Symmetry Induced by Concentration

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1. Symmetry for Dirichelt Problem

Consider

$$\begin{cases} -\Delta u = f(u), \quad u > 0 & \textbf{in } \Omega, \\ u = 0, & \textbf{on } \partial\Omega, \end{cases} \tag{1}$$

where $f(u)$ is Lip-continuous. Typical case: $f(u) = u^p - au$, $a \geq 0$.

Theorem

If Ω is a ball centered at the origin, then any solution of (1) is radial.

This is proved by Gidas, Ni and Nirenberg by using a moving plane method.

Using the same method, one can also prove the following result:

Theorem

If Ω is convex, and is symmetric with respect to all the coordinate planes, then any solution of (1) is symmetric with respect to all the coordinate planes.

Example.

$$\Omega = \left\{x : \frac{x_1^2}{a_1^2} + \cdots + \frac{x_N^2}{a_N^2} < 1\right\}, \tag{2}$$

where $0 < a_1 \leq \cdots \leq a_N$**.**

2. Neumann Problem

Consider

$$\begin{cases} -\Delta u = f(u), \quad u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega. \end{cases} \tag{3}$$

We can find a radial solution for (3) by using the mountain pass lemma in radially symmetric space if

$$f(u) = u^p - \lambda u,$$

where $\lambda > 0$ **and** $1 < p < \frac{N+2}{N-2}$**.**

Question. Is Theorem 0.1 true for any solution of (3)?

Answer. No.

Theorem (Ni and Takagi)

Let Ω be a bounded domain with smooth boundary and $f(u) = u^p - \lambda u$, $1 < p < \frac{N+2}{N-2}$. Then, for $\lambda > 0$ large, mountain pass solution u_λ of (3) has a unique maximum point x_λ and $x_\lambda \in \partial\Omega$. Moreover, as $\lambda \to +\infty$, $x_\lambda \to x_0$, $H(x_0) = \max_{x\in\partial\Omega} H(x)$, where $H(x)$ is the mean curvature of $\partial\Omega$ at x, $u_\lambda(x_\lambda) \geq c_0 > 0$ and $u_\lambda \to 0$ uniformly $\Omega \setminus B_\delta(x_0)$ for any $\delta > 0$.

Remark. **The mountain pass solution has concentration property and it is not radial.**

Question. **If Ω is symmetric with respect to all the coordinate planes, is the mountain pass solution u_λ of (3) symmetric with respect to some the coordinate planes?**

Answer. Yes, under certain condition on the domain Ω, if one can prove the *local uniqueness*.
For simplicity, we assume that

$$\Omega = \Big\{x : \frac{x_1^2}{a_1^2} + \cdots + \frac{x_N^2}{a_N^2} < 1\},\tag{4}$$

where $0 < a_1 \leq \cdots \leq a_N$.

Observation:

1) By Theorem 0.3, mountain pass solution u_λ has a unique blow up point near $(a_1, 0)$.
2) Let $v_\lambda(x) = u_\lambda(x_1, -x_2, x_3, \cdots, x_N)$. The v_λ is also a solution, concentrating at $(a_1, 0)$ as $\lambda \to +\infty$.

Local Uniqueness: Suppose that (3) has two sequence of solutions $u_\lambda^{(1)}$ and $u_\lambda^{(2)}$, which have the same blow-up sets as $\lambda \to +\infty$. The local uniqueness question is to ask whether $u_\lambda^{(1)} = u_\lambda^{(2)}$ if $\lambda > 0$ is sufficiently large.

Conclusion: If we can prove the local uniqueness, then $u_\lambda = v_\lambda$, which gives the symmetry.

The concentration of a sequence of solutions results in the local uniqueness, which implies some kind of symmetry. We call this phenomenon **the symmetry induced by concentration**.

