)^2 + (mR)^2 + \text{osc. modes}$$

We have a duality $R \to 1/R$, such that ST on $M \times S^1$ is equivalent to ST on $M \times \widehat{S}^1$ (or a duality between IIA and IIB ST, for susy ST)

T-duality - Principal S^1-bundles

Suppose we have a pair (P, H), consisting of a principal circle bundle

$$\begin{tikzcd} S^1 \arrow[r] & P \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

and a so-called H-flux H on P, a Čech 3-cocycle.

Topologically, P is classified by an element in $F \in H^2(M, \mathbb{Z})$ while H gives a class in $H^3(P, \mathbb{Z})$

T-duality - Principal S^1-bundles

The (topological) T-dual of (P, H) is given by the pair $(\widehat{P}, \widehat{H})$, where the principal S^1-bundle

$$\begin{array}{ccc} \widehat{S}^1 & \longrightarrow & \widehat{P} \\ & & \Big\downarrow{\scriptstyle \widehat{\pi}} \\ & & M \end{array}$$

and the dual H-flux $\widehat{H} \in H^3(\widehat{P}, \mathbb{Z})$, satisfy

$$\widehat{F} = \pi_* H \,, \qquad F = \widehat{\pi}_* \widehat{H}$$

where $\pi_* : H^3(P, \mathbb{Z}) \to H^2(M, \mathbb{Z})$, is the pushforward map ('integration over the S^1-fibre').

T-duality - Principal S^1-bundles

The ambiguity in the choice of $\widehat{H}$ is (almost) removed by requiring that

$$\widehat{p}^* H - p^* \widehat{H} \equiv 0 \quad \in H^3(P \times_M \widehat{P}, \mathbb{Z})$$

where $P \times_M \widehat{P}$ is the correspondence space

$$P \times_M \widehat{P} = \{(x, \widehat{x}) \in P \times \widehat{P} \mid \pi(x) = \widehat{\pi}(\widehat{x})\}$$

$$\begin{tikzcd}
 & P \times_M \widehat{P} \arrow[dl, "\widehat{p}=1\otimes\widehat{\pi}"'] \arrow[dr, "p=\pi\otimes 1"] & \\
P \arrow[dr, "\pi"'] & & \widehat{P} \arrow[dl, "\widehat{\pi}"] \\
 & M &
\end{tikzcd}$$

T-duality - Principal S^1-bundles

Gysin sequences

$$\cdots \longrightarrow H^3(M) \xrightarrow{\pi^*} H^3(P) \xrightarrow{\pi_*} H^2(M) \xrightarrow{\cup F} H^4(M) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^3(M) \xrightarrow{\widehat{\pi}^*} H^3(\widehat{P}) \xrightarrow{\widehat{\pi}_*} H^2(M) \xrightarrow{\cup \widehat{F}} H^4(M) \longrightarrow \cdots$$

T-duality - Principal S^1-bundles

$$
\begin{tikzcd}
0 \arrow[r, "\cup \widehat{F}"] \arrow[d, "\cup F"] & H^1(M) \arrow[r, "\widehat{\pi}^*"] \arrow[d, "\cup F"] & H^1(\widehat{P}) \arrow[r, "\widehat{\pi}_*"] \arrow[d, "\cup \widehat{\pi}^* F"] & H^0(M) \arrow[r, "\cup \widehat{F}"] \arrow[d, "\cup F"] & H^2(M) \arrow[r] \arrow[d, "\cup F"] & \cdots \\
H^1(M) \arrow[r, "\cup \widehat{F}"] \arrow[d, "\pi^*"] & H^3(M) \arrow[r, "\widehat{\pi}^*"] \arrow[d, "\pi^*"] & H^3(\widehat{P}) \arrow[r, "\widehat{\pi}_*"] \arrow[d, "p^*"] & H^2(M) \arrow[r, "\cup \widehat{F}"] \arrow[d, "\pi^*"] & H^4(M) \arrow[r] \arrow[d, "\pi^*"] & \cdots \\
H^1(P) \arrow[r, "\cup \pi^* \widehat{F}"] \arrow[d, "\pi_*"] & H^3(P) \arrow[r, "\widehat{p}^*"] \arrow[d, "\pi_*"] & H^3(P \times_M \widehat{P}) \arrow[r, "\widehat{p}_*"] \arrow[d, "p_*"] & H^2(P) \arrow[r, "\cup \pi^* \widehat{F}"] \arrow[d, "\pi_*"] & H^4(P) \arrow[r] \arrow[d, "\pi_*"] & \cdots \\
H^0(M) \arrow[r, "\cup \widehat{F}"] \arrow[d] & H^2(M) \arrow[r, "\widehat{\pi}^*"] \arrow[d] & H^2(\widehat{P}) \arrow[r, "\widehat{\pi}_*"] \arrow[d] & H^1(M) \arrow[r, "\cup \widehat{F}"] \arrow[d] & H^3(M) \arrow[r] \arrow[d] & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

