Deforming conformal metrics with negative Bakry-Émery Ricci Tensor on manifolds with boundary

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Bakry-Émery Ricci Tensor

(M^n, g) compact, n-dimensional smooth Riemannian manifold, $n \geq 3$, f a smooth function on M, the Bakry-Émery Ricci tensor is defined as

$$\mathrm{Ric}_f = \mathrm{Ric} + Hess\,(f)$$

It was introduced by Bakry-Émery in 1985 and also by Lichnerowicz in 1970. Bakry-Émery Ricci tensor arises naturally in many different subjects, such as weighted manifold, measured space, Ricci flow and general relativity, etc.

Many important geometric and physical results of this tensor have been obtained, such as the measured Gromov-Hausdorff convergence theorem, volume comparison theorems, the splitting theorem, the rigidity theorem and Hawking-Penrose singularity theorem, etc.

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Let $\sigma_k : \mathbb{R}^n \to \mathbb{R}$ be the k-th elementary symmetric function, namely,

$$\sigma_k(\lambda) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}, \quad \forall \lambda = (\lambda_1, \cdots \lambda_n) \in \mathbb{R}^n.$$

Denote the eigenvalues of Ric_f by $\lambda(g^{-1}\mathrm{Ric}_f)$. Then we call $\sigma_k\left(\lambda\left(g^{-1}\mathrm{Ric}_f\right)\right)$ the k-**curvature** (or σ_k curvature) of Ric_f. The prescribing k-curvature problem of Ric_f is to find a metric $\tilde{g}$ in the conformal class $[g]$ of g satisfying the equation

$$\sigma_k^{\frac{1}{k}}\left(\lambda\left(\tilde{g}^{-1}\widetilde{\mathrm{Ric}}_f\right)\right) = \varphi(x), \tag{1}$$

where φ is a given smooth function on the manifold and $\widetilde{\mathrm{Ric}}_f = \widetilde{\mathrm{Ric}} + \widetilde{Hess}(f)$, $\widetilde{\mathrm{Ric}}$ (resp. $\widetilde{Hess}$) is the Ricci tensor (resp. Hessian) with respect to $\tilde{g}$.

When M is a closed manifold, (1) is exactly the Yamabe problem for $k = 1, f$ and φ are constants, which has been solved by Yamabe, Trudinger, Aubin and Schoen

When $k \geq 2$, (1) corresponds to a fully nonlinear second order partial differential equation which is elliptic when $\lambda \left(g^{-1}\mathrm{Ric}_f\right)$ is restricted to the cone Γ_k^+ in $\mathbb{R}^n$:

$$\Gamma_k^+ := \{\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n |\ \sigma_j(\lambda) > 0, 1 \leq j \leq k\},$$

Denote $\Gamma_k^- := -\Gamma_k^+$. Then there is another elliptic equation for the corresponding prescribing k-curvature problem of Ric_f in the negative cone $\Gamma_k^- \subset \mathbb{R}^n$. The corresponding equation can be written as

$$\sigma_k^{\frac{1}{k}}\left(-\lambda\left(\tilde{g}^{-1}\widetilde{\mathrm{Ric}_f}\right)\right) = \varphi(x), \qquad (2)$$

for $\lambda(g^{-1}\mathrm{Ric}_f) \in \Gamma_k^-$ and $\varphi(x) > 0$, where $\tilde{g} \in [g]$.

L.X. Yuan (2013) studied the equation (2) on closed manifold and proved its solvable for any given positive function φ. Under the same assumption and for a given positive function φ on a manifold with boundary, we proved in 2014 that there exists a unique complete metric $\tilde{g}$ satisfying (2).

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Main Results

In this talk, we consider the influences of conformal change both on Bakry-Émery Ricci tensor and the mean curvature of the boundary ∂M.

Theorem (Theorem 1.1)

Let $\left(\bar{M}^n, g\right), n \geq 3$, be a compact manifold with smooth boundary ∂M and $f \in C^\infty\left(\bar{M}\right)$. Suppose that $\lambda(g^{-1}\mathrm{Ric}_f) \in \Gamma_k^-$ and the mean curvature of ∂M with respect to its inward normal is non-positive. Then for any smooth function $\varphi > 0$, there exists a unique smooth metric $\tilde{g} \in [g]$ satisfying (2) and the boundary ∂M is minimal.

Since Ric_f is just the Ricci tensor Ric when f is a constant function, then by Theorem 1.1, we have

Corollary (Corollary 1.1)

Let $(\bar{M}^n, g), n \geq 3$, be a compact manifold with smooth boundary ∂M. If $\mathrm{Ric} \in \Gamma_k^-$ and the mean curvature of ∂M with respect to its inward normal is non-positive, then for any smooth function $\varphi > 0$, there exists a unique smooth metric $\tilde{g} \in [g]$ such that

$$\sigma_k^{\frac{1}{k}}\left(-\lambda\left(\tilde{g}^{-1}\widetilde{\mathrm{Ric}}\right)\right) = \varphi(x) \tag{3}$$

and the boundary ∂M is minimal.