Consider

$$-\Delta u + \lambda V(|x|)u = u^p, \quad u > 0, \quad u \in H^1(\mathbb{R}^N), \tag{5}$$

where $p \in (1, \frac{N+2}{N-2})$**,** $\lambda > 0$**,** $V(r)$ **is bounded and positive.**

(1) If $V' > 0$**, then any solution of** (5) **is radial.**
(2) Wei and Yan: If $V(r) = 1 + \frac{a}{r^m} + O\Big(\frac{1}{r^{m+1}}\Big)$ **for some** $a > 0$ **and** $m \geq 1$**, then for any fixed** $\lambda > 0$**,** (5) **has infinitely many non-radial solutions.**
(3) If $V''(0) \neq 0$**, then for large** λ**,** (5) **has a solution which has a peak at** $x = 0$ **as** $\lambda \to +\infty$**.**

Question: Is the peak solution at 0 radial?

Let R be any rotation in $\mathbb{R}^N$. Then $v(x) = u(Rx)$ is also a solution of (5) with a peak at 0 as $\lambda \to +\infty$.
If we can prove a local uniqueness result for peak solution at 0, then $v(x) = u(x)$ for $\lambda > 0$ for, which implies that u is radial.

There are many problems whose solutions have concentration properties. Though these problems usually have the symmetry breaking phenomena, one may still ask whether certain type of solutions preserves the symmetry from the problems?

Consider

$$-\Delta u = K(y) u^{\frac{N+2}{N-2}}, u > 0, y \in \mathbb{R}^N. \tag{6}$$

Symmetry Breaking:

Wei and Yan: If $K(y) = K(|y|)$ and $K(r)$ has a local maximum point $r_0 > 0$, then (6) has infinitely many non-radial (fast decay) solutions.

Yan: If $K(y)$ is periodic in y_1 and $K(y)$ has a (very degenerate) local maximum point at 0, then (6) has infinitely many (fast decay) solutions (so it is not periodic in y_1).

A recent result by Y.Y. Li, J. Wei and H. Xu:

Suppose that $K(y)$ is a periodic function in y_1 with period one. Assume that $K(y)$ also satisfies the following expansion: for $y \in B_\delta(0)$,

$$K(y) = 1 - a|y|^m + O(|y|^{m+1}), \quad \nabla K(y) = -am|y|^{m-2}y + O(|y|^m), \tag{7}$$

where $\delta > 0$ is a small constant, $m \in (N-2, N)$, $a > 0$ is some constant.

Denote $x_j = (jL, 0, \cdots, 0)$, $j = 0, \pm 1, \pm 2, \cdots$, where $L > 0$ is a large integer. then (6) has a solution of the form

$$u(y) = \sum_{j=-\infty}^{+\infty} \big(N(N-2)\big)^{\frac{N-2}{4}} \left(\frac{\mu_{j,L}}{1 + \mu_{j,L}^2 |y - x_{j,L}|^2} \right)^{\frac{N-2}{2}} + \omega_L(y), \qquad (8)$$

satisfying that as $L \to +\infty$,

$$x_{j,L} = x_j + o_L(1), \qquad (9)$$

$$\mu_{j,L} = L^{\frac{N-2}{m-N+2}} \big(\bar{B} + o_L(1)\big), \quad \bar{B} > 0, \qquad (10)$$

$$|\omega_L(y)| = o_L(1) \sum_{j=-\infty}^{\infty} \frac{\mu_{j,L}^{\frac{N-2}{2}}}{(1 + \mu_{j,L}|y - x_{j,L}|)^{\frac{N-2}{2}+\tau}}, \qquad (11)$$

where $o_L(1) \to 0$ as $L \to +\infty$, $\tau = 1 + \theta$ and $\theta > 0$ is a small constant.

Outline of the construction of the solutions of the form (8) :

(i) Construct a solution u_k with $2k+1$ bubbles of the following form in the same way as Yan:

$$u_k = \sum_{j=-k}^{k} \big(N(N-2)\big)^{\frac{N-2}{4}} \left(\frac{\mu_{j,L}}{1+\mu_{j,L}^2 |y - x_{j,L}|^2} \right)^{\frac{N-2}{2}} + \omega_{k,L}, \tag{12}$$

where k is a positive integer.
(ii) Carefully estimate u_k in some weighted L^∞ spaces to ensure u_k converges locally in $\mathbb{R}^N$ to a solution u of the form (8) as $k \to +\infty$.