T-duality - Examples

Consider principal S^1-bundles P over $M = S^2$, then

$$H^2(M, \mathbb{Z}) \cong \mathbb{Z}\,, \qquad H^3(P, \mathbb{Z}) \cong \mathbb{Z}$$

and we have, for example,

$$(S^2 \times S^1, 0) \longrightarrow (S^2 \times S^1, 0)$$

$$(S^2 \times S^1, 1) \longrightarrow (S^3, 0)$$

or more generally

$$(L_p, k) \longrightarrow (L_k, p)$$

where $L_p = S^3/\mathbb{Z}_p$ is the lens space.

T-duality - Twisted cohomology

Using $\Omega^k(P)^{\mathrm{inv}} \cong \Omega^k(M) \oplus \Omega^{k-1}(M)$

$$F = dA\,, \qquad H = H_{(3)} + A \wedge H_{(2)}$$

we find

$$\widehat{F} = H_{(2)} = d\widehat{A}\,, \qquad \widehat{H} = H_{(3)} + \widehat{A} \wedge F$$

such that

$$\widehat{H} - H = \widehat{A} \wedge F - A \wedge \widehat{F} = d(A \wedge \widehat{A})\,.$$

Theorem

We have an isomorphism of ($\mathbb{Z}_2$-graded) differential complexes

$$T_* : \; (\Omega(P)^{inv}, d_H) \longrightarrow (\Omega(\widehat{P})^{inv}, d_{\widehat{H}})$$

where $d_H = d + H\wedge$.

T-duality - Twisted cohomology

Proof.

Define

$$T_*\omega = \int_{S^1} e^{A\wedge\widehat{A}}\ \omega$$

then

$$d_H\ T_* = T_*\ d_{\widehat{H}}\,.$$

$\square$

and consequently, we have isomorphisms

$$T_*\ :\ H^{\bar{i}}(P,H) \xrightarrow{\ \cong\ } H^{\overline{i+1}}(\widehat{P},\widehat{H})$$

as well as

$$T_* \ : \ K^i(P,H) \xrightarrow{\cong} K^{i+1}(\widehat{P},\widehat{H})$$

For example,

$$K^i(L_p,k) \cong \begin{cases} \mathbb{Z}_k & i=0 \\ \mathbb{Z}_p & i=1 \end{cases}$$

Much of the above can be generalized to principal SU(2)-bundles:
Gysin sequence for principal SU(2)-bundles $\pi : P \to M$

$$\cdots \longrightarrow H^7(M) \xrightarrow{\pi^*} H^7(P) \xrightarrow{\pi_*} H^4(M) \xrightarrow{\cup c_2(P)} H^8(M) \longrightarrow \cdots$$

where

$$c_2(P) = \frac{1}{8\pi^2} \operatorname{Tr}(F \wedge F) \in H^4(M)$$

is (a de Rham representative of) the 2nd Chern class of P.
However, in this case,

$$[M, BSU(2)] \longrightarrow H^4(M, \mathbb{Z})$$

is, in general, neither surjective nor injective.

SU(2) and quaternions

Recall that

$$\mathrm{SU}(2) = \left\{ U(a,b) = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} : \ a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}$$

can be identified with the unit sphere $S(\mathbb{H}) = \mathrm{Sp}(1) = S^3$ in the quaternions

$$\mathbb{H} = \{\alpha + \beta i + \gamma j + \delta k : \ ij = k = -ji, \text{ cyclic}\}$$

The isomorphism is given explicitly as

$$\mathrm{SU}(2) \ni U(a,b) \mapsto a + jb \in \mathrm{Sp}(1) = S^3$$

The relationship of principal SU(2)-bundles to quaternionic line bundles is analogous to the relationship of principal U(1)-bundles to complex line bundles.