In fact, (3) is the prescribing k-curvature equation of Ricci tensor which has been extensively studied.

- For a manifold $\bar{M}$ with boundary, Guan 2008 and Gursky-Streets-Warran 2011 proved that if $\mathrm{Ric} < 0$, then there exists a complete conformal metric of negative Ricci curvature satisfying (3) for any $\varphi > 0$.
- By a theorem of Lohkamp in 1994, there always exist smooth metrics on $\bar{M}$ with negative Ricci curvature. The results in Guan2008, Gursky2011 imply that the prescribing k-curvature problem (3) of Ricci tensor always has a complete solution for any given positive function φ.
- In the case of $\partial M = \emptyset$, Gursky and Viaclovsky in 2003 found the solution metric $\tilde{g}$ satisfying (3) with $\widetilde{\mathrm{Ric}} < 0$. Li and S. obtained the same result in 2005 by using a parabolic argument.

For the case $\lambda\left(g^{-1}\mathrm{Ric}\right) \in \Gamma_k^+$, the corresponding problem has less results.

When $k = n$, the equation becomes a Monge-Ampère type equation

$$\det\left(\lambda\left(\tilde{g}^{-1}\widetilde{\mathrm{Ric}}\right)\right) = \varphi^n(x). \tag{4}$$

Trudinger and Wang in 2009 solved (4) for $\varphi > 0$ when M is a closed manifold, and not conformally equivalent to the unit sphere.

He and S. 2011 show (4) is solvable on the manifold M with semi-positive Ricci curvature and totally geodesic boundary ∂M, and M is not conformal equivalent to a hemisphere.

In order to prove Theorem 1.1, we write out the corresponding partial differential equation. Let $\tilde{g} = e^{2u}g$. Under this conformal change,

$$\begin{aligned}\widetilde{\mathrm{Ric}}_f &= \mathrm{Ric}_f - (n-2)\nabla^2 u - \triangle u \cdot g + (n-2)\left(du \otimes du - |\nabla u|^2 g\right) \\ &\quad - du \otimes df - df \otimes du + \langle \nabla u, \nabla f\rangle g,\end{aligned}$$

where the covariant derivative is taken with respect to the background metric g. Set

$$\begin{aligned}\hat{W}[u] &= \nabla^2 u + \frac{1}{n-2}\triangle u g - \left(du \otimes du - |\nabla u|^2 g\right) - \frac{1}{n-2}\langle \nabla u, \nabla f\rangle g \\ &\quad + \frac{1}{n-2}(du \otimes df + df \otimes du) - \frac{1}{n-2}\mathrm{Ric}_f.\end{aligned} \tag{5}$$

Then equation (2) becomes

$$\sigma_k^{\frac{1}{k}}\left(\lambda\left(\hat{W}[u]\right)\right) = \frac{\varphi(x)}{n-2}e^{2u}.$$

Under the same conformal change $\tilde{g} = e^{2u}g$, the mean curvature changes as

$$\tilde{\mu} = \left(-\frac{\partial u}{\partial \nu} + \mu\right) e^{-u}, \tag{6}$$

where ν is the unit inward normal on ∂M. Since we want the boundary becomes minimal, that is $\tilde{\mu} \equiv 0$, the equations corresponding to Theorem 1.1 becomes

$$\begin{cases} \sigma_k^{\frac{1}{k}}\left(\lambda\left(\hat{W}\left[u\right]\right)\right) = \frac{\varphi(x)}{n-2}e^{2u} & \text{in} \quad M, \\ \frac{\partial u}{\partial \nu} = \mu & \text{on} \quad \partial M. \end{cases} \tag{7}$$

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In fact, we will consider a more general equation.

Assume that $\Gamma \subset \mathbb{R}^n$ be an open convex symmetric cone with vertex at the origin satisfying $\Gamma_n^+ \subset \Gamma \subset \Gamma_1^+$.

Let $F : \mathbb{R}^n \to \mathbb{R}$ be a general smooth, symmetric, homogeneous function of degree one in Γ normalized with $F(e) = F(1, \cdots, 1) = 1$. Moreover, $F = 0$ on $\partial\Gamma$ and satisfies the following structure conditions in Γ:

(C1) F is positive;

(C2) F is concave (i.e., $\frac{\partial^2 F}{\partial\lambda_i \partial\lambda_j}$ is negative semi-definite);

(C3) F is monotone (i.e., $\frac{\partial F}{\partial\lambda_i}$ is positive).

The elementary symmetric functions $\sigma_k^{1/k}$ satisfy all the structure conditions above on Γ_k^+ (Urbas1991). Moreover if a function F satisfies all the conditions above, then for any $\lambda \in \Gamma$, there have

(S1) $\sum_i \lambda_i \frac{\partial F(\lambda)}{\partial \lambda_i} = F(\lambda)$,

(S2) $\sum_i \frac{\partial F}{\partial \lambda_i}(\lambda) \geq F(e) = 1$.