Question: **Is the solution of the form** (8) **satisfying** (9)–(11) **periodic in** y_1**?**
Remark: **Solution of the form** (12) **is not periodic in** y_1**.**

The blow-up set of the solution of the form (8) **is** $S = \{x_j : \ j = 0, \pm 1, \pm 2, \cdots\}$**. Then** $v(y) = u(y_1 - L, y_2, \cdots y_N)$ **is also a solution of** (6) **with the same blow-up set as** $L \to +\infty$**. If we can prove the local uniqueness result, then** $v = u$**, which implies the periodicity of** u **in** y_1**.**

The main Result (Y. Deng, C.-S. Lin and S. Yan):

Theorem

Assume that $N \geq 5$*. Suppose that* $K(y)$ *is periodic in* y_1 *with period 1 and satisfies* (7)*. If* $u_L^{(1)}$ *and* $u_L^{(2)}$ *are two sequence of solutions of* (6)*, which have the form* (8) *satisfying* (9)–(11)*, then* $u_L^{(1)} = u_L^{(2)}$ *provided* $L > 0$ *is large enough. In particular, solution of the form* (8) *must be periodic in* y_1*.*

Remark: We can derive more information for the solutions from the local uniqueness. For example, if $K(y)$ is even in y_2, then solution of the form (8) is also even in y_2.

Outline of the proof of our main result:

One way to show the local uniqueness of solutions with finite number of bubbles is to prove the uniqueness of solution for the reduced finite dimensional problem by counting the local degree.

Disadvantage of the degree counting method: **Need to estimate the second order derivatives of the error term ω_L. This step is rather technical and lengthy.**

The degree counting method does not work in the present situation, because the reduced problem is still infinitely dimensional.

Our method: Use various types of Pohozaev identities to prove the local uniqueness result. Note that all the terms in those identities involve first order derivatives only.

Suppose that (6) **two different** $u_L^{(1)}$ **and** $u_L^{(2)}$**, which blow up at** x_j**,** $j = 0, \pm 1, \pm 2, \cdots$**. We study the normalization** $\xi_L = \frac{u_L^{(1)} - u_L^{(2)}}{\|u_L^{(1)} - u_L^{(2)}\|_{L^\infty(\mathbb{R}^N)}}$**. Note that** $\|\xi_L\|_{L^\infty(\mathbb{R}^N)} = 1$**.**

(1) $|\xi|$ **attains its maximum at** $\cup_j B_{R\mu_{j,L}^{-1}}(x_{j,L})$ **for some large** $R > 0$**.**

(2) To study ξ_L **in** $\cup_j B_{R\mu_{j,L}^{-1}}(x_{j,L})$**, we define** $\tilde{\xi}_{j,L}(y) = \xi_L\big(\mu_{j,L}^{-1} y + x_{j,L}\big)$**. Then** $\tilde{\xi}_{j,L} \to \xi_j$ **in** $C^1_{loc}(\mathbb{R}^N)$ **and** ξ_j **satisfies**

$$-\Delta \xi_j = (2^* - 1) U_{0,1}^{2^*-2} \xi_j, \quad \mathbb{R}^N, \tag{13}$$

which implies $\xi_j = b_0 \frac{\partial U_{0,\lambda}}{\partial \lambda}\Big|_{\lambda=1} + \sum_{i=1}^N b_i \frac{\partial U_{0,1}}{\partial x_i}$**, where** $U_{x,\mu}(y) = \big(N(N-2)\big)^{\frac{N-2}{4}} \left(\frac{\mu}{1+\mu^2|y-x|^2}\right)^{\frac{N-2}{2}}$**, which is a solution of**

$$-\Delta u = u^{2^*-1}, \ u > 0, \quad \mathbb{R}^N. \tag{14}$$

(3) Use various Pohozaev identities to show all the constants $b_j = 0$**. So we obtain a contradiction.**

