Nodal and singular sets for solutions to some elliptic equations

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Outline

Nodal and singular sets

- The nodal set of a function u is defined by

$$N(u) = \{x : u(x) = 0\}.$$

- The singular set of a harmonic function u is defined by

$$S(u) = \{x : u(x) = 0, Du(x) = 0\}.$$

Nodal and singular sets

- If $\triangle u = 0$ in $B_1 \in R^n$, $u \neq constant$. Then

$$H^{n-1}(N(u) \bigcap B_r) \leq CN < \infty,$$

$$H^{n-2}(S(u) \bigcap B_r) \leq C(N) < \infty,$$

for $0 < r < 1$, where $N = \frac{\int_{B_1} |\nabla u|^2}{\int_{\partial B_1} u^2}$.

Nodal and singular sets

- **Petrovski-Oleinik** Let $f(x), x \in R^n$, be a polynomial of degree N. Suppose $dim_H f^{-1}\{0\} = k$. Then

$$H^k(f^{-1}\{0\} \bigcap B_R) \leq c(n)N^{n-k}R^k.$$

Nodal and singular sets

- To control growth of solutions $\rightleftarrows$ To control the geometry and topology of their level sets.
- Measure estimates.
- Geometric and topological structures.

Previous works

- In 1979, Almgren first gave the definition of frequency for harmonic functions.
- In 1986, Garofalo and Lin established the monotonicity formula for frequencies and the doubling conditions for solutions of a class of uniformly elliptic linear PDEs.
- Main contributions:
  S.Y.Cheng, H.Donnelly, C.Fefferman,Q.Han, R.Hardt, F-H.Lin, L.Simon,$\cdots$, have studied the frequency functions, growth, nodal sets and singular sets of solutions to elliptic/parabolic equations in R^n and Riemannian manifolds.

Bi-harmonic and H-Harmonic Functions

- **Bi-harmonic functions:** $-\Delta^2 u = 0$.
- **H-harmonic functions:** Let $(z,t) = (x,y,t) \in R^{2n+1}, x, y \in R^n$,

$$-\Delta_H u = \Delta_z u + 4|z|^2 \frac{\partial^2 u}{\partial^2 t} + 4\frac{\partial}{\partial t}(Pu) = 0,$$

where $\Delta_z = \sum_{i=1}^n (\frac{\partial^2}{\partial^2 x_i} + \frac{\partial^2}{\partial^2 y_i})$ and $Pu = \sum_{i=1}^n (y_i \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial y_i})u$.

- **Grushin-harmonic functions:** $Pu = 0$.

H-Harmonic Functions

- In another form

$$-\Delta_H u = div(A(z)\nabla u),$$

where

$$A(z) = \begin{pmatrix} I_n & 0_n & (2y)^T \\ 0_n & I_n & (-2x)^T \\ 2y & -2x & 4|z|^2 \end{pmatrix}.$$

H-Harmonic Functions

- $\Delta_H u$ is degenerate.
- The operators Δ_H are hypoelliptic from Hormander's hypoellipticity theorem.
- The classical sub-elliptic theory due to G.B. Folland, L.P. Rothschild, E.M. Stein,...

Motivations

- The operators Δ_H is the sub-Laplacian in the Heisenberg group.
- The Heisenberg group is the simplest model of sub-Riemannian manifolds which are suitable settings of geometric control theory, mathematical physics, CR manifolds and image processing.

- The Heisenberg group: $H^n = (R^{2n+1}, \circ)$, where $\circ$ is the group law given by

$$(x,y,t)\circ(x',y',t') = (x+x', y+y', t+t'+2(<x',y> - <y',x>))$$

where $x, x', y, y' \in R^n$ and $t, t' \in R$.

- H^n is a Lie group with Lie algebra h_n generated by the left-invariant horizontal frame

$$X = \{X_1, \cdots, X_n, X_{n+1}, \cdots, X_{2n}\}$$

given by

$$X_i = \frac{\partial}{\partial x_i} + 2y_i\frac{\partial}{\partial t}, \;\; X_{n+i} = \frac{\partial}{\partial y_i} - 2x_i\frac{\partial}{\partial t}\; (i = 1, \cdots, n).$$

