On the plurisubharmonicity of the solution of the Fefferman type Monge-Ampère equation and application

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Outline of the talk

1. Background, question and answer
2. Rigidity theorem for degenerate Monge-Ampère equation
3. Pseudo-Hermitian CR manifolds
4. Bottom of the Spectrum of Laplace-Beltrami Operators
5. Explicit statement of the main results and sketch of the proof

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Kähler-Einstein metrics

Let (M^n, g) be a Kähler manifold with a Kähler metric

$$g = \sum_{i,j=1}^{n} g_{i\bar{j}} dz_i \otimes d\bar{z}_j$$

Then the Ricci curvature is given by

$$R_{k\bar{j}} = -\frac{\partial^2}{\partial z_k \partial \bar{z}_j} \log \det \left[g_{p\bar{q}} \right]$$

We say that (M^n, g) is Einstein if

$$R_{k\bar{j}} = c\, g_{k\bar{j}}$$

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Monge-Ampère equations

We restrict our attentions to:

1) $M^n = D$, a bounded domain in $\mathbb{C}^n$.

2) Kähler metric $g = g[u]$ is induced by a strictly plurisubharmoinic function u on D, i.e.,

$$g[u]_{i\bar{j}} = \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}$$

Then $(M^n, g[u])$ is Einstein if u is a strictly plurisubharmonic solution of the Monge-Ampère equation:

$$\det H(u) = e^{-cu}$$

where $H(u)$ is $n \times n$ complex Hessian matrix of u.

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BV for Fefferman typed Monge-Ampère equations

- When $c < 0$, by normalization, let $c = -(n+1)$.
- We consider the boundary value problem of:

  (1) Monge-Ampère equation which induces Kähler-Einstein metric

$$\det H(u) = e^{(n+1)u} \text{ in } M, \quad u = +\infty \text{ on } \partial M. \tag{1}$$

  (2) Fefferman Equation

$$J(\rho) = -\det \begin{bmatrix} \rho & \rho_{\bar{j}} \\ \rho_i & H(\rho) \end{bmatrix} = 1, \text{ in } M, \quad \rho = 0 \text{ on } \partial M. \tag{2}$$

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Relation, M-A-Einstein Eq and Fefferman Eq.

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with $u = -\log(-\rho)$ is strictly plurisubharmonic in M.

Assuming that M is a smoothly bounded strictly pseudo convex domain in $\mathbb{C}^n$, he proved:

- Uniqueness: Equation (2) has at most one solution ρ.
- A local approximation formula if (2) has a solution. In particular, for any smooth defining function ρ_0 for M, one has

$$\rho(z) = \frac{\rho_0(z)}{J(\rho_0)^{1/(n+1)}} + O(\rho_0^2)$$

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Cheng and Yau's theorem

When M is a smoothly bounded pseudo convex domain in $\mathbb{C}^n$. Cheng and Yau (1980, CPAM) proved:

$$\det H(u) = e^{(n+1)u} \text{ in } M, \quad u = +\infty \text{ on } \partial M. \tag{1}$$

has a unique strictly plurisubharmonic solution u on M.

Moreover, the metric induces by the solution u:

$$g_u = \sum_{i,j=1}^{m} \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} dz_i \otimes d\bar{z}_j$$

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Cheng and Yau; Mok and Yau's theorems

- Remark: Mok and Yau (1980) proved that the same theorem by replacing $\partial M \in C^\infty$ by a very weak condition.
- When M is strictly pseudoconvex, Cheng and Yau proved that

$$\rho(z) = -e^{-u} \in C^{n+3/2}(\overline{M})$$

- In fact, $\rho \in C^{n+2-\epsilon}(\overline{M})$ by the following theorem of J. Lee and Melrose (Acta Math. 1982).

