# Infinitely many solutions for nonlinear Schrödinger equations involving electromagnetic fields and critical growth

Shuangjie Peng
with Shusen Yan and Chunhua Wang

School of Mathematics and Statistics
Central China Normal University

Dec. 8-12, 2014, National University of Singapore

# Main contents

# The problem

$$\left(\frac{\nabla}{i} - A(|y'|, y'')\right)^2 u + V(|y'|, y'')u = |u|^{\frac{4}{N-2}} u, \; u \in H^1(\mathbb{R}^N, \mathbb{C}), \qquad (1)$$

where $A(|y'|, y'')$ is a bounded map from $\mathbb{R}^+ \times \mathbb{R}^{N-2}$ to $\mathbb{R}^N$, and $V(|y'|, y'')$ is a bounded non-negative function in $\mathbb{R}^+ \times \mathbb{R}^{N-2}$.

$A = (A_1, \cdots, A_N)$ : vector magnetic potential with magnetic field $B = curl A$. $A$ can also be regarded as a $1-$form $A = \sum_{j=1}^N A_j dx^j$, then

$$B = dA = \sum_{j<k} B_{jk} dx^j \wedge dx^k, \;\; B_{jk} := \partial_j A_k - \partial_k A_j;$$

$$\left(\frac{\nabla}{i} - A(|y'|, y'')\right)^2 u = -\Delta u - \frac{1}{i}(divA)u - \frac{2}{i} A\nabla u + |A|^2 u.$$

(Reed-Simon: Methods of Modern Math. Phys.; Lied-Loss: Analysis)

# Background

Nonlinear Schrödinger equation with electromagnetic fields

$$i\hbar\frac{\partial\Psi}{\partial t}=\Big(\frac{\hbar}{i}\nabla-A(y)\Big)^2\Psi+G(y)\Psi-f(y,|\Psi|)\Psi,\ (t,y)\in\mathbb{R}\times\mathbb{R}^N,\quad (2)$$

where $\Psi(y,t)$ takes on complex values, $\hbar$ is the Planck constant, $i$ is the imaginary unit, $A:\mathbb{R}^N\to\mathbb{R}^N$ is a magnetic potential, $G:\mathbb{R}^N\to\mathbb{R}$ represents an electric potential, and $N\geq 2$.

This type of nonlinear Schrödinger equations arises in different physical theories, e.g. the description of Bose-Einstein condensates, plasma physics and nonlinear optics. See [C. Sulem, P. L. Sulem, Applied Mathematical Sciences, 139. Springer-Verlag, New York, 1999.] The presence of many particles leads one to consider nonlinear terms which simulate the interaction effect among them.

We are interested in standing wave solutions for (2), i.e. solutions of the form $\Psi(y,t) = e^{-\frac{iEt}{\hbar}} u(y)$ for some function $u : \mathbb{R}^N \mapsto \mathbb{C}$. Thus, one is led to solve the complex equation in $\mathbb{R}^N$,

$$\Big(\frac{\hbar}{i}\nabla - A(y)\Big)^2 u + (G(y) - E)u = f(y, |u|)u. \tag{3}$$

Set $V(y) = G(y) - E$. We consider $f(y, |u|) = |u|^{\frac{4}{N-2}}$. i.e.

$$\Big(\frac{\nabla}{i} - A(|y'|, y'')\Big)^2 u + V(|y'|, y'')u = |u|^{\frac{4}{N-2}} u, \;\; u \in H^1(\mathbb{R}^N, \mathbb{C})$$

## Some known results

The case $A(y) \equiv 0$.

In this case one is led to look for solutions $u : \mathbb{R}^N \to \mathbb{R}$ of the semilinear equation

$$-\hbar^2 \Delta u(y) + V(y)u(y) = f(y, |u|)u. \tag{4}$$

The study of (4) for fixed $\hbar > 0$ and subcritical nonlinearities dates back to the 1970s. See A. Bahri, Berestiky, T. Bartsch, D. Cao, Dancer, G. Li, Y .Y. Li, Z. Liu, P. L. Lions, Z.-Q. Wang, J. Wei, M. Willem, S. Yan, $\cdots$.

The variational frame was employed.

In the semiclassical case $\hbar \to 0$, peak solutions for (4):

- A. Ambrosetti, J. Byeon, D. Cao, A. Floer, Y. G. Oh, M. del Pino, P. Felmer, A. Malchiodi, S. Peng, P. H. Rabinowitz, S. Secchi, Y. Y. Li, Z. Liu, E. S. Noussair, S. Yan, Z.-Q. Wang, J. Wei, A. Weinstein, H. Zhou, $\cdots$.

The solutions present concentration phenomenon as $\hbar \to 0$.

For (3) with $A(x) \not\equiv 0$: existence of different kinds of solutions for (3) with sub-critical non-linearities was proved. See

- M. Esteban, P. . Lions (1989): the first one with the variational argument for fixed $h$.

