Vanishing on Witten Genus

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The classical $\widehat{A}$-form and Hirzebruch $\widehat{L}$-form are defined as follows

$$\widehat{A}(M,\nabla^{TM}) = \det \sqrt{\frac{\frac{\sqrt{-1}}{4\pi}R^{TM}}{\sinh\left(\frac{\sqrt{-1}}{4\pi}R^{TM}\right)}}$$

and

$$\widehat{L}(M,\nabla^{TM}) = \det \sqrt{\frac{\frac{\sqrt{-1}}{2\pi}R^{TM}}{\tanh\left(\frac{\sqrt{-1}}{4\pi}R^{TM}\right)}},$$

where $R^{TM} = (\nabla^{TM})^2$ is the curvature form.

Here we list some classical results from Index theory.
For a spin manifold M^{4k}, we have

$$\int_{M^{4k}} \widehat{A}(M^{4k}, \nabla^{TM}) ch(V_{\mathbb{C}}, \nabla^{V_{\mathbb{C}}}) \in \mathbb{Z}$$

and $$\int_{M^{8k+4}} \widehat{A}(M^{4k}, \nabla^{TM}) ch(V_{\mathbb{C}}, \nabla^{V_{\mathbb{C}}}) \in 2\mathbb{Z}$$

The signature σ of a manifold M^{4k} can be expressed as

$$\sigma(M^{4k}) = \int_{M^{4k}} \widehat{L}(M^{4k}, \nabla^{TM}).$$

Introduction III

Relations between A roof genus and Hirzebruch $\widehat{L}$ genus are given by the following examples,

$$\left\{\widehat{L}(M, \nabla^{TM})\right\}^{(4)} = -8\left\{\widehat{A}(M, \nabla^{TM})\right\}^{(4)}$$

and

$$\left\{\widehat{L}(M, \nabla^{TM})\right\}^{(12)} = \left\{8\widehat{A}(M, \nabla^{TM})ch(T_{\mathbb{C}}M, \nabla^{T_{\mathbb{C}}M}) - 32\widehat{A}(M, \nabla^{TM})\right\}^{(12)}$$

More generally we can deduce the following Ochanine divisibility from this procedure combined with the above results from Index theory.

Theorem (Ochanine)

For an $8k+4$ dimensional closed spin manifold, its signature $\sigma(M^{8k+4})$ is divisible by 16.

Witten Genus I

Question

What should be the $\widehat{A}$-genus on loop space LM of a manifold M?

We have a very good candidate which is called **Witten Genus**–a q-deformed $\widehat{A}$-genus.

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Witten Genus II

Let M be a $4k$ dimensional closed oriented smooth manifold and $\left\{\pm 2\pi\sqrt{-1}z_j; 1 \leq j \leq 2k\right\}$ denote the formal Chern roots of $T_{\mathbb{C}}M$.

The famous Witten genus is defined as follows

$$W(M^{4k}) = \int_{M^{4k}} \left(\prod_{j=1}^{2k} z_j \frac{\theta'(0,\tau)}{\theta(z_j,\tau)} \right) \in \mathbb{Q}[q],$$

where the Jacobi theta function $\theta(z,\tau) = 2q^{\frac{1}{4}} \sin(\pi z) \prod_{j=1}^{+\infty} (1-q^{2j})(1-e^{2\pi\sqrt{-1}z}q^{2j})(1-e^{-2\pi\sqrt{-1}z}q^{2j})$ and $q = e^{\pi\sqrt{-1}\tau}$.

Witten Genus III

Equivalently, Witten genus can be defined via Witten bundle

$$W(M^{4k}) = \int_{M^{4k}} \widehat{A}(M^{4k}, \nabla^{TM^{4k}}) ch(\Theta(T_{\mathbb{C}}M^{4k}), \nabla^{\Theta(T_{\mathbb{C}}M^{4k})}),$$

where the Witten bundle $\Theta(T_{\mathbb{C}}M^{4k}) = \bigotimes_{m=1}^{+\infty} S_{q^{2m}}(\widetilde{T_{\mathbb{C}}M^{4k}})$,

$S_t(E) = \mathbb{C}|_M + tE + t^2S^2(E) + \cdots$ and $\widetilde{T_{\mathbb{C}}M^{4k}} = T_{\mathbb{C}}M^{4k} - \mathbb{C}^{4k}$.