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Asymptotic expansion of Lee and Melrose

The following theorem was proved by Lee and Melrose (Acta Math, 1982)

Theorem

Let D be smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$. Let $\rho(z)$ be the solution of Fefferman equation (2) with $u = -\log(-\rho)$ being strictly plurisubharmonic. Let ρ_0 be any smooth defining function for D. Then

$$\rho(z) = \rho_0(z)\Big[\sum_{j=0}^{\infty} a_j(z)\Big(\rho_0(z)^{n+1}\log(-\rho_0(z))\Big)^j\Big] \qquad (5)$$

where $a_j \in C^\infty(\overline{D})$ and $a_0(z) > 0$ on ∂D.

Remarks, Some results of R. Graham

- Notice that there is log terms in Lee and Melrose's formula:

(5) $$\rho(z) = a_0(z)\rho_0(z) + a_1(z)\rho_0(z)^{n+2}\log(-\rho_0) + \cdots$$

- In general, $\rho \notin C^{n+2}(\overline{D})$.

- A natural question is: What can one say about D if $\rho(z) \in C^\infty(\overline{D})$?

- When $n = 2$, R. Graham (1987) proved that if $\rho \in C^\infty(\overline{D})$ then ∂D is spherical.

- He also provided an iteration formula to compute a_0 and a_1 based on the idea of Feffereman.

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An example of explicit ρ

- In general, it is very difficult to solve the equation:

$$J(\rho) = -\det \begin{bmatrix} \rho & \rho_{\bar{j}} \\ \rho_i & H(\rho) \end{bmatrix} = 1, \text{ in } D, \quad \rho = 0 \text{ on } \partial D. \qquad (2)$$

with $u = -\log(-\rho)$ being strictly plurisubharmonic **explicitly.**

- However, if $D = B_n$, the unit ball in $\mathbb{C}^n$, then

$$\rho(z) = |z|^2 - 1, \quad z \in B_n.$$

- In particular, $\rho(z) = |z|^2 - 1$ is strictly plurisubharmonic in B_n.

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Main question and answer

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Question: Is ρ strictly plurisubharmonic in D?

Answer: No in general, this is main result of this talk (The detail will be given later.) The question was asked based on:

1) Question of P.-M. Wong on a rigidity theorem;
2) Positivity of pseudo Ricci curvature for a pseudo-Hermitian manifold (M^{2n+1}, θ);
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Domain with real ellipsoid boundary

Let $A = (A_1, \cdots, A_n)$ with $A_j \in (-1, 1)$ and let

$$D(A) = \{z \in \mathbb{C}^n : |z|^2 + \mathrm{Re} \sum_{j=1}^{n} A_j z_j^2 - 1 < 0\}.(6)$$

Then

(i) $D(A)$ is strictly convex
(ii) $\partial D(A)$ is real elliposid
(iii) Every real ellipsoid E in $\mathbb{C}^n$ is the form $\partial D(A)$ after a complex linearly change of variables.

The result on Ellipsoid

- The following theorem was proved by L. (CAG, 2010)

Theorem

Let u be the potential function of Kähler-Einstein metric on $D(A)$. Then $\rho(z) = -e^{-u}$ is strictly plurisubharmonic in $\overline{D(A)}$

- **Remark.** For the case $n = 2$, Chanillo, Chiu and Yang (2013) proved that ρ is strictcy pseudoconvex near and on $E = \partial D(A)$ from different point of view.

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Model Example for Rigidity

Let $M = B_n$ be the unit ball in $\mathbb{C}^n$. The map

$$\tau(z) = |z|^2 : B_n \to [0, 1)$$

is onto and is strictly plurisubharmonic. Moreover,

$$\log \tau(z) = \log |z|^2$$

satisfies the degenerate Monge-Ampère equation

$$\det H(\log \tau(z)) = 0, \quad z \in B_n \setminus \tau^{-1}(0).$$

More generally, if $\phi : M \to B_n$ is a biholomorphic mapping and

$$\tau(z) =: |\phi(z)|^2,$$

then

(i) τ is strictly plurisubharmonic in M;
(ii) $\tau : M \to [0, 1)$ is onto;

$$\text{(iii)} \quad \det H(\log \tau)(z) = 0, \quad \text{if } \ \tau(z) > 0.$$

In fact,

$$\det \left[\frac{\delta_{ij}}{|z|^2} - \frac{\overline{z}_i z_j}{|z|^4} \right] = 0, \quad \text{if } z \neq 0$$