  For semiclassical case: $h$ small

- K. Kurata (2000): the least energy solutions:

$$u_h \sim e^{i(\omega + A(x_h)\frac{x-x_h}{h})} u(\frac{x-x_h}{h}),$$

where $u$ is a least energy solution of

$$-\Delta u(y) + V(x_h)u(y) = f(y, |u|)u.$$

- S. Cingolani, J. Differential Equats. 188 (2003).
- S. Cingolani, S. Secchi, J. Math. Phys. 46 (2005).
- T. Bartsch, E. N. Dancer, S. Peng, Adv. Differential Equations 11 (2006).
- D. Cao, Z. Tang, J. Differential Equations 222 (2006).
- S. Cingolani, M. Clapp, Nonlinearity 22 (2009), 2309-2331.

Helffer (1994,1997) studied asymptotic behavior of the eigenfunctions of the Schrödinger operators with magnetic fields in the semiclassical limit.

More results can also be found

- B. Helffer, J. Sjöstrand, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 14 (1987), 625-657 (1988).
- Z. Tang, J. Differential Equations 245 (2008).
- S. Cingolani, L. Jeanjean, S. Secchi, ESAIM Control Optim. Calc. Var. 15 (2009), 653-675.
- H. Pi, C. Wang, ESAIM Control Optim. Calc. Var. 19 (2013), 91-111.

Not much is known for (3) or (4) with the critical nonlinearity $f(y,|u|)u = |u|^{2^*-2}u$. Arioli and Szulkin (2003) proved the existence of a ground state solution under the assumption that $N \geq 4$ and $V$ is negative somewhere. They also proved that if $V \geq 0$ and $V \neq 0$, then (3) has no ground state solution.

Concerning the existence of solution for (4) with $V \geq 0$, Benci and Cerami (1990) showed that (4) with $\hbar = 1$ has a solution if $\|V\|_{L^{\frac{N}{2}}}$ is sufficiently small. Note that the assumption $V \in L^{\frac{N}{2}}(\mathbb{R}^N)$ excludes the case $V \geq c_0 > 0$.

There is no other result for (4) until the work of Chen, J. Wei and S. Yan (2012), where they prove (4) had infinitely many nonradial solutions if $N \geq 5$, $V(x)$ is radially symmetric and $r^2V(r)$ has a local maximum point, or a local minimum point $r_0 > 0$ with $V(r_0) > 0$.

Other results for (3) involving critical non-linearity: J. Chabrowski, A. Szulkin (2005): sign-changing $V(y)$; S. Barile, S. Cingolani, S. Secchi (2006): small $A(x)$ and small $V(x)$.

Some interesting results on (2) with isolated singularity of the electromagnetic potential: V. Felli, A. Ferrero, S. Terracini (2011), asymptotic behavior of solutions to (2) with singular magnetic and electric potentials; L. Abatangelo, S. Terracini(2011); V. Felli, E. M. Marchini, S. Terracini(2009).

# Our aim

Firstly, we weaken the symmetry condition imposed on $V$ in [W. Chen, J. Wei, S. Yan, JDE. 252 (2012)].

Secondly, we intend to detect the influence of the magnetic potential $A$ on the solutions.

So we consider the following problem:

$$\Big(\frac{\nabla}{i} - A(|y'|, y'')\Big)^2 u + V(|y'|, y'')u = |u|^{\frac{4}{N-2}} u, \;\; u \in H^1(\mathbb{R}^N, \mathbb{C}), \tag{5}$$

where $V(r, y'') \geq 0$, and $y = (y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2}$.

# Main result

**Theorem 1.1**

Suppose that both $A$ and $V \geq 0$ are bounded, and belong to $C^1$. If $N \geq 5$ and $r^2V(r, y'')$ has an isolated local minimum point or local maximum point $(r_0, y_0'')$ with $r_0 > 0$ and $r_0^2V(r_0, y_0'') > 0$, then problem (5) has infinitely many complex-valued solutions, whose energy can be made arbitrarily large.

We do not assume the condition

$$V(|y|) \geq V_0 > 0, \ \text{for } |y| \text{ large},$$

which is essential to obtain the existence results for Schrödinger equation with subcritical growth.

The symmetry condition for $V$ is weaker than that in [Chen-Wei-Yan, JDE. 252 (2012)].

When $A \not\equiv 0$, (5) is a complex-valued problem and cannot be reduced to a real-valued single equation. Note that if we write (5) in real form, it corresponds to a system instead of a single equation. So we may expect that some estimates in this paper are quite different from those used in [Wei-Yan, JFA. 258 (2010); Wei-Yan, JMPA. (9) 96 (2011); Chen-Wei-Yan, JDE. 252 (2012); Peng-Wang, ARMA(2013) ].

$$\begin{cases} -\Delta v - div(A)w - A\nabla w + |A|^2 v + Vv = (v^2+w^2)^{\frac{2}{N-2}} v \\ -\Delta w + div(A)v + A\nabla w + |A|^2 w + Vw = (v^2+w^2)^{\frac{2}{N-2}} w \end{cases}$$

# Difficulties

- Since $V \geq 0$, (5) has no ground state solution. So it is extremely hard to use the variational techniques to obtain existence results for (5).
- Some estimates should be established for complex-valued functions.
- The appearance of magnetic potential $A$ brings us linear term $div(A)u$ and gradient term $A\nabla u$, which need more technique and fine estimates.