If M is spin ($w_1(TM) = 0$ and $w_2(TM) = 0$), then we know by Index theory

$$W(M^{4k}) = Ind(D \bigotimes \Theta(T_{\mathbb{C}}M^{4k})) \in \mathbb{Z}[q].$$

String Manifold

Although Witten genus is defined in a way that integration is over base manifold M, but the original idea to introduce Witten genus is to find out the counterpart of of "A roof genus" on loop space LM. As the index theory tell us the integrality property of A roof genus when we assume that M satisfy the spin condition.

Question

Is there any nice property of Witten genus when we assume certain condition of the loop space LM?

The answer is "Yes". We need to introduce the following concept of "string manifold" first.

A spin manifold M is called string if one more condition $\frac{1}{2}p_1(TM) = 0$ is satisfied, where p_1 is the first Pontryagin class.

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Modular Form

Theorem (Zagier)

If M is a $4k$ dimentional string (or even weaker: "spin" + "$p_1(TM) = 0$") manifold, then Witten genus $W(M^{4k})$ is a modular form of weight $2k$ over $SL(2, \mathbb{Z})$.

Question

What is a modular form?

Definition

Let Γ be a subgroup of $SL(2, \mathbb{Z})$. A modular form over Γ is a holomorphic function $f(\tau)$ on upper half plane H s.t.

$$f(g\tau) := f\left(\frac{a_1\tau + a_2}{a_3\tau + a_4}\right) = \chi(g)(a_3\tau + a_4)^k f(\tau)\ \forall g = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in \Gamma,$$

where $\chi : \Gamma \to \mathbb{C}^*$ is a character of Γ and k is called the weight of f.

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Positive scalar curvature and $\widehat{A}$-vanishing

Theorem (Lichnerowicz)

If M is a $4k$ dimensional spin manifold admitting a Riemannian metric with positive scalar curvature, then A roof genus $\widehat{A}(M) = 0$.

Question

Is there any similar vanishing results for Witten genus?

For the vanishing of Witten genus, we have the famous Höhn-Stolz conjecture.

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Positive Ricci curvature and Witten genus vanishing-Höhn-Stolz conjecture

Conjecture(Höhn-Stolz) *If M^{4k} is a string manifold admitting a Riemannian metric with positive Ricci curvature, then Witten genus $W(M^{4k}) = 0$.*

Theorem (Landweber-Stone)

If M^{4k} is a string complete intersection in a complex projective space, then Witten genus $W(M^{4k}) = 0$.

Remark

This confirms the Höhn-Stolz conjecture in a kind of special cases, because string complete intersections admits *a Riemannian metric with positive Ricci curvature.*

Generalization of Landweber-Stone Theorem

Theorem (Chen-Han)

If M^{4k} is a string complete intersection in a product of complex projective space, then Witten genus $W(M^{4k}) = 0$.

Remark

Our paper also gave a systematic machinery to produce string manifolds, which was also recommended by somebody on MathOverflow.

(generalized) complete intersection in a product of complex projective spaces

Let $V_{(d_{pq})}$ be a nonsingular $4k$-dimensional (generalized) complete intersection in a product of complex projective spaces $\mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2} \times \cdots \times \mathbb{C}P^{n_s}$, which is dual to $\prod_{p=1}^{t} \left(\sum_{q=1}^{s} d_{pq} x_q \right) \in H^{2t}(\mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2} \times \cdots \times \mathbb{C}P^{n_s}, \mathbb{Z})$, where $x_q \in H^2(\mathbb{C}P^{n_q}, \mathbb{Z})$ for $1 \le q \le s$ is the generator of $H^*(\mathbb{C}P^{n_q}, \mathbb{Z})$ and d_{pq} for $1 \le p \le t$, $1 \le q \le s$ are integers.

Let $P_q : \mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2} \times \cdots \times \mathbb{C}P^{n_s} \to \mathbb{C}P^{n_q}$ for $1 \le q \le s$ be the q-th projection. Actually $V_{(d_{pq})}$ is the intersection of the zero loci of smooth global sections of line bundles $\bigotimes_{q=1}^{s} P_q^*(\mathcal{O}(d_{pq}))$ for $1 \le p \le t$, where $\mathcal{O}(d_{pq}) = \mathcal{O}(1)^{d_{pq}}$ is the d_{pq}-th power of the canonical line bundle $\mathcal{O}(1)$ over $\mathbb{C}P^{n_q}$.