- dilation: $\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t), \quad \lambda > 0.$
- The Hausdorff dimension of H^n is $Q = 2n + 2$, while the Topology dimension of H^n is $N = 2n + 1$.
- The gauge norm:

$$\rho(z, t) = [(\sum_{i=1}^{n}(x_i^2 + y_i^2))^2 + t^2]^{\frac{1}{4}} \equiv (|z|^4 + t^2)^{\frac{1}{4}},$$

- The horizontal gradient of a function u

$$\nabla_H u = Xu = (X_1 u, \cdots, X_n u, X_{n+1} u, \cdots, X_{2n} u).$$

- $$-\Delta_H u = -\sum_{i=1}^{2n} X_i^2 u = 0,$$

  which is the Euler equation of (the sub-elliptic) Dirichlet energy functional

$$\frac{1}{2}\int |Xu|^2.$$

Definition and Properties of Frequency

Definition

Let u be a bi-harmonic functions in B_1 and let $v = \triangle u$. Then we define

$$N(r) = r\frac{\int_{B_r} |\nabla u|^2 + |\nabla v|^2 + uv}{\int_{\partial B_r} u^2 + v^2},$$

$$M(r) = r\frac{\int_{B_r} |\nabla v|^2}{\int_{\partial B_r} v^2}.$$

Definition and Properties of Frequency

- We denote by

$$D_1(r)=\int_{B_r}|\nabla u|^2,\quad D_2(r)=\int_{B_r}|\nabla v|^2,\quad D_3(r)=\int_{B_r}uv,$$

$$H_1(r)=\int_{\partial B_r}u^2,\quad H_2(r)=\int_{\partial B_r}v^2,$$

- The functions $N(r), M(r)$ are the frequencies of u and v at the origin with radius r. Similarly, we can define the frequencies of u and v at any point p with radius r, which are denoted by $N(p,r), M(p,r)$.
- Since v is harmonic, $M(r)$ has a lot of interesting properties.

Definition and Properties of Frequency

Lemma

Let v be a harmonic function, then
(1) if v is a homogeneous harmonic polynomial of degree k then

$$M(r) \equiv k;$$

(2)(Monotonicity Formula) $M(r)$ is nondecreasing of $r \in (0,1)$;
(3) the limit of $M(r)$ as $r \longrightarrow 0^+$ exists and is equal to the vanishing order of v at the origin.

Definition and Properties of Frequency

Lemma

(4)(Doubling Conditions) for any $R \in (0, 1/2)$ *and* $\eta \in (1, 2]$

$$\fint_{\partial B_{\eta R}} v^2 \le \eta^{2M(1)} \fint_{\partial B_R} v^2,$$

$$\fint_{B_{\eta R}} v^2 \le \eta^{-1}\eta^{2M(1)} \fint_{B_R} v^2;$$

(5) for any $p \in B_R$ *with* $R < 1$*, we have*

$$M(p, \frac{1}{2}(1 - R)) \le C_1 M(1) + C_2,$$

where C_1 *and* C_2 *are positive constants depending only on* n *and* R.

Definition and Properties of Frequency

Now let us focus on the frequency of u. We can obtain the following properties.

Lemma

(1) If the vanishing order of u at the origin is $k \geq 2$, then

$$\lim_{r \longrightarrow 0} N(r) \geq k - 2.$$

(2)

$$N(r) \geq -C$$

in B_1, where C is a positive constant depending only on n.

Definition and Properties of Frequency

Lemma

(3)(Monotonicity Formula) There exists a constant C_0 such that if $N(r) \geq C_0$, then

$$\frac{N'(r)}{N(r)} \geq -C_3 r,$$

where C_0 and C_3 are two positive constants depending only on n.
(4) For any $p \in B_R$ with $R < 1/2$, we have

$$N(p, \frac{1}{2}(1-R)) \leq C_4 N(1) + C_5,$$

where C_4 and C_5 are constants depending only on n and R.

Doubling Conditions

From the monotonicity formula of $N(r)$, we can directly get the following doubling conditions.

Lemma

$$\fint_{\partial B_{4r}} u^2 + v^2 \le 2^{C_6 N(1)+C_7} \fint_{\partial B_r} u^2 + v^2,$$

and

$$\fint_{B_{4r}} u^2 + v^2 \le 2^{C_8 N(1)+C_9} \fint_{B_r} u^2 + v^2,$$

where $r < 1/4$ and C_6, C_7, C_8 and C_9 are positive constants depending only on n.