Let $\theta, \gamma \in \mathbb{R}$, $s(x)$, $t(x)$, $a(x)$, $b(x)$ be smooth functions, and T be a smooth symmetric (0,2)-tensor defined on $\bar{M}$. Denote

$$\begin{aligned} W_g[u] := {} & \theta\nabla^2 u + \gamma\triangle u g + s(x)\, du \otimes du \\ & - \frac{t(x)}{2}|\nabla u|^2 g + a(x)\langle \nabla u, \nabla f\rangle g \\ & + b(x)(du \otimes df + df \otimes du) + T. \end{aligned} \tag{8}$$

We call a function $v \in C^2(M)$ **admissible** if $\lambda\left(g^{-1}W_g[v]\right) \in \Gamma$. Let $\Phi(x,z) \in C^\infty\left(\bar{M} \times \mathbb{R}\right)$ and $\psi(x) \in C^\infty(\partial M)$ be two given functions, we consider the equation

$$\begin{cases} F\left(g^{-1}W_g[u]\right) = \Phi(x,u) & \text{in } M, \\ \frac{\partial u}{\partial \nu} = \mu - e^u \psi(x) & \text{on } \partial M. \end{cases} \tag{9}$$

Theorem (Theorem 1.2)

Let $\left(\bar{M}^n, g\right), n \geq 3$, *be a compact manifold with smooth boundary* ∂M, $f \in C^\infty\left(\bar{M}\right)$. *Suppose* γ *and* θ *are constants and* $\gamma + \theta > 0$, $\gamma > 0$, *the* $(0,2)$ *tensor* $T \in \Gamma$, *the mean curvature of the boundary* μ *is non-positive with respect to the inward normal, and the function* $\Phi\left(x, z\right) \in C^\infty\left(\bar{M} \times \mathbb{R}\right)$ *satisfies that*

$$\Phi > 0, \quad \partial_z \Phi > 0, \qquad \lim_{z \to +\infty} \Phi\left(x, z\right) = +\infty, \qquad \lim_{z \to -\infty} \Phi\left(x, z\right) = 0, \tag{10}$$

then for any function $s, t, a, b \in C^\infty(\bar{M})$, $\psi \in C^\infty(\partial M)$ *and* $\psi \leq 0$, *there exists a unique admissible function* $u \in C^\infty\left(\bar{M}\right)$ *solving the Equation (9).*

Remark

- *Different with the results of Sheng-Trudinger-Wang JDG 2007, in this theorem, we need not add any restriction on* $s\,(x)\,,t\,(x)\,,a(x)$, $b(x)$. *Moreover,* $W_g\,[u]$ *is different with the expression in Yuan 2013, where* $\theta = 1$, s *and* t *are constants.*
- *Another important point is that we needn't impose any condition on the boundary except the mean curvature is non-positive with respect to the inward normal.*

We may apply Theorem 1.2 to the modified Schouten tensor

$$A_g^\tau = \frac{1}{n-2}\left(\mathrm{Ric} - \frac{\tau}{2(n-1)} R \cdot g\right), \quad \tau \in \mathbb{R},$$

which was introduced by Gursky-Viaclovsky 2003 and A.Li-YY Li 2003. There are many interesting results about the prescribing k-curvature problems of this tensor.

Under the conformal change of metric $\tilde{g} = e^{2u} g$, the modified Schouten tensor changes according to the formula

$$A_{\tilde{g}}^\tau = \frac{\tau-1}{n-2} \triangle u g - \nabla^2 u + du \otimes du + \frac{\tau-2}{2} |\nabla u|^2 g + A_g^\tau. \tag{11}$$

We have the following corollary.

Corollary (Corollary 1.2)

Let $(\bar{M}^n, g)$, $n \geq 3$, be a compact manifold with smooth boundary ∂M. If $\tau > n-1$, $A_g^\tau \in \Gamma_k^+$ (or $\tau < 1$, $A_g^\tau \in \Gamma_k^-$) , and the mean curvature of ∂M with respect to its inward normal is non-positive, then for any smooth function $\varphi > 0$, there exist a unique smooth metric $\tilde{g} \in [g]$ such that ∂M is minimal and satisfies

$$\sigma_k^{\frac{1}{k}}\left(\lambda\left(\tilde{g}^{-1}A_{\tilde{g}}^\tau\right)\right) = \varphi(x)$$

or

$$\sigma_k^{\frac{1}{k}}\left(-\lambda\left(\tilde{g}^{-1}A_{\tilde{g}}^\tau\right)\right) = \varphi(x),$$

respectively.

Aobing Li and Huan Zhu has got the same result for the modified Schouten tensor for $\tau < 1, A_g^\tau \in \Gamma_k^-$ in arXiv 2011.

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The proof of Theorem 1.2 is based on the continuity method. So we need to study the a priori estimates of the solutions to equation (9) .

At first, we use the compactness of the manifold $\bar{M}$ to get the global C^0 estimate of the solution to Equation (9). We consider the upper and lower bounds of u respectively.