Then for z with $\phi(z) \neq 0$ we have

$$\det H(\log \tau)(z) = |\det \phi'(z)|^2 \det H(\log |w|^2)(\phi(z)) = 0$$

Stoll's theorem

Conversely, the following theorem was proved by W. Stoll (Ann. Scuola Norm. Sup. Pisa Cl. Sci.,1980), by D. Burns (Ann. of Math., 1982) and by P. M. Wong (Invent. Math. 1982)

Theorem

Let N be a complex manifold of dimension n. Let $\tau \in C^\infty(N)$ be such that
(i) τ is strictly plurisubharmonic in N;
(ii) $\tau : N \to [0, R)$ is onto;

$$(iii) \quad \det H(\log \tau) = 0, \quad z \in N \text{ with } \tau(z) > 0.$$

Then N is biholomorphic to the ball $B(0, R)$ in $\mathbb{C}^n$.

Kähler-Einstein potential function

Assume that $\phi : D \to B_n$ is a biholomorphic map.

- Let $\rho(z) = |\phi|^2 - 1$. Then
  (i) $J(\rho) = |\det \phi'(z)|^2$;
  (ii) $\log J(\rho) = \log |\det \phi'(z)|^2$ is pluriharmonic in D.
- Let

$$\tau(z) =: 1 + \rho(z) = |\phi(z)|^2.$$

Then
  (i) $\tau : D \to [0, 1)$ is strictly plurisubharmonic and onto;
  (ii) $\log \tau$ is plurisubharmonic in D and

$$\det H(\log \tau) = 0.$$

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- Conversely, we have the following theorem.

Theorem (Li, CAG, 2005)

Assume that ρ is a smooth defining function for D with $v =: -\log(-\rho)$ being strictly plursubharmonic in D. If
(i) $-\log J(\rho)$ is plurisubharmonic in D;
(ii) $\log \tau(z)$ is is plurisubharmonic near ∂D, where

$$\tau(z) = \rho(z) + m \quad \text{and } m = -\min\{\rho(z) : z \in D\} > 0,$$

then D is biholomorphic to B_n.

New relations

Assume that ρ is a defining function for a bounded pseudoconvex domain D in $\mathbb{C}^n$. Let

$$\tau(z) =: \rho(z) + m, \quad m = -\min\{\rho(z) : z \in D\} > 0$$

Then

$$\tau(z)^{n+1} \det H(\log \tau) = -m \det H(\rho) + J(\rho).$$

In fact,

$$\begin{aligned} &\tau(z)^{n+1} \det H(\log \tau) \\ &= J(\tau) \\ &= -[\det H(\tau)(\tau(z) - (\bar\partial \tau)^* H(\tau)^{-1}(\bar\partial \tau))] \\ &= -[m \det H(\rho) + \det H(\rho)(\rho(z) - (\bar\partial \rho)^* H(\rho)^{-1}(\bar\partial \rho)] \\ &= -m \det H(\rho) + J(\rho) \end{aligned}$$

This identity:

$$\tau(z)^{n+1} \det H(\log \tau) = J(\tau) = -m \det H(\rho) + J(\rho).$$

implies that

$$\det H(\log \tau) = 0 \iff J[\tau] = 0 \iff \frac{\det H(\rho)}{J(\rho)} = \left(\frac{1}{m}\right)$$

These relations give following equivalent statements of the theorem of W. Stoll, Burns and Wong:

Equivalent statements of Stoll' s theorem

Theorem

Let N be a complex manifold of dimension n. Let $\tau = m + \rho \in C^\infty(N)$ *be such that*
(i) τ *is strictly plurisubharmonic in N;*
(ii) $\tau : N \to [0, m)$ *is onto; and*

$$\frac{\det H(\rho)}{J(\rho)} \equiv \frac{1}{m} > 0 \text{ on } N.$$

Then N is biholomorphic to the ball $B(0, m)$ *in* $\mathbb{C}^n$.