Spin (generalized) complete intersection

Let's compute Stiefel-Whitney class of $V_{(d_{pq})}$

Let $i : V_{(d_{pq})} \to \mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2} \times \cdots \times \mathbb{C}P^{n_s}$ be the inclusion. So we have

$$i^* T_{\mathbb{R}}(\mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2} \times \cdots \times \mathbb{C}P^{n_s}) \cong TV_{(d_{pq})} \bigoplus i^* \left(\bigoplus_{p=1}^{t} \left(\bigotimes_{q=1}^{s} P_q^* \mathcal{O}(d_{pq}) \right) \right)$$

and furthermore we have the following equation of Stiefel-Whitney class

$$w(TV_{(d_{pq})}) = \prod_{q=1}^{s} (1 + i^* x_q)^{n_q+1} \left(\prod_{p=1}^{t} \left(1 + \sum_{q=1}^{s} d_{pq} i^* x_q \right) \right)^{-1} \quad \mod 2$$

Thus we can easily obtain that

$$w_1(TV_{(d_{pq})}) = 0 \text{ and } w_2(TV_{(d_{pq})}) = \sum_{q=1}^{s} (n_q + 1 - \sum_{p=1}^{t} d_{pq}) i^* x_q \quad \mod 2$$

String (generalized) complete intersection

Similarly, we can also compute the Pontryagin class in the following way

$$p(TV_{(d_{pq})}) = \prod_{q=1}^{s} \left(1+(i^*x_q)^2\right)^{n_q+1} \left(\prod_{p=1}^{t}\left(1+\left(\sum_{q=1}^{s} d_{pq} i^* x_q\right)^2\right)\right)^{-1} \mod 2$$

Thus we can easily obtain that

$$p_1(TV_{(d_{pq})}) = \sum_{q=1}^{s}(n_q+1)(i^*x_q)^2 - \sum_{p=1}^{t}\left(\sum_{q=1}^{s} d_{pq} i^* x_q\right)^2 \mod 2$$

Let $D = \begin{pmatrix} d_{11} & d_{12} & \cdots & d_{1s} \\ d_{21} & d_{22} & \cdots & d_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ d_{t1} & d_{t2} & \cdots & d_{ts} \end{pmatrix}$ and let be m_q the number of nonzero elements in the q-th column of D.

Example of string (generalized) complete intersection

Theorem (Chen-Han)

We assume $m_q + 2 \le n_q$ *for* $1 \le q \le s$. *If* $p_1(V_{(d_{pq})}) = 0$, *then* $V_{(d_{pq})}$ *is spin. Therefore* $V_{(d_{pq})}$ *is string if and only if any of the following holds*

- $p_1(V_{(d_{pq})}) = 0$
- $$\begin{cases} \sum_{p=1}^{t} d_{pq}^2 = n_q + 1 & \text{for } 1 \le q \le s, \\ \sum_{p=1}^{t} d_{pu} d_{pv} = 0 & \text{for } 1 \le u, v \le s \text{ and } u \ne v. \end{cases}$$
- $D^t D = diag(n_1 + 1, ..., n_s + 1)$

Example

Consider $D = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$, $n_1 = n_2 = 4$, i.e. Let V_D be a nonsingular 12-dimensional (generalized) complete intersection in $\mathbb{C}P^4 \times \mathbb{C}P^4$, which is dual to $(x_1 + 2x_2)(2x_1 - x_2) \in H^4(\mathbb{C}P^4 \times \mathbb{C}P^4, \mathbb{Z})$.

Positive scalar curvature and α-invariants

The A roof genus $\widehat{A}$ is determined by

$$\widehat{A} : \Omega_{4k}^{spin} \to \mathbb{Z} \text{ via } \widehat{A}(M) := IndD$$

Now we consider the $\mod 2$ analogue to A roof genus $\widehat{A}$, which is called alpha invariant $\alpha(M)$ by Atiyah and determined by

$$\alpha : \Omega_{8k+1/8k+2}^{spin} \to \mathbb{Z}_2 \text{ via } \alpha(M) := Ind_2 D$$

We state a vanishing result considering alpha invariant $\alpha(M)$ due to Hitchin, which could be seen as the analogue of Lichnerowicz Theorem.