Lemma (Lemma 3.1)

Let the constants γ and θ satisfy $\gamma > 0, \gamma + \theta > 0$, the function $\psi \leq 0$ on ∂M, $T \in \Gamma$, and the positive function $\Phi(x, z) \in C^\infty\left(\bar{M} \times \mathbb{R}\right)$ satisfies condition (10). Then for any admissible solution $u \in C^2\left(\bar{M}\right)$ to Equation (9), we have

$$u \leq C_0 \quad \text{on } \bar{M},$$

where the constant C_0 depends only on n, g, θ, γ, $|\nabla f|_{C^0(\bar{M})}$, $|s|_{L^\infty(\bar{M})}$, $|t|_{L^\infty(\bar{M})}$, $|a|_{L^\infty(\bar{M})}$, $|b|_{L^\infty(\bar{M})}$, $|T|_{C^0(\bar{M})}$ and $|\Phi|_{C^0(\bar{M})}$.

Proof of Lemma 3.1.

Since (M^n, g) is compact manifold with boundary, $n \geq 3$, by a result of Escobar1992 JDG: $\exists\ \breve{g} = e^{2w} g$ s.t. $R_{\breve{g}}$ does not change sign and the boundary is minimal, i.e. $\breve{\mu} = 0$ on ∂M.
Note that $\tilde{g} = e^{2u} g = e^{2(u-w)} \breve{g}$. Denote $\breve{u} = u - w$. Since $\breve{g} = e^{2w} g$, then

$$\begin{aligned} W_{\breve{g}}[\breve{u}] &= W_g[u] - W_g[w] - (\theta + s)(dw \otimes d\breve{u} + d\breve{u} \otimes dw) \\ &\quad + (\theta + \gamma(n-2) + t)\langle \nabla w, \nabla \breve{u} \rangle_g \cdot g + t\,|\nabla w|_g^2 \cdot g + T \\ &= W_g[u] - \check{W}. \end{aligned}$$

Then $\breve{u}$ satisfies the equation

$$\begin{cases} F\left(g^{-1}\left(W_{\breve{g}}[\breve{u}] + \check{W}\right)\right) = \Phi(x, u) & \text{in} \quad M, \\ \frac{\partial \breve{u}}{\partial \breve{\nu}} = -e^{\breve{u}} \psi(x) & \text{on} \quad \partial M, \end{cases} \tag{12}$$

Let x_0 is the maximum point of $\check{u}$ on $\bar{M}$.
Case 1. If x_0 is an interior point of M, then $\nabla_{\check{g}}\check{u}\left(x_0\right) = 0$ and $\nabla^2_{\check{g}}\check{u}\left(x_0\right) \leq 0$.
Therefore

$$W_{\check{g}}\left[\check{u}\right]\left(x_0\right) \leq T, \qquad \check{W} \leq C' g,$$

where the constant C' depends on $n, g, \theta, \gamma, |s|_{L^\infty}, |t|_{L^\infty}, |a|_{L^\infty}, |b|_{L^\infty}$ and $|\nabla f|$. Then we obtain

$$\begin{aligned}\Phi\left(x\left(x_0\right), u\left(x_0\right)\right) &= F\left(g^{-1}\left(W_{\check{g}}\left[\check{u}\right] + \check{W}\right)\right)\left(x_0\right) \leq F\left(g^{-1}\left(T + C' g\right)\right)\left(x_0\right) \\ &\leq \max_{\bar{M}} F\left(g^{-1}\left(T + C' g\right)\right) \leq C.\end{aligned}$$

Then by the condition $\partial_z \Phi > 0$ and $\lim_{z \to +\infty} \Phi\left(x, z\right) = +\infty$ in (10), we know that $u\left(x_0\right) \leq C$. So we have

$$\check{u}\left(x_0\right) \leq C.$$

Case 2. If $x_0 \in \partial M$, then $\frac{\partial \breve{u}}{\partial \breve{\nu}}(x_0) \leq 0$. Since $\psi(x_0) \leq 0$, then $\psi(x_0) = 0$, which means that $\frac{\partial \breve{u}}{\partial \breve{\nu}}(x_0) = 0$. So $\nabla_{\breve{g}} \breve{u}(x_0) = 0$ and therefore $\nabla^2_{\breve{g}} \breve{u}(x_0) \leq 0$. We can proceed as in Case 1 to obtain $\breve{u}(x_0) \leq C$.
By the arguments in Case 1 and Case 2, we have $u \leq C$ on the whole manifold $\bar{M}$.

Lemma (Lemma 3. 2)

Let the constants γ and θ satisfy $\gamma > 0, \gamma + \theta > 0$, $\psi \leq 0$ on ∂M, $T \in \Gamma$, and the function $\Phi(x, z) \in C^\infty\left(\bar{M}\right) \times \mathbb{R}$ satisfies (10). Suppose the mean curvature of the boundary with respect to the inward normal is non-positive, i.e. $\mu \leq 0$. Then for any admissible solution $u \in C^2\left(\bar{M}\right)$ to Equation (9), we have

$$u \geq -C_0 \quad \text{on } \bar{M},$$

where the constant C_0 depends only on n, g,θ, γ, $|\nabla f|_{C^0(\bar{M})}$, $|s|_{L^\infty(\bar{M})}$, $|t|_{L^\infty(\bar{M})}$, $|a|_{L^\infty(\bar{M})}$, $|b|_{L^\infty(\bar{M})}$, $|T|_{C^0(\bar{M})}$, $|\Phi|_{C^0(\bar{M})}$ and $|\psi|_{L^\infty(\partial M)}$.