Example

Assume that $\phi : N \to B_n$ is a biholomorphic mapping. Let

$$\rho(z) = |\phi(z)|^2 - 1, \quad \tau(z) = |\phi(z)|^2$$

Then

(i) $\tau : N \to [0, 1)$ is onto and strictly plurisubharmonic in N;

(ii) $J(\tau) = J(|\phi|^2) = |\det \phi'(z)|^2 J(|w|^2)(\phi(z)) = 0$;

(iii) $\det H(\rho) = J[\rho](z) = |\det \phi'(z)|^2$ or $\det H(\rho)/J[\rho] \equiv 1$.

Question: Boundary value problem

Theorem of Stoll: *Let N be a complex manifold of dimension n. Let* $\tau = m + \rho \in C^\infty(N)$ *be such that*

(i) τ *is strictly plurisubharmonic in N; (ii)* $\tau : N \to [0, m)$ *is onto; and* $\frac{\det H(\rho)}{J(\rho)} \equiv \frac{1}{m} > 0$ *on N. Then N is biholomorphic to the ball* $B(0, m)$ *in* $\mathbb{C}^n$*.*

Question: Can one replaces $\frac{\det H(\rho)}{J(\rho)} \equiv \frac{1}{m} > 0$ on N by $\frac{\det H(\rho)}{J(\rho)} \equiv \frac{1}{m} > 0$ on ∂N *with some good condition or no condition* ?

Remark: When ∂N is pseudo-Hermitian CR manifold, $n(n-1)\frac{\det H(\rho)}{J(\rho)}$ is pseudo scalar curvature for $(\partial N, \theta)$ with $\theta = -i(\partial\rho - \partial\rho)/2$.

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A theorem of L., CAG, 2009

The problem was studied by L. (CAG, 2009), he proved:

Theorem

Let D be pseudoconvex domain in $\mathbb{C}^n$ with defining function $\rho \in C^\infty(D) \cap C^3(\overline{D})$ so that
(i) $u = -\log(-\rho)$ *is strictly plurisubharmonic in D;*
(ii) $-\log J[\rho]$ *is plurisubharmonic;*
(iii) $\det H(\rho)/J[\rho] \equiv c > 0$ *on* ∂D*.*
Then D is biholomorphic to the unit ball in $\mathbb{C}^n$.

Remark: $-\log J[\rho]$ is plurisubharmonic if and only the Ricci curvature $R_{i\bar{j}}$ for the Kähler metric g_u induced by u satisfying:

$$R_{i\bar{j}} \geq -(n+1)(g_u)_{i\bar{j}}.$$

Question of P. Wong

Question of Wong: When $J(\rho) \equiv 1$ on D. Can one replace the condition (iii): det $H(\rho) \equiv c > 0$ on ∂D by the condition: det $H(\rho) > 0$ on ∂D?

Answer: No, in general. The counterexample is $D(A)$. By the previous theorem of Li, ρ is strictly plurisubharmonic in $\overline{D(A)}$. So det $H(\rho) > 0$ on $\partial D(A)$. However, $D(A)$ is not biholomorphic to B_n if $A \neq 0$ by the following theorem of Webster (JDG, 1978).

Theorem

$D(A)$ is biholomorphic to the unit ball in $\mathbb{C}^n$ if and only $A_j = 0$ for $1 \leq j \leq n$.

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Theorem

$D(A)$ is biholomorphic to the unit ball in $\mathbb{C}^n$ if and only $A_j = 0$ for $1 \leq j \leq n$.