Principal SU(2)-bundles and quaternionic line bundles

Recall that a **quaternionic line bundle** over a manifold M is a complex rank 2 vector bundle $V \to M$ together with a reduction of structure group to $\mathbb{H} \setminus \{0\}$. Note that the unit sphere bundle $S(V) \to M$ is an S^3-bundle together with the inherited group structure, i.e. a principal SU(2)-bundle.

Conversely, given a principal SU(2)-bundle $P \to M$, then the associated vector bundle

$$V = P \times_{\mathrm{SU}(2)} \mathbb{H} \to M$$

is a quaternionic line bundle.

Principal SU(2)-bundles on S^4

Principal SU(2)-bundles on S^4 are described by smooth maps $g : \mathrm{SU}(2) \to \mathrm{SU}(2)$. Let $g(z) = z$, $z \in \mathrm{SU}(2)$, which is a degree 1 map. Then $g(z) = z^r$, $r \in \mathbb{Z}$ is a degree r map. Let $P(r) \to S^4$ be the corresponding principal SU(2)-bundle on S^4. Then $c_2(P(r)) = r \in \mathbb{Z} \cong H^4(S^4, \mathbb{Z})$.

The principal SU(2)-bundle $S^7 = P(1) \to S^4$ is known as the **Hopf bundle.**

Principal SU(2)-bundles on M^4

Let M be a compact, connected, oriented 4-dimensional manifold. Then one can show fairly easily that isomorphism classes of principal SU(2)-bundles P on M is canonically identified with homotopy classes $[M, S^4] \cong H^4(M; \mathbb{Z})$ given by $c_2(P)$.

More precisely, given a degree 1 map $h : M \to S^4$, then $h^*(P(r)) \to M$ is a principal SU(2)-bundle on M with $c_2(h^*(P(r))) = r \in \mathbb{Z} \cong H^4(M, \mathbb{Z})$.

Spherical T-duality

Recall the Gysin sequence for principal SU(2)-bundles $\pi : P \to M$

$$\cdots \longrightarrow H^7(M) \xrightarrow{\pi^*} H^7(P) \xrightarrow{\pi_*} H^4(M) \xrightarrow{\cup c_2(P)} H^8(M) \longrightarrow \cdots$$

We consider pairs of the form (P, H) consisting of a principal SU(2)-bundle $P \to M$ and a 7-cocycle H on P.

The Gysin sequence implies that π_* is a canonical isomorphism $H^7(P, \mathbb{Z}) \cong H^4(M, \mathbb{Z}) \cong \mathbb{Z}$, and intuitively spherical T-duality exchanges H with the second Chern class c_2

Spherical T-duality

More precisely, the **spherical T-dual** bundle $\widehat{\pi} : \widehat{P} \to M$ is defined by $c_2(\widehat{P}) = \pi_* H$ while the dual 7-cocycle $\widehat{H} \in H^7(\widehat{P})$ is related to $c_2(P)$ by the isomorphism $\widehat{\pi}_*$, via a similar Gysin sequence for $\widehat{P} \to M$.

Isomorphism of 7-twisted cohomology

Let M be a connected compact, oriented, 4 dimensional manifold, and consider the principal SU(2)-bundle $P(r)$ over M with $c_2(P(r)) = r \in \mathbb{Z} \cong H^4(M, \mathbb{Z})$, together with the 7-cocycle $H = s \,\mathrm{vol}$ on $P(r)$.