# The main idea in the proof of Theorem 1.1

It is well known that the functions

$$U_{x,\lambda}(y)=[N(N-2)]^{\frac{N-2}{4}}\Big(\frac{\lambda}{1+\lambda^2|y-x|^2}\Big)^{\frac{N-2}{2}},\ \ \lambda>0,\ x\in\mathbb{R}^N,$$

are the only solutions to the problem

$$-\Delta u=u^{\frac{N+2}{N-2}},\ \ u>0\ \text{ in } \mathbb{R}^N. \tag{6}$$

One can check that $e^{i\sigma}U_{x,\lambda}(y),\sigma\in\mathbb{R}^1$ are solutions of

$$-\Delta u=|u|^{\frac{4}{N-2}}u,\ u\in D^{1,2}(\mathbb{R}^N,\mathcal{C}). \tag{7}$$

It was proved that $e^{i\sigma}U_{x,\lambda}(y)$ is non-degenerate. More precisely, define the functional corresponding to (7) as follows

$$f_0(u)=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2-\frac{1}{2^*}\int_{\mathbb{R}^N}|u|^{2^*},\ \ u\in D^{1,2}(\mathbb{R}^N,\mathcal{C}).$$

$f_0$ possesses a finite-dimensional manifold $Z$ of least energy critical points, given by

$$Z = \{e^{i\sigma}U_{x,\lambda} : \ \sigma \in \mathbb{R}^1, \ \lambda > 0, \ x \in \mathbb{R}^N\}.$$

Moreover,

$$ker\, f_0''(z) = span_{\mathbb{R}}\Big\{\frac{\partial z}{\partial x^1}, \cdots, \frac{\partial z}{\partial x^N}, \frac{\partial z}{\partial \lambda}, \frac{\partial z}{\partial \sigma}\Big\}, \ \forall\ z = e^{i\sigma}U_{x,\lambda} \in Z.$$

We use the least energy solutions of problem

$$\Big(\frac{\nabla}{i} - A_0\Big)^2 u = |u|^{\frac{4}{N-2}}u, \ u \in H^1(\mathbb{R}^N, \mathcal{C}), \tag{8}$$

where $A_0 = (A_{0,1}, A_{0,2}, ..., A_{0,N})$ is a constant vector, to construct approximate solutions for (5). The functional corresponding to (8) is

$$I_0(u) = \frac{1}{2}\int_{\mathbb{R}^N}\Big|\frac{\nabla}{i}u - A_0 u\Big|^2 - \frac{1}{2^*}\int_{\mathbb{R}^N}|u|^{2^*}, \ \forall\ u \in H^1(\mathbb{R}^N, \mathbb{C}).$$

## Remark

We can infer that $e^{i\sigma+iA_0\cdot(y-x)}U_{x,\lambda}$ is non-degenerate. That is

$$ker I_0''(e^{i\sigma+iA_0\cdot(y-x)}U_{x,\lambda}) = span_{\mathbb{R}}\left\{\partial(e^{i\sigma+iA_0\cdot(y-x)}U_{x,\lambda})\right\},$$

where $\partial(e^{i\sigma+iA_0\cdot(y-x)}U_{x,\lambda})$ represents all the partial derivatives with respect to $x_1, \cdots, x_N$, $\lambda$ and $\sigma$.

Define

$$\begin{aligned} H_s = \Big\{u : \ & \int_{\mathbb{R}^N} \Big(\Big|\frac{\nabla}{i}u - Au\Big|^2 + V|u|^2\Big) < +\infty, \\ & u(y_1, -y_2, y'') = u(y_1, y_2, y''), \\ & u(r\cos\theta, r\sin\theta, y'') = u\Big(r\cos\Big(\theta + \frac{2\pi j}{m}\Big), r\sin\Big(\theta + \frac{2\pi j}{m}\Big), y''\Big)\Big\} \end{aligned}$$

Let

$$x_j = \Big(r\cos\frac{2(j-1)\pi}{m}, r\sin\frac{2(j-1)\pi}{m}, \bar{y}''\Big), \quad j = 1, 2, ..., m,$$

where $\bar{y}''$ is a vector in $\mathbb{R}^{N-2}$.