- (M^{2n+1}, θ) is a $(2n+1)$-dimensional CR manifold with CR dimension n.
- $H(M)$ is holomorphic tangent bundle of M.
- θ is real, no-where vanishing 1-form (contact or hermitian form) on M satisfying

$$\theta(X) = 0, \quad X \in H(M). \tag{1}$$

- $H^*(M)$ is holomorphic cotangent bundle of M.
- $\{\theta^1, \cdots, \theta^n\}$ is a local basis for $H^*(M)$ such that

$$d\theta = \sqrt{-1} \sum_{\alpha,\beta=1}^{n} h_{\alpha\overline{\beta}} \theta^\alpha \wedge \theta^{\overline{\beta}}. \tag{2}$$

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Strictly pseudoconvex

- M is strictly pseudoconvex

  $\Longleftrightarrow$ $(h_{\alpha\overline{\beta}})$ is positive definite on M

  $\Longleftrightarrow$ Levi-form $L_\theta = -\sqrt{-1}d\theta$ is positive definite on $H(M)$.

  Here
  $$L_\theta(w, \overline{v}) = -\sqrt{-1}d\theta(w, \overline{v})$$

- From now on, we assume (M, θ) is strictly pseudoconvex.

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Example

M is a real hypersurface in $\mathbb{C}^{n+1}$ given by a defining function ρ:

$$M = \{z \in C^{n+1} : \rho(z) = 0\}, \quad \nabla\rho(z) \neq 0, \text{ for all } z \in M.$$

A contact form θ on M can be determined as:

$$\theta = -\frac{1}{2}\sqrt{-1}(\partial\rho - \overline{\partial}\rho)$$

The Levi-form is:

$$L_\theta(w, \overline{v}) = -\sqrt{-1}d\theta(w, \overline{v}) = \langle H(\rho)(z)w, v\rangle,$$

$$w, v \in H_z = \{w \in \mathbb{C}^{n+1} : \sum_{j=1}^{n+1} \rho_j(z)w_j = 0\}, \quad \rho_j = \frac{\partial\rho}{\partial z_j}, \quad z \in M.$$

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Webster-Tanaka pseudo-Hermitian manifolds

- $\{\theta^1, \cdots, \theta^n\}$ is a local basis for $H(M)^*$ so that.

$$d\theta = \sqrt{-1} \sum_{\alpha,\beta=1}^{n} h_{\alpha\overline{\beta}} \theta^\alpha \wedge \theta^{\overline{\beta}} \tag{3}$$

Moreover,

$$d\theta^\beta = \theta^\alpha \wedge \omega_\alpha^\beta + \theta \wedge \tau^\beta, \tag{4}$$

where ω_α^β is 1-form, and τ^β is $(0,1)$-form.

- In general ω_β^α is not unique.
- Webeter (78', JDG) showed a way to choose 1-forms ω_α^β with

$$dh_{\alpha\overline{\beta}} - h_{\gamma\overline{\beta}}\omega_\alpha^\gamma - h_{\alpha\overline{\gamma}}\omega_{\overline{\beta}}^{\overline{\gamma}} = 0. \tag{5}$$

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Write

$$\tau_\alpha = \sum_{\beta=1}^{n} h_{\alpha\overline{\beta}}\tau^{\overline{\beta}} = \sum_{\beta=1}^{n} A_{\alpha\beta}\theta^\beta. \tag{6}$$

Then

$$\tau_\alpha \wedge \theta^\alpha = 0 \quad \text{or} \quad A_{\alpha\gamma} = A_{\gamma\alpha}.$$

The torsion is defined:

$$\mathrm{Tor}_z(w, v) = 2\mathrm{Im}\,(A_{\alpha\beta} w_\alpha v_\beta). \tag{7}$$

Pseudo curvature tensor

- The curvature 2-form (defined by Webster, JDG, 1978):

$$\Omega_{\alpha\overline{\beta}} = \mathcal{R}_{\alpha\overline{\beta}\gamma\overline{\ell}}\theta^{\gamma} \wedge \theta^{\overline{\ell}} + \theta \wedge \lambda_{\alpha\overline{\beta}},$$

where $\mathcal{R}_{\alpha\overline{\beta}\gamma\overline{\ell}}$ is Webster-Tanaka pseudo curvature tensors;
$\Omega_{\beta}^{\alpha} = d\omega_{\beta}^{\alpha} - \omega_{\beta}^{\gamma} \wedge \omega_{\gamma}^{\alpha} - i\theta_{\beta} \wedge \tau^{\alpha} + i\tau_{\beta} \wedge \theta^{\alpha}$ and $\theta_{\beta} = h_{\beta\overline{\gamma}}\theta^{\overline{\gamma}}$.