Proof of Lemma 3.2

Let ρ be a smooth function on $\bar{M}$ such that ρ is the distance function to ∂M near the boundary, and $0 \leq \rho \leq 1$. Then $\rho_\nu \equiv 1$ on ∂M. Denote $\hat{g} = e^{2\varepsilon\rho} g$, where $\varepsilon > 0$ being a constant to be chosen later. Let $\hat{u} = u - \varepsilon\rho$. Then $\tilde{g} = e^{2\hat{u}}\hat{g}$. By a similar calculation to Lemma 1, we have $W_{\hat{g}}[\hat{u}] = W_g[u] - \hat{W}$, where

$$\begin{aligned} \hat{W} :=& \varepsilon\left(\theta\nabla^2\rho + \gamma\triangle\rho\cdot g\right) + \varepsilon^2\left(sd\rho\otimes d\rho - \frac{3t}{2}\left|\nabla\rho\right|_g^2\cdot g\right) \\ &+ \varepsilon a\langle\nabla\rho,\nabla f\rangle_g\cdot g + \varepsilon b\left(d\rho\otimes df + df\otimes d\rho\right) \\ &+ \varepsilon\left(\theta+s\right)\left(d\rho\otimes d\hat{u} + d\hat{u}\otimes d\rho\right) - \varepsilon\left(\theta+\gamma\left(n-2\right)+t\right)\langle\nabla\rho,\nabla\hat{u}\rangle_g\cdot g. \end{aligned} \tag{13}$$

Then $\hat{u}$ satisfies the following equation

$$\begin{cases} F\left(g^{-1}\left(W_{\hat{g}}[\hat{u}] + \hat{W}\right)\right) = \Phi(x,u) & \text{in } M, \\ \frac{\partial\hat{u}}{\partial\hat{\nu}} = \hat{\mu} - e^{\hat{u}}\psi(x) & \text{on } \partial M. \end{cases} \tag{14}$$

Let y_0 is the minimum point of $\hat{u}$ on $\bar{M}$.
Case 1. If y_0 is an interior point of M, then we have $\nabla_{\hat{g}}\hat{u}\left(y_0\right)=0$, $\nabla^2_{\hat{g}}\hat{u}\left(y_0\right)\geq 0$. Therefore $W_{\hat{g}}\left[\hat{u}\right]\geq T$ and $\hat{W}\left(y_0\right)\geq -\varepsilon\hat{C}\cdot g$. Since $T\in\Gamma$, we may choose $\varepsilon=\varepsilon_0$ small enough such that

$$T-\varepsilon_0\hat{C}\cdot g\in\Gamma.$$

Therefore, we have

$$\begin{aligned}\Phi\left(y_0,u\left(y_0\right)\right)&=F\left(g^{-1}\left(W_{\hat{g}}\left[\hat{u}\right]+\hat{W}\right)\right)\left(y_0\right)\\&\geq F\left(g^{-1}\left(T-\varepsilon_0\hat{C}\cdot g\right)\right)\left(y_0\right)\geq C>0.\end{aligned}$$

Hence, by $\partial_z\Phi>0$ and $\lim_{z\to-\infty}\Phi\left(x,z\right)=0$ in (10), we know that $u\left(y_0\right)\geq -C$ which means $\hat{u}\left(y_0\right)\geq -C$.

Case 2. If y_0 is a boundary point of M, then $\frac{\partial \hat{u}}{\partial \hat{\nu}}(y_0) \geq 0$. Note that $\mu \leq 0$, we have

$$\hat{\mu}(y_0) = \left(-\varepsilon \frac{\partial \rho}{\partial \nu} + \mu\right) e^{-\varepsilon \rho}(y_0) \leq -\varepsilon \ . \tag{15}$$

It follows from (14) and (15) that

$$-\varepsilon \geq \hat{\mu}(y_0) = \frac{\partial \hat{u}}{\partial \hat{\nu}}(y_0) + e^{\hat{u}} \psi(x)(y_0) \geq e^{\hat{u}} \psi(x)(y_0) \geq -Ce^{\hat{u}}(y_0).$$

Then we obtain

$$\hat{u}(y_0) \geq -C.$$

By the arguments in Case 1 and Case 2, we have $u \geq -C$ on the whole manifold $\bar{M}$.

Combining Lemmas 3.1 and 3.2, we have the following proposition.

Proposition (Proposition 1)

Let the constants γ and θ satisfy $\gamma > 0, \gamma + \theta > 0$, $\psi \leq 0$ on ∂M, $T \in \Gamma$, and the function $\Phi(x, z) \in C^\infty \left(\bar{M}\right) \times \mathbb{R}$ satisfies (10). Suppose the mean curvature of the boundary with respect to the inward normal is non-positive, i.e. $\mu \leq 0$. Then for any admissible solution $u \in C^2 \left(\bar{M}\right)$ to Equation (9), we have

$$\sup_{\bar{M}} |u| \leq C_0,$$

where the constant C_0 depends only on $n, g, \theta, \gamma, f, s, t, a, b, T$, Φ and ψ.