We will use $e^{i\sigma+iA(x_j)\cdot(y-x_j)}U_{x_j,\lambda}$ as an approximate solution. To deal with the slow decay of this function when $N$ is not big, we define $\xi(y) = \xi(|y'|, y'')$ be a smooth function satisfying $\xi = 1$ if $|(|y'|, y'') - (r_0, y_0'')| \le \delta$, $\xi = 0$ if $|(|y'|, y'') - (r_0, y_0'')| \ge 2\delta$, and $0 \le \xi \le 1$. Denote

$$\eta_j(y) = i\sigma + iA(x_j)\cdot(y - x_j), \quad Z_{x_j,\lambda,\sigma}(y) = \xi e^{\eta_j}U_{x_j,\lambda}$$

$$Z_{r,\bar{y}'',\lambda,\sigma}(y) = \sum_{j=1}^{m} Z_{x_j,\lambda,\sigma}(y).$$

$\lambda \in [L_0 m^{\frac{N-2}{N-4}}, L_1 m^{\frac{N-2}{N-4}}]$ for some constants $L_1 > L_0 > 0$ and $|(r, \bar{y}'') - (r_0, y_0'')| \leq \theta$.

To prove Theorem 1.1, we will show the following result.

## Proposition 1.2

Under the assumptions of Theorem 1.1, there exists a positive integer $m_0 > 0$, such that for any integer $m \geq m_0$, (2) has a solution $z_m$ of the form

$$z_m = Z_{r_m, \bar{y}_m'', \lambda_m, \sigma}(y) + \omega_m = \sum_{j=1}^{m} \xi e^{i\sigma + iA(x_j)\cdot(y - x_j)} U_{x_j, \lambda_m}(y) + \omega_m,$$

where $\sigma$ is any real number, and $\omega_m \in H_s$. Moreover, as $m \to +\infty$, $\lambda_m \in [L_0 m^{\frac{N-2}{N-4}}, L_1 m^{\frac{N-2}{N-4}}]$, $(r_m, \bar{y}_m'') \to (r_0, y_0'')$, and $\lambda_m^{-\frac{N-2}{2}} \|\omega_m\|_{L^\infty} \to 0$.

Proposition 1.2 implies that (5) has solutions with large number of bubbles near $(r_0, y_0'')$. It is also worth pointing out that the magnetic vector fields do not affect the locations of the bubbles of the solutions.

Let us point out a simple fact which implies that some estimates in the complex case are harder to obtain than the real-value case. To estimate the derivative of the energy of $z_m$, we need to control some terms involving $|Z_{r_m,\bar{y}''_m,\lambda_m,\sigma}|^{2^*-3}\frac{\partial Z_{x_1,\lambda,\sigma}}{\partial\lambda}$. If $N \geq 7$, $2^*-3<0$. In the real case $A \equiv 0$, we have $|Z_{r_m,\bar{y}''_m,\lambda_m,\sigma}| \geq \frac{1}{2}U_{x_1,\lambda}$, which gives $|Z_{r_m,\bar{y}''_m,\lambda_m,\sigma}|^{2^*-3}\left|\frac{\partial Z_{x_1,\lambda,\sigma}}{\partial\lambda}\right| \leq \frac{C}{\lambda}U_{x_1,\lambda}^{2^*-2}$. However, in the complex case, it is not easy to prove the weaker inequality $|Z_{r_m,\bar{y}''_m,\lambda_m,\sigma}| \geq c_0 U_{x_1,\lambda}$ for some small $c_0>0$.

Let

$$\|u\|_* = \sup_{y\in\mathbb{R}^N}\Big(\sum_{j=1}^m \frac{1}{(1+\lambda|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{-1}\lambda^{-\frac{N-2}{2}}|u(y)| \tag{9}$$

and

$$\|f\|_{**} = \sup_{y\in\mathbb{R}^N}\Big(\sum_{j=1}^m \frac{1}{(1+\lambda|y-x_j|)^{\frac{N+2}{2}+\tau}}\Big)^{-1}\lambda^{-\frac{N+2}{2}}|f(y)|, \tag{10}$$

where $\tau = \frac{N-4}{N-2}$. For this choice of $\tau$, we find that

$$\sum_{j=2}^m \frac{1}{|\lambda(x_j-x_1)|^\tau} \le \frac{Cm^\tau}{\lambda^\tau}\sum_{j=2}^m \frac{1}{j^\tau} \le \frac{Cm}{\lambda^\tau} \le C'.$$

Denote

$$Z_{j,1} = \frac{\partial Z_{x_j,\lambda,\sigma}}{\partial r}, \quad Z_{j,2} = \frac{\partial Z_{x_j,\lambda,\sigma}}{\partial \sigma}, \quad Z_{j,N+1} = \frac{\partial Z_{x_j,\lambda,\sigma}}{\partial \lambda},$$
$$Z_{j,k} = \frac{\partial Z_{x_j,\lambda,\sigma}}{\partial \bar{y}''_k}, \quad k = 3, ..., N.$$