Theorem (Hitchin)

If M is a $8k+1/8k+2$ dimensional spin manifold admitting a Riemannian metric with positive scalar curvature, then the alpha invariant $\alpha(M) = 0$.

Current Picture

Now it's time to summarize the current situation.

spin	*string*
$\left[\begin{array}{c}\text{Vanishing of } \widehat{A}(M^{4k}) : \\ \textbf{Lichnerowicz}\text{ Theorem}\end{array}\right]$	$\left[\begin{array}{c}\text{Vanishing of } W(M^{4k}) : \\ \textbf{Höhn-Stolz}\text{ Conjecture} \\ \text{(Supporting Examples:} \\ \textbf{Landweber-Stone}\text{ Theorem)}\end{array}\right]$
$\left[\begin{array}{c}\text{Vanishing of } \alpha(M^{8k+2}) : \\ \textbf{Hitchin}\text{ Theorem}\end{array}\right]$	$??$

$\bmod 2$ Witten genus–"q-deformed α-invariant"

So it is very natural to introduce the following $\bmod 2$ Witten genus.

Definition (Chen-Han-Zhang)

Let B be a $8k+2$ dimensional compact spin manifold, then the $\bmod 2$ Witten genus $\phi(B)$ is given by

$$\phi(B) := Ind_2(\Theta(TB)) \in \mathbb{Z}_2[[q]],$$

where Ind_2 is the $\bmod 2$ index.

The next step is to formulate $\bmod 2$ Höhn-Stolz conjecture in a reasonable way.

Conjecture ($\mod 2$ Höhn-Stolz, Chen-Han-Zhang) *If M^{8k+2} is a string manifold admitting a Riemannian metric with positive Ricci curvature, then $\mod 2$ Witten genus $\phi(M^{8k+2}) = 0$.*

Theorem ($\mod 2$ Landweber-Stone type, Chen-Han-Zhang)

If M^{8k+2} is a string complete intersection in a complex projective space, then $\mod 2$ Witten genus $\phi(M^{8k+2}) = 0$.

Remark

This confirms the $\mod 2$ Höhn-Stolz conjecture in a kind of special cases, because string complete intersections admits *a Riemannian metric with positive Ricci curvature.*

Example

Let $V_{2,2}$ be a nonsingular 10-dimensional (generalized) complete intersection in $\mathbb{C}P^7$, which is dual to $(2x)\,(2x) \in H^4(\mathbb{C}P^7, \mathbb{Z})$.

$\bmod 2$ Höhn-Stolz Conjecture

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Sketch of proof of $\mod 2$ Landweber-Stone type Theorem

1. To prove the following

Main Theorem (Chen-Han-Zhang)

If $m_\beta + 2 \leq n_\beta$ for $1 \leq \beta \leq s$, $\dim_{\mathbb{R}} V_{(d_{\alpha\beta})} = 8k+2$, $V_{(d_{\alpha\beta})}$ is string and one of the generalized hypersurfaces that generate $V_{(d_{\alpha\beta})}$ is of even degree (i.e. one of the rows consists of only even numbers), then the $\mod 2$ Witten genus $\phi(V_{(d_{\alpha\beta})})$ vanishes.

2. If we set $s = 1$, then the main theorem reduce to classical complete intersection in a complex projective space. For the dimension reason, any string complete intersection must have at least one even degree.

How to Compute $\mod 2$ index?

Atiyah and Singer pointed out that in practice $\mod 2$ index is difficult to compute in their celebrated paper.

Now we introduce a very powerful theorem proved by Weiping Zhang, which tells one how to compute identities involving $\mod 2$ index.