Doubling Conditions

The above doubling conditions are for the integration

$$\fint_{\partial B_r} u^2 + v^2 \quad and \quad \fint_{B_r} u^2 + v^2.$$

But what we need is the doubling conditions for

$$\fint_{\partial B_r} u^2 \quad and \quad \fint_{B_r} u^2.$$

In order to do this, we need the following lemma.

Doubling Conditions

Lemma

For any $r < \frac{1}{4}$, we have

$$r^4 \fint_{B_r} v^2 \leq C_{10} 2^{C_{11} M(1)} \fint_{B_{2r}} u^2,$$

where C_{10} and C_{11} are positive constants depending only on n.

Doubling Conditions

Sketch of the Proof. For any $\psi \in C_0^\infty(B_1)$,

$$\int_{B_1} \triangle u \triangle \psi = 0.$$

Choose $\psi = u\phi^2$, where ϕ satisfies
(1)$\phi \equiv 1$ in B_r and $\phi \equiv 0$ outside B_{2r};
(2)$|\nabla\phi| \leq \frac{C}{r}$ and $|\triangle\phi| \leq \frac{C}{r^2}$.
Then by putting this ψ into the above equation, using the Hölder inequality, integration by parts, Caccippoli inequality of harmonic functions, and doubling conditions of harmonic functions, one can obtain the desired result.

□

Doubling Conditions

From this lemma and the above doubling conditions, we can obtain the doubling conditions about $\int_{\partial B_r} u^2$ and $\int_{B_r} u^2$.

Lemma

For any $r < \frac{1}{4}$,

$$r^4 \int_{B_{2r}} u^2 \leq 2^{C_{12}(N(1)+M(1))+C_{13}} \int_{B_r} u^2,$$

where C_{12} and C_{13} are positive constants depending only on n.

Measure Estimates of Nodal Sets

The measure estimates of bi-harmonic functions are as follows.

Theorem

(L. Tian-Y) *Let u be a bi-harmonic function in B_1 and let $v = \triangle u$. Then we have*

$$\mathcal{H}^{n-1}\{x \in B_{1/16} | u(x) = 0\} \le C_{15}(N(1) + M(1)) + C_{16},$$

where C_{15} and C_{16} are positive constants depending only on n.

Measure Estimates of Nodal Sets

Lemma

For any $r \le \frac{1}{2}$,

$$\sup_{B_r} |u| \le C_{17}(\|u\|_{L^2(B_{2r})} + r^2\|v\|_{L^2(B_{2r})}),$$

where C_{17} is a positive constant depending only on n.

Measure Estimates of Nodal Sets

Sketch of the Proof of Theorem

- **Step 1.** From the doubling conditions to show

$$\fint_{B_{1/16}(p)} u^2 \geq 4^{-C_{19}(N(1)+M(1))-C_{20}}$$

for any $p \in \partial B_{1/4}$, where C_{19} and C_{20} are positive constants depending only on n. So there are points $x_p \in B_{1/16}(p)$ such that

$$|u(x_p)| \geq 2^{-C_{19}(N(1)+M(1))-C_{20}}.$$

Measure Estimates of Nodal Sets

- **Step 2.** We choose p_j to be the points on $\partial B_{1/4} \bigcap j - axis$, $j = 1, 2, \cdots, n$. Then for any $w \in \mathbb{S}^{n-1}$ and $j = 1, 2, \cdots, n$, define

$$f_j(w; t) = u(x_{p_j} + tw) \quad for \quad t \in (-\frac{5}{8}, \frac{5}{8}).$$

Then $f_j(w; t)$ is an analytic function of t. Thus we can extend it to be an analytic function $f_j(w; z)$ of z. Then we obtain

$$\mathcal{H}^0 \left\{ |t| < \frac{1}{2} | u(x_{p_j} + tw) = 0 \right\} \leq C_{23}(N(1) + M(1)) + C_{24}.$$

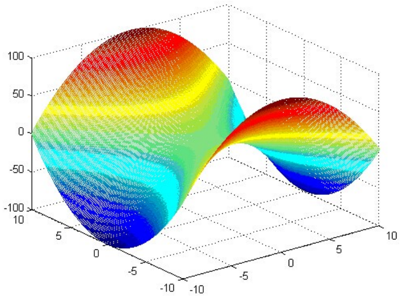
Measure Estimates of Nodal Sets

- **Step 3.** Finally from the integral geometric formula, we have

$$\begin{aligned}\mathcal{H}^{n-1}\{x \in B_{1/16} | u(x) = 0\} &\leq C_{25} \sum_{j=1}^{n} \int_{\mathbb{S}^{n-1}} N_j(w) dw \\ &\leq C_{15}(N(1) + M(1)) + C_{16}.\end{aligned}$$