- Webster pseudo Ricci curvature:

$$\mathcal{R}_{\alpha\overline{\beta}} = h^{\gamma\overline{\ell}}\mathcal{R}_{\alpha\overline{\beta}\gamma\overline{\ell}},$$

- Webster pseudo scalar curvature:

$$\mathcal{R} = h^{\alpha\overline{\beta}}\mathcal{R}_{\alpha\overline{\beta}}.$$

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$$\mathcal{R} = h^{\alpha\bar\beta}\mathcal{R}_{\alpha\bar\beta}.$$

L. and Luk's formula (2006, CAG)

Theorem

Let (M^{2n+1}, θ) be a strictly pseudo convex pseudo-Hermitian manifold with $M = \{z \in \mathbb{C}^{n+1} : \rho = 0\}$ and $\theta = \frac{1}{2i}(\partial\rho - \overline{\partial}\rho)$. Then pseudo Ricci curvature

$$\mathcal{R}ic(w, \overline{v}) = -\sum_{i,j=1}^{n+1} \frac{\partial^2 \log J(\rho)}{\partial z_i \partial \overline{z}_j} w_i \overline{v}_j + (n+1)\frac{\det H(\rho)}{J(\rho)} L_\theta(w, \overline{v})$$

where $w, v \in H_z(M), z \in M$.

Asymptotic Einstein

- Let D be a bounded pseudoconvex domain in $\mathbb{C}^{n+1}$. Let ρ be a smooth defining function for D such that $u = -\log(-\rho)$ is strictly plurisubharmonic in D. We say that (D, g_u) is **asymptotic Einstein** if

$$J(\rho) = 1 + O(\text{dist}(z, \partial D)^2)$$

Corollary

If (D, g_u) is asymptotic Einstein then the pseudo Ricci curvature for $(\partial D, i(\overline{\partial} - \partial \rho)/2)$ is:

$$\mathcal{R}ic(w, \overline{v}) = (n+1)\det H(\rho) L_\theta(w, \overline{v})$$

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Remarks on pseudo Einstein

• If D is a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^{n+1}$. u is the potential function of Kähler-Einstein metric. Then

(i) $(\partial D, \theta)$ is pseudo-Einstein, where

$$\theta = i(\overline{\partial}\rho - \partial\rho)/2.$$

(ii) $\det H(\rho) > 0$ if and only if $(\partial D, \theta)$ has positive pseudo Ricci curvature.

(iii) Pseudo scalar curvature

$$\mathcal{R} = n(n+1)\det H(\rho).$$

Remarks for Existence of pseudo-Einstein

• Existence of Pseudo Einstein structure was studied by J. Lee (AMJ, 1988). He proved several results and made a conjecture.

When (M, θ) is torsion-free, his conjecture was partially solved

• by D-C Chang, S-C Chang and J-Z. Tie (JDG, divergence of torsion is free).

• by X-D. Wang's recent work on generalization of Jerison-Lee's theorem (torsion-free).

Spectrum for compact complete manifolds

• If (M, g) is complete and compact, then spectrum of Δ_g is discrete (all are eigenvalues) and the first positive eigenvalue is

$$\lambda_1(\Delta_g) = \inf\left\{4\int_M \sum_{i,j=1}^n g^{i\bar{j}} \frac{\partial f}{\partial z_i}\frac{\partial f}{\partial \bar{z}_j} dv_g : f \in C_0^\infty(M), \|f\|_{L^2} = 1\right\},$$

where dv_g is the volume measure on M with respect to the Kähler metric g.