Now we consider

$$\Big(\frac{\nabla}{i}-A(|y'|,y'')\Big)^2(Z_{r,\bar y'',\lambda,\sigma}+\varphi)+V(|y'|,y'')(Z_{r,\bar y'',\lambda,\sigma}+\varphi)$$
$$= |Z_{r,\bar y'',\lambda,\sigma}+\varphi|^{2^*-2}(Z_{r,\bar y'',\lambda,\sigma}+\varphi)+\sum_{l=1}^{N+1}c_l\sum_{j=1}^{m}|Z_{x_j,\lambda,\sigma}|^{2^*-2}Z_{j,l}, \quad \text{in } \mathbb{R}^N$$
$$\varphi\in H_s,\ \ \langle\textstyle\sum_{j=1}^{m}|Z_{x_j,\lambda,\sigma}|^{2^*-2}Z_{j,l},\varphi\rangle=0,\ l=1,2,...,N+1. \qquad (11)$$

## Proposition 2.3

There exists a positive integer $m_0$ such that for each $m \geq m_0, \lambda \in [L_0 m^{\frac{N-2}{N-4}}, L_1 m^{\frac{N-2}{N-4}}], \sigma \in \mathbb{R}, r \in [r_0 - \theta, r_0 + \theta], \bar{y}'' \in B_\theta(y_0''),$ where $\theta > 0$ small, (11) has a unique solution $\varphi = \varphi_{r,\bar{y}'',\lambda,\sigma} \in H_s$ satisfying

$$\|\varphi\|_* \leq C \left(\frac{1}{\lambda}\right)^{1+\epsilon}, \quad |c_l| \leq C \left(\frac{1}{\lambda}\right)^{1+n_l+\epsilon}, \tag{12}$$

where $\epsilon > 0$ is a small constant.

Moreover, $\varphi_{r,\bar{y}'',\lambda,\sigma} = e^{i\sigma} \varphi_{r,\bar{y}'',\lambda,0}$ for any $\sigma \in \mathbb{R}$.

Rewrite (11) as

$$
\begin{aligned}
&\Big(\frac{\nabla}{i}-A(|y'|,y'')\Big)^2\varphi+V(|y'|,y'')\varphi\\
&-[(2^*-2)|Z_{r,\bar y'',\lambda,\sigma}|^{2^*-4}Z_{r,\bar y'',\lambda,\sigma}\mathrm{Re}(\bar Z_{r,\bar y'',\lambda,\sigma}\varphi)+|Z_{r,\bar y'',\lambda,\sigma}|^{2^*-2}\varphi]\\
=\ &N(\varphi)+l_m+\sum_{l=1}^{N+1}c_l\sum_{j=1}^{m}|Z_{x_j,\lambda,\sigma}|^{2^*-2}Z_{j,l},\quad \text{in } \mathbb{R}^N,\\
&\qquad\varphi\in H_s,\ \ \langle\textstyle\sum_{j=1}^m|Z_{x_j,\lambda,\sigma}|^{2^*-2}Z_{j,l},\varphi\rangle=0,\ l\ =1,2,\cdots,N+1,
\end{aligned}
\tag{13}
$$

where

$$
\begin{aligned}
N(\varphi)\ =\ &\Big(|Z_{r,\bar y'',\lambda,\sigma}+\varphi|^{2^*-2}(Z_{r,\bar y'',\lambda,\sigma}+\varphi)-|Z_{r,\bar y'',\lambda,\sigma}|^{2^*-2}Z_{r,\bar y'',\lambda,\sigma}\\
&-\Big[(2^*-2)|Z_{r,\bar y'',\lambda,\sigma}|^{2^*-4}\mathrm{Re}(\bar Z_{r,\bar y'',\lambda,\sigma}\varphi)Z_{r,\bar y'',\lambda,\sigma}\\
&+|Z_{r,\bar y'',\lambda,\sigma}|^{2^*-2}\varphi\Big]\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
& l_m \\
= \; & \Big(|Z_{r,\bar{y}'',\lambda,\sigma}|^{2^*-2} Z_{r,\bar{y}'',\lambda,\sigma} - \sum_{j=1}^{m} |Z_{x_j,\lambda,\sigma}|^{2^*-2} Z_{x_j,\lambda,\sigma}\Big) - V(|y'|, y'') Z_{r,\bar{y}'',} \\
& + \frac{1}{i} \mathrm{div} A Z_{r,\bar{y}'',\lambda,\sigma} + |A(r,\bar{y}'') - A(|y'|, y'')|^2 Z_{r,\bar{y}'',\lambda,\sigma} \\
& + 2A(r,\bar{y}'') \cdot (A(r,\bar{y}'') - A(|y'|, y'')) Z_{r,\bar{y}'',\lambda,\sigma} \\
& - \frac{2}{i} (A(r,\bar{y}'') - A(|y'|, y'')) \cdot \nabla Z_{r,\bar{y}'',\lambda,\sigma} \\
& + \Big(\frac{\nabla}{i} - A(|y'|, y'')\Big)^2 (1-\xi) \Big[\sum_{j=1}^{m} e^{\eta_j} U_{x_j,\lambda}\Big] \\
=: \; & \sum_{j=0}^{6} J_j.
\end{aligned}
$$