Outline

We denote

$$\Psi(x,u) := \mu - e^u \psi$$

on ∂M. By extending μ and ψ to a neighborhood of ∂M smoothly, we may assume Ψ is defined on a neighborhood of the boundary ∂M. Let $x \in \partial M$ and $\bar{B}_r^+ = \left\{x_n \geq 0, \left(\sum_{i=1}^n x_i^2\right) \leq r^2\right\}$ be a half-ball centered at x of radius r (w.r.t. g) in the Fermi coordinates $\left\{x^i\right\}_{1\leq i\leq n}$. Denote the boundary of $\bar{B}_r^+$ on ∂M by $\sum_r = \left\{x_n = 0, \sum_i x_i^2 \leq r^2\right\}$. Choosing a cutoff function $\eta \in [0,1]$ satisfies $\eta = 1$ in $\bar{B}_{r/2}^+$ and $\eta = 0$ outside $\bar{B}_r^+$. Moreover,

$$|\nabla \eta| \leq b_0 \frac{\eta^{1/2}}{r}, \qquad \left|\nabla^2 \eta\right| \leq \frac{b_0}{r^2}.$$

Since η only depends on r, we have

$$\frac{\partial \eta}{\partial x^n} = 0 \text{ on } \partial M.$$

Let $\phi : \mathbb{R} \to \mathbb{R}$ is a function defined on $\bar{M}$ by

$$\phi(u) = \alpha_1 (\alpha_2 + u)^p,$$

where $\alpha_1 = \frac{1}{p^2 (3C_0)^p}$, $\alpha_2 = 2C_0$ and the positive constants p to be fixed. It is clear that

$$0 < \phi \le \frac{1}{p^2}, \qquad \frac{1}{3^{p+1} C_0 p} < \phi' \le \frac{1}{C_0 p}.$$

Lemma (Lemma 4.1)

Let $\gamma > 0$ and $\gamma + \theta > 0$. Suppose $u \in C^3(\bar{B}_r^+)$ is an admissible solution to Equation (9). Then there is a positive constant C depending only on $n, \gamma, \theta, g, r, |\mu|_{C^2(\Sigma_r)}, |f|_{C^2(\bar{B}_r^+)}, |s|_{C^1(\bar{B}_r^+)}, |t|_{C^1(\bar{B}_r^+)}, |a|_{C^1(\bar{B}_r^+)}, |b|_{C^1(\bar{B}_r^+)}$ $|T|_{C^1(\bar{B}_r^+)}, |\Phi|_{C^1(\bar{B}_r^+) \times [-C_0, C_0]}$ and $|\psi|_{C^2(\Sigma_r)}$ such that

$$\sup_{\bar{B}_{r/2}^+} |\nabla u|_g \le C.$$

Proof of Lemma 4.1

Consider the auxiliary function

$$G := \frac{1}{2}\eta e^{\beta} \left|\nabla u - \Psi\left(x, u\right)\nabla\rho\right|^2,$$

where $\beta := qx_n + \phi\left(u\right)$, q is a large constant which will be fixed later, ρ is a smooth function on $\bar{M}$ such that ρ is the distance function to ∂M near the boundary and $0 \leq \rho \leq 1$. By the definition of Fermi coordinates, $\rho = x^n$ near the boundary.

Let x_0 be the maximum point of G on $\bar{B}_r^+$. We can show the point x_0 must be an interior point of $\bar{B}_r^+$. Otherwise we can prove $G_n(x_0) > 0$. By use of the maximum principle at the interior point x_0, we the gradient estimate.

Remark (Remark 4.1)

If $\bar{B}_r^+$ and $\bar{B}_{r/2}^+$ are replaced by two local geodesic open balls in the interior of M, $q = 0$ and $\rho = 0$ in the auxiliary function G, we can get the interior gradient estimates for Equation (9) by the proof of Lemma 4.1.

by Lemma 4.1 and Remark 4.1, we can derive the global gradient estimate of the solutions to Equation (9).

Proposition (Proposition 2)

Let $\gamma > 0$ and $\gamma + \theta > 0$. Suppose $u \in C^3(\bar{M})$ is an admissible solution of Equation (9). Then there is a positive constant C_1 depending only on $n, g, \mu, \theta, \gamma, f, s, t, a, b, T, \Phi, \psi$ and C_0 such that

$$\sup_{\bar{M}} |\nabla u|_g \leq C_1.$$

Outline

To get the Hessian estimates of the solution to Equation (9), we first control its bound of double normal derivatives on the boundary.