Spectrum for complete non-compact manifolds

• If (M, g) is complete and noncompact, then spectrum of Δ_g is not discrete anymore.

However, the bottom of the spectrum $\lambda_1(\Delta_g)$ remains to be:

$$\lambda_1(\Delta_g) = \inf \left\{ 4 \int_M \sum_{i,j=1}^{n} g^{i\bar{j}} \frac{\partial f}{\partial z_i} \frac{\partial f}{\partial \bar{z}_j} dv_g : f \in C_0^\infty(M), \|f\|_{L^2} = 1 \right\},$$

with dv_g is the volume measure on M with respect to the Kähler metric g.

Estimating bottom of spectrum for Δ_g

Question. How to estimate λ_1 in term of curvature of (M^n, g) ?

The problem have been studied by many people, including: S-Y. Cheng, J. Lee, Ji, S-L. Kong, P. Li, Munteanu, J-P. Wang, X.-D. Wang, D-T. Zhou, etc. Here, I only state a few results which directly related to my talk (recent works).

Mnteanu's upper bound estimate (JDG, 2009)

Theorem

Let (M^n, g) be a complete, non-compact manifold such the Ricci curvature $R_{i\bar{j}} \geq -(n+1)g_{i\bar{j}}$. Then

$$\lambda_1(\Delta_g) \leq n^2.$$

The upper bound is sharp when $M^n = B_n$ with g being Kähler-Einstein metric.

Remark: The theorem was proved by P. Li and J. Wang with stronger assumption: BK≥ -1.

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Example

- When $D = B_n$ is the unit ball in $\mathbb{C}^n$ with Kähler-Einstein metric

$$g = \sum_{i,j=1}^{n} \frac{1}{1-|z|^2}(\delta_{ij} - \frac{\bar{z}_i z_j}{1-|z|^2}) dz_i \otimes d\bar{z}_j$$

In this case:

- Holomorphic bisectional curvature $BK = -1$.
- $$R_{i\bar{j}} = -(n+1) g_{i\bar{j}}$$
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More examples for $\lambda_1(\Delta_g) = n^2$

• It seems that the only known example (M^n, g) where $\lambda_1(\Delta_g) = n^2$ is $M = B_n$ and g K-E. metric.

• A joint work with M.A. Tran (2009), we try to find many Kähler manifolds where $\lambda_1(\Delta_g) = n^2$.

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Setting

Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. Let $\rho \in C^3(\mathbb{C})$ be a defining function for D:

(i) $D = \{z \in \mathbb{C}^n : \rho(z) < 0\}$;
(ii) $\partial D = \{z \in \mathbb{C}^n : \rho(z) = 0\}$;
(iii) $\nabla \rho(z) \neq 0$ on ∂D.

- Assume that

$$u(z) = -\log(-\rho(z))$$

is strictly plurisubharmonic in D. Then u induces a Kähler metric

$$g = g[u] = \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial z_i \partial \overline{z}_j} dz_i \otimes d\overline{z}_j.$$

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A result of L. and Tran

Theorem

Let $M = D$, a bounded pseudoconvex domain in $\mathbb{C}^n$ with a plurisubharmonic defining function $\rho \in C^2(\overline{D})$. If $u = -\log(-\rho)$ is strictly plurisubharmonic in D and induce a Kähler metric $g = u_{i\bar{j}} dz_i \otimes d\bar{z}_j$. Then $\lambda_1(\Delta_g) = n^2$.
In particular, if ρ is strictly plurisubharmonic, then $\lambda_1(\Delta_g) = n^2$.

$\lambda_1(\Delta_g) =?$ when g is K.-E.