Remark: (i) The kernel $\frac{\partial U_{0,1}}{\partial x_i}$ comes from the translation invariance of the limit problem (14). We have the following Pohozaev identity generating from translation:

$$
\begin{aligned}
&-\int_{\partial B_\delta(x_{j,L})} \frac{\partial u_L}{\partial \nu}\frac{\partial u_L}{\partial y_i} + \frac{1}{2}\int_{\partial B_\delta(x_{j,L})} |\nabla u_L|^2 \nu_i \\
=&\frac{1}{2^*}\int_{\partial B_\delta(x_{j,L})} K(y) u_L^{2^*}\nu_i - \frac{1}{2^*}\int_{B_\delta(x_{j,L})} \frac{\partial K(y)}{\partial y_i} u_L^{2^*},
\end{aligned} \tag{15}
$$

which we use to kill the kernel $\frac{\partial U_{0,1}}{\partial x_i}$, where ν is the outward unit normal of $\partial B_\delta(x_{j,L})$. In fact, we can prove

$$
\text{LHS of (15)} = o(1), \quad \text{LHS of (15)} = B(b_i + o(1)),
$$

for some $B \neq 0$.

(ii) The kernel $\frac{\partial U_{0,\lambda}}{\partial\lambda}\Big|_{\lambda=1}$ is from the scaling invariance of the limit problem (14). We have the following Pohozaev identity generating from scaling:

$$
\begin{aligned}
&-\int_{\partial B_\delta(x_{j,L})}\frac{\partial u_L}{\partial\nu}\Big\langle y-x_{j,L},\nabla u_L\Big\rangle+\frac{1}{2}\int_{\partial B_\delta(x_{j,L})}|\nabla u_L|^2\Big\langle y-x_{j,L},\nu\Big\rangle\\
&+\frac{2-N}{2}\int_{\partial B_\delta(x_{j,L})}\frac{\partial u_L}{\partial\nu}u_L\\
=&\frac{1}{2^*}\int_{\partial B_\delta(x_{j,L})}K(y)u_L^{2^*}\Big\langle y-x_{j,L},\nu\Big\rangle-\frac{1}{2^*}\int_{B_\delta(x_{j,L})}\Big\langle\nabla K(y),y-x_{j,L}\Big\rangle u_L^{2^*},
\end{aligned}
\tag{16}
$$

which we use to to kill $\frac{\partial U_{0,\lambda}}{\partial\lambda}\Big|_{\lambda=1}$. We can prove

$$
\text{LHS of (16)}=o(1),\quad \text{LHS of (15)}=B'(b_i+o(1)),
$$

for some $B'\neq 0$.

Main Task: **To obtain accurate point-wise estimates for ξ_L, as well as the estimates for the quantities $|x_{j,L} - x_j|$ and $\mu_{j,L}$, so that all the terms appearing in the Pohozaev identities can be controlled.**

(i) Use Pohozaev identities again to obtain the following estimates

$$\mu_{j,L} = L^{\frac{N-2}{m-N+2}}\Big(\bar{B} + O\Big(\frac{1}{L^{\frac{N-2}{m-N+2}}}\Big)\Big), \quad |x_{j,L} - x_j| = O\Big(\frac{1}{\mu_{j,L}^2}\Big).$$

where $\bar{B} > 0$ is a constant.

(ii) Use the techniques from potential theories to obtain accurate point-wise estimates for ξ_L.

Further Result (with Y. Guo and S. Peng): The result in Theorem 4 has been extended to the following problem

$$(-\Delta)^m u = K(y) u^{\frac{N+2m}{N-2m}}, \quad \text{in } \mathbb{R}^N.$$

Main Difficulty in the Proof of Local Uniqueness: For large integer m, there are many terms in the Pohozaev identities. It is impossible to estimate each term in those identities. Better understanding of the Pohozaev identities is needed.

Thank You Very Much.

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