Lemma (Lemma 5.1)

Let $\gamma > 0$ and $\gamma + \theta > 0$. Suppose $u \in C^3(\bar{M})$ is an admissible solution to Equation (9). Then there is a positive constant C depending only on $n, g, \gamma, \theta, r,$ $|\mu|_{C^2(\Sigma_r)}$, $|f|_{C^2(\bar{M})}$, $|s|_{C^1(\bar{M})}$, $|t|_{C^1(\bar{M})}$, $|a|_{C^1(\bar{M})}$, $|b|_{C^1(\bar{M})}$ $|T|_{C^1(\bar{M})}$, $|\Phi|_{C^2(\bar{M})\times[-C_0,C_0]}$ and $|\psi|_{C^2(\Sigma_r)}$ such that

$$|u_{nn}| \leq C \quad \text{on} \quad \partial M.$$

Proof of Lemma 5.1

For any $x_0 \in \partial M$, let $r = dist(x, x_0)$ and $\bar{B}_r^+$ be the half ball centered at x_0. Consider a barrier function

$$h\left(x\right) := Ar^2\left(x\right) - v\left(x\right) \pm \left(u_n\left(x\right) - \Psi\left(x, u\right)\right),$$

where A is a large constant to be fixed and the function v is defined by

$$v := p\left(q\rho^2 - \rho\right)$$

for some positive constant p and q, and $\rho = dist\left(x, \partial M\right)$ near the boundary, $\Psi(x, u) = \mu(x) - e^u\psi(x)$ is defined in a neighborhood of ∂M as in Lemma 4.1. Then $v < 0$ on $\bar{B}_r^+ \setminus \Sigma_r$ for $r < \frac{1}{q}$.

For any function w on $\bar{M}$, define a linear elliptic differential operator $\mathcal{P}$ by

$$\mathcal{P}w = P^{ij}w_{ij} + 2sF^{ij}u_iw_j - tu_lw_l\mathcal{T} + a\langle\nabla w, \nabla f\rangle\mathcal{T} + 2bF^{ij}w_if_j.$$

Claim 5.1. For p and q large enough, we have

$$\mathcal{P}h \leq 0. \tag{16}$$

Since $u_n = \Psi(x, u)$ on ∂M, $h(x) \geq 0$ on $\sum_r$. We can choose A large enough such that

$$h(x) \geq Ar^2(x) - |u_n(x) - \Psi(x, u)| > Ar^2(x) - C > 0$$

on $\partial \bar{B}_r^+ \cap \{x_n > 0\}$ .Then by the maximum principle and (16), we have $h \geq 0$ on $\bar{B}_r^+$. Note that $h(x_0) = 0$, then $h_n(x_0) \geq 0$, which implies that

$$|u_{nn}|(x_0) < C.$$

Lemma (Lemma 5.2)

Let $\gamma > 0$ *and* $\gamma + \theta > 0$. *Suppose* $u \in C^4(\bar{B}_r^+)$ *is an admissible solution to Equation (9). Then there is a positive constant* C *depending only on* $n, g, \gamma, \theta, r, |\mu|_{C^2(\Sigma_r)}, |f|_{C^3(\bar{M})}, |s|_{C^2(\bar{B}_r^+)}, |t|_{C^2(\bar{B}_r^+)}, |a|_{C^2(\bar{B}_r^+)}, |b|_{C^2(\bar{B}_r^+)}, |T|_{C^2(\bar{B}_r^+)}, |\Phi|_{C^2(\bar{B}_r^+)\times[-C_0,C_0]}$ *and* $|\psi|_{C^2(\Sigma_r)}$ *such that*

$$u_{\zeta\zeta} \leq C \text{ on } \bar{B}_r^+ \tag{17}$$

for any vector $\zeta \in span\{e_1, \cdots, e_{n-1}\}$ *with* $|\zeta| = 1$.

Proof of Lemma 5.2

By rotating the coordinates around the inward unit normal e_n, it suffices to establish (17) for $\zeta = e_1$. Consider the function

$$G := \eta e^{qx_n} \left(u_{11} + m \left|\nabla u\right|^2\right), \tag{18}$$

where η is a cut-off function, q and m are larger positive constants to be fixed. Suppose x_0 is the maximum point of G on $\bar{B}_r^+$. Without loss of generality, we may assume $r = 1$,

$$u_{11}(x_0) >> 1, \quad \left|u_{\alpha\alpha}\right|(x_0) \leq Cu_{11}(x_0) \ \text{ for } \ 2 \leq \alpha \leq n-1. \tag{19}$$

We can show the maximum point x_0 of G does not belong to $\sum_r$ at first.

Since x_0 is in $\bar{B}_r^+ \setminus \Sigma_r$, we can use the maximum principle at x_0 to get the estimate (17) directly.

Proposition (Proposition 3)

Let $\gamma > 0$ and $\gamma + \theta > 0$. Suppose $u \in C^4(\bar{M})$ is an admissible solution to equation (9). Then there is a positive constant C_2 depending only on $n, g, \mu, \gamma, \theta,\ f, s, t, a, b, T,\ \Phi,\ \psi, C_0$ and C_1, such that

$$\sup_{\bar{M}} \left|\nabla^2 u\right|_g \leq C_2.$$

Proof of Proposition 3

Since $\Gamma \subset \Gamma_1^+$, we have $u_{\xi\xi} \geq -C$ on $\bar{M}$ for any unit vector $\xi \in \mathbb{S}^n$. Therefore, we only need to get the upper bound of $u_{\xi\xi}$.