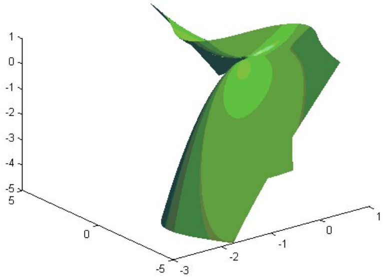
Nodal sets of H-harmonic polynomials

- The nodal set of H-harmonic polynomial $x^2 - y^2 + t$



Nodal sets of H-harmonic polynomials

- The nodal set of H-harmonic polynomial $x^3 + xy^2 + 2ty$.



Nodal domains of spherical harmonic polynomials in R^3

- Courant's Nodal Line Theorem: an upper bound for numbers of nodal domains for spherical harmonics.
- H. Lewy's Theorem: minimal number(lower bound) of nodal domains for spherical harmonics, i.e., 2 resp. 3 domains for odd degree k resp. even degree k.

Nodal lines of $\mathcal{L}$-harmonic polynomials in R^3

The following homogenous polynomials of degree k are Grushin-harmonic

$$u_{k,l} =: \rho^k \sin^{l/2}\phi \mathbb{C}^{\frac{l+1}{2}}_{\frac{k-l}{2}}(\cos\phi)e^{il\theta}, l = 0, 1, \cdots, k,$$

- There are $\frac{k+l}{2}$ nodal curves of these spherical Grushin-harmonics $u_{k,l}$.

Nodal domains of spherical Grushin-harmonics

Theorem

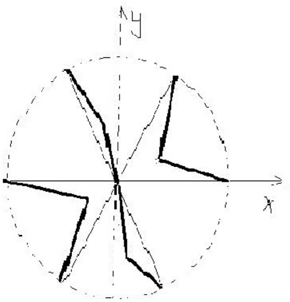
(Liu-Tian-Y)

1)For $k \neq 4m$, $m \in \mathbb{N}$, there exists a spherical Grushin harmonic function of degree k, such that the nodal lines of this function divide the gauge sphere S^2 into two domains.

2)For $k = 4m$, $m \in \mathbb{N}$, there are no spherical Grushin harmonic functions F of degree k such that the nodal curve of F divides the gauge sphere S^2 into two parts.

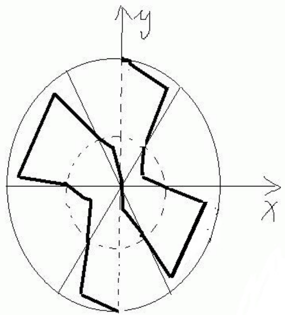
Nodal domains of G-harmonic polynomials

- The nodal domains of G-harmonic polynomials of degree three (two parts)



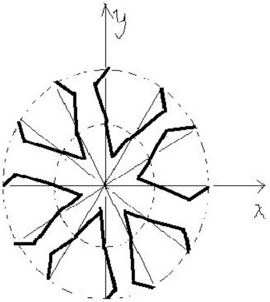
Nodal domains of G-harmonic polynomials

- The nodal domains of G-harmonic polynomials of degree five (two parts)



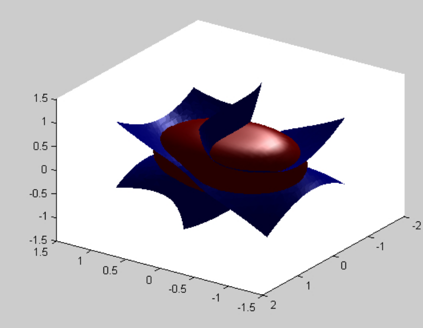
Nodal domains of G-harmonic polynomials

- The nodal domains of G-harmonic polynomials of degree six (two parts)