Since $H \cup H = 0$ for dimension reasons, we can define **integer-valued H-twisted cohomology** as

$$H^\bullet(P(r), H; \mathbb{Z}) \equiv H^\bullet(H^\bullet(P(r); \mathbb{Z}), H\cup).$$

Isomorphism of 7-twisted cohomology

Use the Gysin sequence to calculate the cohomology groups $H^{even/odd}(F(p);\mathbb{Z})$, and obtain for $p \neq 0$

$$H^j(P(r);\mathbb{Z}) = H^{4-j}(M;\mathbb{Z}),\ j = 0,1,2,3$$
$$H^4(P(r);\mathbb{Z}) = \mathbb{Z}_r \oplus H^1(M;\mathbb{Z})$$
$$H^{7-j}(P(r);\mathbb{Z}) = H^{4-j}(M;\mathbb{Z}),\ j = 0,1,2,3$$

Therefore there is an isomorphism of 7-twisted cohomology groups over the integers with a parity change,

Theorem

$$H^{even}(P(r),s;\mathbb{Z}) \cong H^{odd}(P(s),r\,;\mathbb{Z})\,,$$
$$H^{odd}(P(r),s;\mathbb{Z}) \cong H^{even}(P(s),r\,;\mathbb{Z})\,.$$

There is a similar isomorphism of 7-twisted K-theories.

Spherical T-duality beyond dimension 4

Beyond dimension 4 the situation becomes more complicated as not all integral 4-cocycles of M are realized as c_2 of a principal SU(2)-bundle $\pi : P \to M$ and moreover multiple bundles can have the same $c_2(P)$.

More precisely, principal SU(2)-bundles are classified upto isomorphism by homotopy classes of maps into the classifying space $M \to B\mathrm{SU}(2)$. However, the complete homotopy type of $S^3 = \mathrm{SU}(2)$ is still unknown, and therefore also for $B\mathrm{SU}(2)$.

However Serre's theorem tells us that $\pi_j(B\mathrm{SU}(2)) \otimes \mathbb{Q} \cong \pi_j(K(\mathbb{Z}, 4)) \otimes \mathbb{Q}$, i.e. the homotopy groups of degree higher than 4 are all torsion.

Spherical T-duality beyond dimension 4

For example, recall that principal SU(2)-bundles over S^5 are classified by $\pi_4(\mathrm{SU}(2)) \cong \mathbb{Z}_2$, while $H^4(S^5, \mathbb{Z}) = 0$.

By a theorem of Granja, there is a natural number $N(d)$ where $d = \dim(M)$, such that if $\alpha \in N(d) \times H^4(M, \mathbb{Z})$, then it is the 2nd Chern class of a principal SU(2)-bundle over M. Therefore a pair (P, H) is spherical T-dualizable if $\pi_*(H) \in N(d) \times H^4(M; \mathbb{Z})$. Then $\pi_*(H) = c_2(\widehat{P})$ where $\widehat{P}$ is a principal SU(2)-bundle over M. However, this does not necessarily uniquely specify $\widehat{P}$. But at most, there are finitely many choices.

We will simply assert that a spherical T-dual $\widehat{\pi} : \widehat{P} \to M$ be any SU(2)-bundle with $c_2(\widehat{P}) = \pi_* H$, with $\widehat{H}$ defined such that $\widehat{\pi}_* \widehat{H} = c_2(P)$ with $\widehat{p}^* H = p^* \widehat{H}$ on the correspondence space $P \times_M \widehat{P}$.

Spherical T-duality beyond dimension 4

T-duality induces an isomorphism on twisted cohomologies with real or rational coefficients.

Theorem

$$H^{even}(P, H; \mathbb{Q}) \cong H^{odd}(\widehat{P}, \widehat{H}\,; \mathbb{Q}),$$
$$H^{odd}(P, H; \mathbb{Q}) \cong H^{even}(\widehat{P}, \widehat{H}\,; \mathbb{Q}).$$

There is a similar isomorphism of 7-twisted K-theories with parity shift, upto $\mathbb{Z}_2$-extensions.

Comments and open questions

1. T-duality for non-principal SU(2)-bundles (non-uniqueness, even for S^4)
2. A generalised geometry counterpart of spherical T-duality?
3. What is the physics behind spherical T-duality?
4. What are useful geometric realisations of integral 7-cocycles?
5. Is there a useful geometric description of 7-twisted K-theory?
6. When $\dim M \geq 4$, then it is known that not every spherical pair (P, H) has a spherical T-dual. Can the missing spherical T-duals be obtained some other way?
7. Is there a C*-algebra version of spherical T-duality?

THANK YOU