Consider the function

$$\bar{G} = \eta(x) e^{qx_n} \left(\nabla^2 u + m |\nabla u|^2 \cdot g\right)$$

over the set $(x, \xi) \in \left(\bar{M}, \mathbb{S}^n\right)$. Suppose $\bar{G}$ attains its maximum at some point $x_0 \in \bar{M}$ in the direction $\xi \in T_{x_0}\bar{M} \cap \mathbb{S}^n$.

If x_0 is an interior point of $\bar{M}$. Let B_r and $B_{r/2}$ be two local geodesic balls in the interior of M, and $q = 0$ in the auxiliary function G. Since $\bar{M}$ is compact, by similar proof of Lemma 5.2 one can show that $\bar{G}(x_0, \xi) \leq C$ for some universal constant C.

If x_0 is belong to the boundary ∂M, we may write $\xi = ke_n + l\zeta$, where ζ is a unit vector in span $\{e_1, \cdots, e_{n-1}\}$, and k, l are two numbers satisfying $k^2 + l^2 = 1$. Then

$$u_{\xi\xi} = k^2 u_{nn} + l^2 u_{\zeta\zeta} + 2kl u_{\zeta n},$$

which is also bounded by Lemma 5.1 and Lemma 5.2 and a direct computation on ∂M

$$u_{n\alpha} = L_{\alpha\gamma} u_\gamma - e^u \psi u_\alpha + \mu_\alpha - e^u \psi_\alpha,$$

where $L_{\alpha\gamma}$ is the second fundamental form of the boundary. Hence $\bar{G}(x_0, \xi) \leq C$.
The above argument shows that $u_{\xi\xi} \leq C$ on $\bar{M}$ for any unit vector $\xi \in \mathbb{S}^n$.
Therefore $\left|\nabla^2 u\right| \leq C$ on $\bar{M}$.

Proof of Theorem 1.2

We consider the uniqueness at first. Let u_1 and u_2 be two different solutions to Equation (9). Then they satisfy

$$\begin{cases} F\left(g^{-1}W_g\left[u_i\right]\right) = \Phi\left(x, u_i\right) & \text{in} \quad M, \\ \frac{\partial u_i}{\partial \nu} = \mu \ - e^{u_i}\psi\left(x\right) & \text{on} \quad \partial M, \end{cases} \qquad i = 1, 2. \tag{20}$$

Let $v := u_1 - u_2$. We can show $\max_{x \in \bar{M}} v \leq 0$ and $\min_{x \in \bar{M}} v \geq 0$. Hence, $v \equiv 0$ on $\bar{M}$. So the solution to Equation (9) is unique.

Now, we use the continuity method to prove the existence of Equation (9).
For $\beta \in [0,1]$, we consider the equation

$$F\left(g^{-1}\left(\begin{array}{c}\theta\nabla^2 u+\gamma\triangle u g+s(x)\,du\otimes du-\frac{t(x)}{2}|\nabla u|^2 g\\ +a(x)\langle\nabla u,\nabla f\rangle g+b(x)\,(du\otimes df+df\otimes du)+T_\beta\end{array}\right)\right)=\Phi_\beta(x,u), \tag{1.10$_\beta$}$$

where

$$T_\beta=\beta T+\frac{1-\beta}{F(e)}g,\ \text{and}\ \Phi_\beta(x,u)=(1-\beta)\,e^{2u}+\beta\Phi(x,u).$$

Clearly, T_β and Φ_β satisfy the following conditions:

- ◇ $T_\beta\in\Gamma$ and $|T_\beta|_{C^4(M)}\le C$, where the constant C is independent of β.
- ◇ $\Phi_\beta(x,u)>0$, $\partial_z\Phi_\beta>0$, $\lim\limits_{z\to+\infty}\Phi_\beta(x,z)\to+\infty$, and $\lim\limits_{z\to-\infty}\Phi_\beta(x,z)\to 0$.
- ◇ $|\Phi_\beta|_{C^2\left(\bar{M}\times[-C_0,C_0]\right)}\le C$, where C is independent of β.

For any function v on M, define

$$\mathcal{B}[v] := F(\cdots) - \Phi_\beta(x, v).$$

Then any solution u of Equation (1.10_β) satisfies $\mathcal{B}[u] = 0$. We can show easily that the linearized operator of $\mathcal{B}$ is invertible.

It follows from Propositions 1, 2 and 3 that for each β, the admissible solution to (1.10_β) has a uniform priori C^2 estimates (independent of β). Then we obtain the uniform $C^{2,\alpha}$ estimates by Evans-Krylov's Theory and classical Schauder theory. Define

$$I = \{\beta \in [0,1] \mid (1.10_\beta) \text{ has admissible solution}\}.$$

Clearly, $u \equiv 0$ is the unique admissible solution of (1.10_0). Hence, $I \neq \varnothing$. Since the linearized operator is invertible, $I \subset [0,1]$ is open. By the a uniform priori $C^{2,\alpha}$ estimates and the standard degree theory, we conclude that I is also closed. Therefore $I = [0,1]$. This shows that Equation (9) is solvable.

Thank you!