Stability and Rigidity

in

Positive Scalar Curvature

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curvature spaces by Killing and Hopf",
the first rigidity (or isometry) result
goes ~~t~~ here back to Toponogov.

<u>Thm (Toponogov 1959)</u>

$$\sec_M \geq 1 \;\cancel{\text{and}} \Rightarrow \operatorname{diam}_M \leq \pi,$$

$$\text{and } \operatorname{diam}_M = \pi \Leftrightarrow M^n \equiv S^n(1)$$

<u>isometric</u>

<u>Rigidity</u>!

There is a corresponding <u>stability</u> statement:

<u>Thm (Grove-Shiohama 1977)</u>

$$\sec_{M^n} \geq 1, \; \operatorname{diam} > \frac{\pi}{2} \Rightarrow M^n \approx S^n.$$

<u>homeomorphic</u>

So what could be a substitute for the diameter here?

<u>Leon Green 1963</u> $\int_M \mathrm{scal}_M \, d\mathrm{vol}_M \geq n(n-1) \mathrm{vol}$

$\Rightarrow$ $\mathrm{conj}_M \leq \pi$, and

" $= \pi$, $\Leftrightarrow$ M^n is isometric to a spherical space form S^n/Γ with $\sec_M = 1$

<u>Cor</u> $\mathrm{scal}_M \geq n(n-1)$, ~~[illegible]~~

$\mathrm{inj} = \pi$ $\Rightarrow$ $M^n \equiv S^n(1)$

Thus, as with Toponogov and Cheng, in $\mathrm{scal} > 0$ there is also <u>rigidity</u>, when the diameter is replaced by the inj or conj radius!

· What about <u>stability</u> here?

Thm $\forall n \in \mathbb{N}$ $\forall C, \lambda_0, \lambda_1, i_0 > 0$ $\forall 0 \leq \beta \leq 1$

$\exists$ $\varepsilon = \varepsilon(n, C, \lambda_0, \lambda_1, i_0, \beta) > 0$ s.t.

any closed n-mfd M admitting a metric w/

$$\int_M \mathrm{scal}_M \, d\mathrm{vol}_M \geq n(n-1) \operatorname{Vol} M$$

$\mathrm{conj} \geq \pi - \varepsilon$,

$\|\mathrm{Ric}\| \leq \lambda_0$, $\|\nabla \mathrm{Ric}\| \leq \lambda_1$

$$\operatorname{Vol} M \leq \frac{C}{(\pi - \mathrm{conj}\, M)^{\beta}}, \quad \mathrm{inj} \geq i_0$$

is diffeomorphic to a spherical space form.

In particular, one has

Cor 1 $\forall n$ $\forall$ $\lambda_0, \lambda_1, d > 0$

$\exists$ $\varepsilon = \varepsilon(n, \lambda_0, \lambda_1, d) > 0$ s.t.

every closed Riemannian n-mfd M w/

$\int_M \mathrm{scal}\, d\mathrm{vol}_M \geq n(n-1)$, $\mathrm{inj} \geq \pi - \varepsilon$,

$\mathrm{diam} \leq d$, $\|\mathrm{Ric}\| \leq \lambda_0$, $\|\nabla \mathrm{Ric}\| \leq \lambda_1$

is diffeomorphic to S^n.

Another sample application of the theorem consists in a stability or <u>isolation result</u> for the standard sphere among all Einstein mfds with positive Einstein constant:

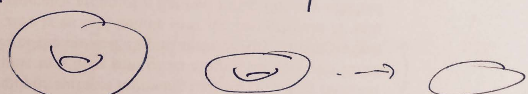
<u>Cor 2</u> $\forall n \quad \exists \quad \varepsilon = \varepsilon(n) > 0$

s.t. every closed simply connected n-dim. Einstein mfd M with Einstein constant $n-1$ and $\mathrm{conj} \geq \pi - \varepsilon$ is diffeomorphic to S^n.

Convergence theory for mfds under Ricci curvature bounds

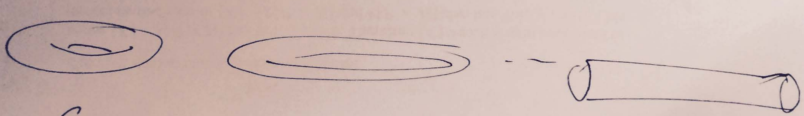
- Need, though maybe only technical, ∇ Ric to be bounded ($\to C^{2,\alpha}$)

- Need $\mathrm{inj} \geq i_0$ to prevent collapsing



(flat tori to lowerdimensional tori)

- Need upper bound on Volume to prevent noncompact limits



(flat tori to infinite cylinder)