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Result and Conjecture of Li and Wang

Let D have a defining function ρ such that $u = -\log(-\rho)$ is strictly plurisubharmonic such that the Kähler metric $g[u]$ induced by u satisfies the following

(i) Sub-Einstein: $R_{i\bar{j}} \geq -(n+1)g_{i\bar{j}}$

(ii) $(D, g[u])$ is asymptotic Einstein ($J(\rho) = 1 + O(\rho^2)$) and

(iii) $\det H(\rho) \geq 0$ on ∂D (Pseudo scalar curvature is non-negative on ∂D).

Then $\lambda_1(\Delta_g) = n^2$.

• **Conjecture:** They conjecture that Condition (iii) can be replaced by a weaker condition: Nonnegative Yamabe invariant.

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Supper-pseudoconvex domain in $\mathbb{C}^n$

Definition

Let D be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$. We say that D is strictly super-pseudoconvex (super-superconvex) if there is a strictly plurisubharmonic defining function $r \in C^4(\overline{D})$ such that $\mathcal{L}_2[r] > 0$ ($\mathcal{L}_2[r] \geq 0$) on ∂D, respectively. Here

$$\mathcal{L}_2[r] =: 1 + \frac{|\partial r|_r^2}{n(n+1)} \tilde{\Delta} \log J(r) - \frac{2\mathrm{Re}\, R \log J(r)}{n+1} - |\partial r|_r^2 |\tilde{\nabla} \log J(r)|^2,$$

$$\tilde{\Delta} = a^{i\bar{j}}[r] \frac{\partial^2}{\partial z_i \partial \bar{z}_j}, \quad R = \sum_{j=1}^n r^j \frac{\partial}{\partial z_j}, \quad |\tilde{\nabla} f|^2 = a^{i\bar{j}}[r] \frac{\partial f}{\partial z_i} \frac{\partial f}{\partial \bar{z}_j} \text{ and}$$

$$r^i = \sum_{j=1}^n r^{i\bar{j}} r_{\bar{j}}, \quad \left[r^{i\bar{j}} \right]^t = H(r)^{-1}, \quad a^{i\bar{j}}[r] =: r^{i\bar{j}} - \frac{r^i r^{\bar{j}}}{-r + |\partial r|_r^2}.$$

Startmemt of Main Theorem 1

The main theorem 1 of this talk is:

Theorem

Let D be smoothly bounded super-psudoconvex domain in $\mathbb{C}^n$ *with* $C^{4-\epsilon}(\overline{D})$ *defining function* ρ *such that* $u = -\log(-\rho)$ *is strictly plurisubharmonic in D. If* (D, g_u) *is Asymptotic Einstein and sub-Einstein* ($R_{i\bar{j}} \geq -(n+1)g_{i\bar{j}}$)*, then* ρ *is strictly plurisubharmonic in D.*
In particular, then ρ *is the solution of the Fefferman equation (2), then* ρ *is strictly plurisubharmonic in D.*

Startmemt of Main Theorem 2

The main theorem 2 of this talk is:

Theorem

Let D be smoothly bounded psudoconvex domain in $\mathbb{C}^n$ with $C^{4-\epsilon}(\overline{D})$. Then
(i) When $n = 1$, D is (strictly) super-pseudoconvex if and only if D is (strictly) convex;
(ii) There is a strictly convex domain in $\mathbb{C}^n$ with $n > 1$ is not super-pseudoconvex;
(iii) There is a strictly super-pseudoconvex domain in $\mathbb{C}^n$ with $n > 1$ is not convex.

Corollary

As an application, we have the following theorem which solves the conjecture posed by L. and Wang:

Corollary

Let D be smoothly bounded super-psudoconvex domain in $\mathbb{C}^n$ *with* $C^{4-\epsilon}(\overline{D})$ *defining function* ρ *such that* $u = -\log(-\rho)$ *is strictly plurisubharmonic in D. If* (D, g_u) *is Asymptotic Einstein and sub-Einstein* ($R_{i\bar{j}} \geq -(n+1)g_{i\bar{j}}$)*, then*

$$\lambda_1(\Delta_{g_u}) = n^2.$$

Thank You!