Theorem (Zhang)

Let M be a $8k+4$ dimensional spin^c manifold and $B \subset M$ is dual to $w_2(TM)$. Then $M \backslash B$ is spin. Let $i_B : B \hookrightarrow M$ denote the canonical embedding of B into M and $e \in H^2(M, \mathbb{Z})$ be the dual of $[B] \subset H_{8k+2}(M, \mathbb{Z})$. The following identities holds

$$< \widehat{A}(M)ch(E_{\mathbb{C}})\exp(\frac{e}{2}), [M] > \equiv Ind_2(i_B^* E) \mod 2,$$

where E is a real vector bundle over M and $i_B^* E$ is a real vector bundle over spin manifold B.

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Sketch of proof of Main Theorem I

1. Choose a suitable 8k+4 dimensional spin^c manifold V_t, s.t. $V_{(d_{\alpha\beta})} \subset V_t$ is dual to $w_2(TV_t)$
2. Apply Zhang's theorem
3. Express the identity in terms of Jacobi-theta functions.
4. By using the property of modular invariance and Residue theorem, we obtain the final result.

Sketch of proof of Main Theorem II

In order to successfully apply modular invariance, we need to use the subtlety in our assumption that entries in one row of degree matrix D are all even. Without lost of generality, we assume $d_{t\beta}$, $1 \leq \beta \leq s$ are even numbers. Let V_t be the generalized complete intersection determined by $\bigotimes_{\beta=1}^{s} P_\beta^*(\mathcal{O}(d_{\alpha\beta}))$ for $1 \leq \alpha \leq t-1$. Denote $\bigotimes_{\beta=1}^{s} P_\beta^*(\mathcal{O}(d_{t\beta}))$ by ζ. Let $i : V_t \hookrightarrow \mathbb{C}P^{n_1} \times \cdots \times \mathbb{C}P^{n_s}$ and $i_{V_{(d_{\alpha\beta})}} : V_{(d_{\alpha\beta})} \hookrightarrow V_t$ be the embeddings.

Clearly, $[V_{(d_{\alpha\beta})}] \in H_{8k+2}(V_t, \mathbb{Z}_2)$ is dual to $c_1(i^*\zeta) \mod 2$ and we have

$$
\begin{aligned}
c_1(i^*\zeta) &= \sum_{\beta=1}^{s} d_{t\beta} i^* x_\beta \equiv \sum_{\beta=1}^{s} d_{t\beta}^2 i^* x_\beta = \sum_{\beta=1}^{s} \left(n_\beta + 1 - \sum_{\alpha=1}^{t-1} d_{\alpha\beta}^2 \right) i^* x_\beta \\
&\equiv \sum_{\beta=1}^{s} \left(n_\beta + 1 - \sum_{\alpha=1}^{t-1} d_{\alpha\beta} \right) i^* x_\beta \equiv w_2(TV_t) \mod 2,
\end{aligned}
$$

Sketch of proof of Main Theorem III

Then we can apply the Zhang's theorem

$$\phi(V_{(d_{\alpha\beta})}) \equiv Ind_2(\Theta(TV_{(d_{\alpha\beta})}))$$

$$\begin{aligned}\phi(V_{(d_{\alpha\beta})}) &\equiv Ind_2(\Theta(TV_{(d_{\alpha\beta})})) \\ &\equiv \left\langle A(TV_t)\cosh\left(\frac{\sum_{\beta=1}^{s} d_{t\beta} i^* x_\beta}{2}\right) ch(\Theta(T_C V_t - i^*\zeta_{\mathbb{R}} \otimes \mathbb{C})), [V_t] \right\rangle \\ &\quad \bmod 2\mathbb{Z}[[q]],\end{aligned}$$

where we have used $i^*_{V_{(d_{\alpha\beta})}}(TV_t - i^*\zeta_{\mathbb{R}}) = TV_{(d_{\alpha\beta})}$.

Complete Picture

Now we have a complete understanding of this story.

spin	*string*
Vanishing of $\widehat{A}(M^{4k})$: **Lichnerowicz** Theorem	Vanishing of $W(M^{4k})$: **Höhn-Stolz** Conjecture (Supporting Examples: **Landweber-Stone** Theorem)
Vanishing of $\alpha(M^{8k+2})$: **Hitchin** Theorem	Vanishing of $\phi(M^{8k+2})$: $mod2$ **Höhn-Stolz** Conjecture (Chen-Han-Zhang) Supporting Examples: **Landweber-Stone** type Theorem (Chen-Han-Zhang)

Thank you!