### Lemma 2.4

If $N \geq 5$, then

$$||N(\varphi)||_{**} \leq C\|\varphi\|_{*}^{\min(2^*-1,2)}.$$

### Lemma 2.5

If $N \geq 5$, then there is a constant $\epsilon > 0$, such that

$$\|l_m\|_{**} \leq C\Big(\frac{1}{\lambda}\Big)^{1+\epsilon}.$$

Let $\tilde{F}(r,\bar{y}'',\lambda,\sigma)=I(Z_{r,\bar{y}'',\lambda,\sigma}+\varphi_{r,\bar{y}'',\lambda,\sigma})$, where $r=|x_1'|$, and $\varphi_{r,\bar{y}'',\lambda,\sigma}$ is obtained in Proposition 2.3, and

$$I(u)=\frac{1}{2}\int\Big(\Big|\Big(\frac{\nabla}{i}-A(|y'|,y'')\Big)u\Big|^2+V(|y'|,y'')|u|^2\Big)-\frac{1}{2^*}\int|u|^{2^*}.$$

We will choose $(r,\bar{y}'',\lambda,\sigma)$ such that $Z_{r,\bar{y}'',\lambda,\sigma}+\varphi_{r,\bar{y}'',\lambda,\sigma}$ is a solution of (5). We only need to prove that $(r,y'',\lambda,\sigma)$ is a critical point of function $\tilde{F}(r,\bar{y}'',\lambda,\sigma)$.

Since $\varphi_{r,\bar{y}'',\lambda,\sigma}=e^{i\sigma}\varphi_{r,\bar{y}'',\lambda,0}$, it is easy to check that

$$\tilde{F}(r,\bar{y}'',\lambda,\sigma)=\tilde{F}(r,\bar{y}'',\lambda,0).$$

That is, $\frac{\partial\tilde{F}(r,\bar{y}'',\lambda,\sigma)}{\partial\sigma}=0$ for all $(r,y'',\lambda,\sigma)$. So, we just need to find a critical point $(r,y'',\lambda)$ for the function

$$F(r,\bar{y}'',\lambda):=\tilde{F}(r,\bar{y}'',\lambda,0).$$

## Lemma 3.1

We have

$$F(r,\bar y'',\lambda) = I(Z_{r,\bar y'',\lambda}) + O\Big(\frac{m}{\lambda^{2+\epsilon}}\Big)$$

$$= \quad m\Big(A + \frac{B_1}{\lambda^2}V(r,\bar y'') - \sum_{j=2}^{m}\frac{B_2}{\lambda^{N-2}|x_1-x_j|^{N-2}} + O\Big(\frac{1}{\lambda^{2+\epsilon}}\Big)\Big),$$

where $B_i > 0, i = 1,2$ are some constants.

## Lemma 3.2

We have

$$\frac{\partial F(r,\bar y'',\lambda)}{\partial\lambda} = m\Big(-\frac{2B_1}{\lambda^3}V(r,\bar y'') + \sum_{j=2}^{m}\frac{B_2(N-2)}{\lambda^{N-1}|x_1-x_j|^{N-2}} + O\Big(\frac{1}{\lambda^{3+\epsilon}}\Big)\Big),$$

where $B_i > 0, i = 1,2$ are the same as in Lemma 3.1.

From [J. Wei, S. Yan, JFA. 258 (2010)], $\exists$ a constant $B_3 > 0$ S.T.

$$\sum_{j=2}^{m} \frac{1}{|x_1 - x_j|^{N-2}} = \frac{B_3 m^{N-2}}{|x_1|^{N-2}} + O\Big(\frac{m}{|x_1|^{N-2}}\Big).$$

Therefore,

$$F(r, \bar{y}'', \lambda) = m\Big(A + \frac{B_1}{\lambda^2} V(r, \bar{y}'') - \frac{B_4 m^{N-2}}{\lambda^{N-2} r^{N-2}} + O\Big(\frac{1}{\lambda^{2+\epsilon}}\Big)\Big), \tag{14}$$

and

$$\frac{\partial F(r, \bar{y}'', \lambda)}{\partial \lambda} = m\Big(-\frac{2B_1}{\lambda^3} V(r, \bar{y}'') + \frac{B_4 (N-2) m^{N-2}}{\lambda^{N-1} r^{N-2}} + O\Big(\frac{1}{\lambda^{3+\epsilon}}\Big)\Big). \tag{15}$$

For each fixed $(r, \bar{y}'')$, let $\Lambda_0(r, \bar{y}'')$ be the solution of

$$-\frac{2B_1}{\Lambda^3}V(r, \bar{y}'') + \frac{B_4(N-2)}{\Lambda^{N-1}r^{N-2}} = 0.$$

Then

$$\Lambda_0(r, \bar{y}'') = \Big(\frac{B_4(N-2)}{2B_1V(r, \bar{y}'')r^{N-2}}\Big)^{\frac{1}{N-4}}.$$

Note that $\Lambda_0(r, \bar{y}'')$ is the unique maximum point of the function

$$\frac{B_1}{\Lambda^2}V(r, \bar{y}'') - \frac{B_4}{\Lambda^{N-2}r^{N-2}},$$

for any fixed $(r, \bar{y}'')$.

Define

$$\Omega = \Big\{(r, \bar{y}'', \lambda) : \quad (r, \bar{y}'') \in \overline{B_\theta(r_0, y_0'')}, \ \lambda = \Lambda m^{\frac{N-2}{N-4}}, \qquad \Lambda \in \Big[\Lambda_0(r, \bar{y}'') - \frac{1}{\lambda^\theta}, \Lambda_0(r, \bar{y}'') + \frac{1}{\lambda^\theta}\Big]\Big\}, \tag{16}$$

where $\theta > 0$ is a small constant, satisfying $0 < \theta \ll \epsilon$.

For any $(r, \bar{y}'', \lambda) \in \Omega$, we have

$$\begin{aligned} & \frac{B_1}{\lambda^2} V(r, \bar{y}'') - \frac{B_4 m^{N-2}}{\lambda^{N-2} r^{N-2}} \\ =: \ & \Big(B'\Big(r^2 V(r, \bar{y}'')\Big)^{\frac{N-2}{N-4}} - \frac{2(N-4) B_1 V(r, \bar{y}'')}{\Lambda_0^4} (\Lambda - \Lambda_0(r, \bar{y}''))^2 \\ & + O\Big(\frac{1}{\lambda^{3\theta}}\Big)\Big) m^{-\frac{2(N-2)}{N-4}}. \end{aligned} \tag{17}$$

Hence

$$
\begin{aligned}
&F(r,\bar y'',\lambda)\\
=&m\Big(A+\Big[B'\Big(r^2V(r,\bar y'')\Big)^{\frac{N-2}{N-4}}-\frac{2(N-4)B_1V(r,\bar y'')}{\Lambda_0^4}(\Lambda-\Lambda_0(r,\bar y''))^2\Big]\\
&m^{-\frac{2(N-2)}{N-4}}\Big)+O\Big(\frac{m}{\lambda^{2+3\theta}}\Big).
\end{aligned}
\tag{18}
$$

**Proof of Theorem 1.3** We want to prove that $F(r,\bar y'',\lambda)$ has a critical point in $\Omega$.

Suppose that $(r_0,y_0'')$ is an isolated maximum point of $r^2V(r,y'')$. We consider

$$\max_{(r,\bar y'',\lambda)\in\Omega}F(r,\bar y'',\lambda). \tag{19}$$

Let $(r_m,\bar y_m'',\lambda_m)\in\Omega$ be a maximum point. We can prove that $(r_m,\bar y_m'',\lambda_m)\notin\partial\Omega$.

Now we consider the case that $(r_0, y_0'')$ is an isolated minimum point of $F(r, \bar{y}'', \lambda)$. Let

$$\tilde{F}(r, \bar{y}'', \lambda) = -F(r, \bar{y}'', \lambda), \ (r, \bar{y}'', \lambda) \in \Omega.$$

Denote

$$\begin{aligned}
\alpha_1 &= m\left(-A - B'\left(r_0^2 V(r_0, y_0'')(1+\beta)\right)^{\frac{N-2}{2(N-4)}} m^{-\frac{2(N-2)}{N-4}}\right),\\
\alpha_2 &= m(-A+\beta),
\end{aligned}$$

where $\beta > 0$ is a small constant, and $B' > 0$ is the constant in (18).

For $c \in \mathbb{R}$, define

$$\tilde{F}^c = \big\{(r, \bar{y}'', \lambda) \in \Omega, \;\; \tilde{F}(r, \bar{y}'', \lambda) \le c\big\}.$$

Consider

$$\begin{cases} \dfrac{dr}{dt} = -D_r \tilde{F}, & t > 0; \\ \dfrac{d\bar{y}_j''}{dt} = -D_{\bar{y}_j''} \tilde{F} \;\; (j = 3, 4, ..., N), & t > 0; \\ \dfrac{d\lambda}{dt} = -D_\lambda \tilde{F}, & t > 0; \\ (r, \bar{y}'', \lambda) \in \tilde{F}^{\alpha_2}. \end{cases}$$

Using (15) and (18), we can prove that the flow $(r(t), \lambda(t), \bar{y}''(t))$ does not leave $\Omega$ before $(r(t), \lambda(t), \bar{y}''(t))$ reaches $\tilde{F}^{\alpha_1}$.

Define

$$\Gamma = \Big\{\gamma : \ \gamma(r, \bar{y}'', \lambda) = (\gamma_1(r, \bar{y}'', \lambda), \gamma_2(r, \bar{y}'', \lambda)) \in \Omega, \quad (r, \bar{y}'', \lambda) \in \Omega, \\ \gamma(r, \bar{y}'', \lambda) = (r, \bar{y}'', \lambda), \text{ if } (r, \bar{y}'') \in \partial B_\theta((r_0, y_0''))\Big\}.$$

Denote

$$c = \inf_{\gamma \in \Gamma} \max_{(r, \bar{y}'', \lambda) \in \Omega} \tilde{F}(\gamma(r, \bar{y}'', \lambda)).$$

We have the following estimates for this constant $c$.

$$(i) \quad \alpha_1 < c < \alpha_2;$$
$$(ii) \quad \sup_{\partial B_\theta((r_0, y_0''))} \tilde{F}(\gamma(r, \bar{y}'', \lambda)) < \alpha_1, \ \ \forall \gamma \in \Gamma.$$

From these estimates, we can use the standard deformation argument to prove that $c$ is a critical point of $\tilde{F}$.

In order to prove (ii), let $\gamma \in \Gamma$. Then for any $(r, \bar{y}'') \in \partial B_\theta((r_0, y_0''))$, we have $\gamma(r, \bar{y}'', \lambda) = (r, \bar{y}'', \lambda)$. Hence, from (18), we obtain

$$\tilde{F}\left(\gamma(r, \bar{y}'', \lambda)\right) = \tilde{F}(r, \bar{y}'', \lambda) < \alpha_1.$$

Now we prove (i). It is obvious that

$$c < \alpha_2.$$

For any $\gamma = (\gamma_1, \gamma_2) \in \Gamma$, define

$$\tilde{\gamma}_1(r, \bar{y}'') = \gamma_1 \left( r, \bar{y}'', \Lambda_0(r, \bar{y}'') m^{\frac{N-2}{N-4}} \right), \quad \forall\, (r, \bar{y}'') \in B_\theta((r_0, y_0'')).$$

Then

$$\tilde{\gamma}_1(r, \bar{y}'') = (r, \bar{y}''), \qquad \text{if } (r, \bar{y}'') \in \partial(B_\theta((r_0, y_0''))). \tag{20}$$

Hence there is a $(\tilde{r}, \tilde{y}'') \in B_\theta^\circ((r_0, \bar{y}_0''))$ such that

$$\tilde{\gamma}_1(\tilde{r}, \tilde{y}'') = (r_0, y_0''). \tag{21}$$

Let $\tilde{\lambda} = \gamma_2\left(\tilde{r}, \tilde{y}'', \Lambda_0(\tilde{r}, \tilde{y}'')m^{\frac{N-2}{N-4}}\right)$. Then by (18) and (21)

$$
\begin{aligned}
& \max_{(r,\bar{y}'',\lambda)\in\Omega} \tilde{F}(\gamma(r, \bar{y}'', \lambda)) \\
\geq\ & \tilde{F}\left(\gamma\left(\tilde{r}, \tilde{y}'', \Lambda_0(\tilde{r}, \tilde{y}'')m^{\frac{N-2}{N-4}}\right)\right) = \tilde{F}(r_0, y_0'', \tilde{\lambda}) \\
=\ & m\left(-A - \left(B'\left(r_0^2 V(r_0, y_0'')\right)^{\frac{N-2}{2(N-4)}} + O(\tilde{\lambda}^{-2\theta})\right)m^{-\frac{2(N-2)}{N-4}}\right) > \alpha_1.
\end{aligned}
$$

## Lemma B.1

Suppose $a = a_1 + ia_2$. For any complex number $z$, we have that if $t \in (2, 3]$, then

$$\mathrm{Re}\Big[|1+z|^{t-2}(1+z)\bar{a} - \bar{a} - \big((t-2)(\mathrm{Re}\, a)z + z\bar{a}\big)\Big] = O(|z|^2|a|), \quad (22)$$

and

$$\mathrm{Re}\Big[|1+z|^{t-2}(1+z)\bar{a} - \bar{a} - \big((t-2)(\mathrm{Re}\, a)z + z\bar{a}\big)\Big] = O(|z|^{t-1}|a|). \quad (23)$$

If $t > 3$, then

$$\mathrm{Re}\Big[|1+z|^{t-2}(1+z)\bar{a} - \bar{a} - \big((t-2)(\mathrm{Re}\, a)z + z\bar{a}\big)\Big] = O(|z|^2|a| + |z|^{t-1}|a|). \quad (24)$$

## Lemma B.2

For $t \in (2, 3]$,

$$\Big||1+z|^{t-2}(1+z)\bar{a} - \bar{a} - \big((t-2)(\mathrm{Re}\,a)z + z\bar{a}\big)\Big| = O(|z|^2|a|), \tag{25}$$

and

$$\Big||1+z|^{t-2}(1+z)\bar{a} - \bar{a} - \big((t-2)(\mathrm{Re}\,a)z + z\bar{a}\big)\Big| = O(|z|^{t-1}|a|). \tag{26}$$

For $t > 3$, we have

$$\Big||1+z|^{t-2}(1+z)\bar{a} - \bar{a} - \big((t-2)(\mathrm{Re}\,a)z + z\bar{a}\big)\Big| = O(|z|^2|a| + |z|^{t-1}|a|). \tag{27}$$

Thank you !