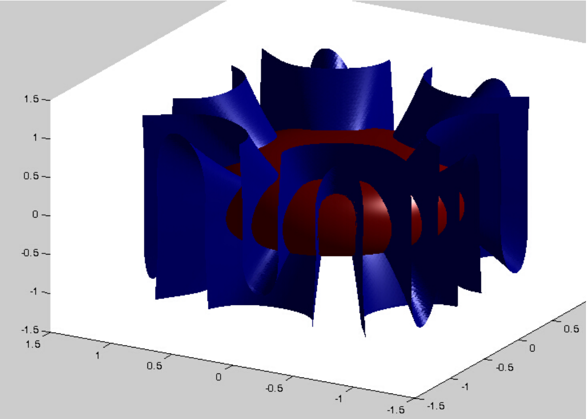
Nodal domains of G-harmonic polynomials

- The nodal domains of a H-harmonic polynomial of degree five (two parts)



Nodal sets of G-harmonic polynomials

- The nodal domain of of G-harmonic polynomial of degree 12 (three parts)



Measure Estimates

Theorem (Tian-Y)

Suppose that u is a nontrivial $H-$harmonic function in $B_d(0,1) \subseteq H^n$, and $Pu = 0$. Then there exist positive constants $\widetilde{r} < 1$, r_0 and $\widetilde{r} < r_0$ depending only on Q such that

$$\mathcal{H}^{2n}\{p \in B_d(0,\widetilde{r}) : \ u(p) = 0\} \leq C(N(0,r_0) + 1).$$

Sketch of Proof

Step1. We claim that

$$N(p, r) \leq CN(0, r_0) + C$$

for $r < cr_0$, where C and c are positive constants depending on Q. This claim comes from Lemma of Changing Centers, the monotonicity formula of frequency and the doubling conditions.

Step 2. We first assume that

$$\fint_{B_d(0,r_0)} u^2 \psi dzdt = 1.$$

Under this assumption, The doubling condition implies that one can find p_j on the axis, and $p_j \in \partial B_d(0, \frac{r_0}{4})$, $j = 1, 2, \cdots, 2n+1$,

$$\fint_{B_d(p_j, \frac{r_0}{16})} u^2 \psi dzdt \geq 4^{-CN(0,r_0)-C},$$

Finally one can show that there exists $\widetilde{p_j} \in B_d(p_j, \frac{r_0}{16})$ such that

$$|u(\widetilde{p_j})| \geq 2^{-CN(0,r_0)-C}.$$

Step 3. Define $f_j(\omega;\xi) = u(\widetilde{p_j} + \xi\omega)$ for ξ belongs to suitable interval and ω be any unit vector of $\mathbb{R}^{2n+1}$. Then f_j are all analytic with respect to ξ. Then we do the complexification of f_j. By using a theorem of H.Donnelly-C.Fefferman, we can have

$$\mathcal{H}^0\left\{|\xi| < \frac{5r_0}{8} : \quad u(\widetilde{p_j} + \xi\omega) = 0\right\} \leq CN(0, r_0) + C.$$

Step 4. From the integral geometric formula, the desired result can be derived.

Definition of Horizontal Singular Sets

Definition

Let u be a smooth function from H^n to $\mathbb{R}$. The horizontal singular set of u is defined as

$$\mathcal{S}(u) = \left\{ p \in H^n : u(p) = 0, \sum_{i=1}^{2n} |X_i u|^2(p) = 0 \right\}.$$

In the Heisenberg group, Franchi, Serapioni and Serra Cassano established the implicit function theorem.

We also denote

$$\mathcal{S}_k(u) = \left\{ p \in H^n : X^\alpha u(p) = 0, \forall \alpha \in \bigcup_{m=0}^{k-1} \mathcal{I}_m, \exists \alpha_0 \in \mathcal{I}_k, X^{\alpha_0} u(p) \neq 0 \right\},$$

where

$$X^\alpha = X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_{2n}^{\alpha_{2n}} X_{2n+1}^{\alpha_{2n+1}},$$

$$\mathcal{I}_m = \left\{ \alpha = (\alpha_1, \cdots, \alpha_{2n}, \alpha_{2n+1}) : \quad \sum_{i=1}^{2n} \alpha_i + 2\alpha_{2n+1} = m \right\},$$

and call it the $k-$horizontal singular set of u.

H–homogeneous polynomial and Horizontal Singular Set

Lemma

Let P be a homogeneous polynomial of degree k. Then, either (i)$\mathcal{S}_k(P)$ is a linear subspace of $\mathbb{R}^{2n+1}$, and all points on t-axis are in $\mathcal{S}_k(P)$.

or

(ii)$\mathcal{S}_k(P)$ is a linear subspace of $\mathbb{R}^{2n+1}$, and $t-$axis is orthogonal to $\mathcal{S}_k(P)$. Moreover, in this case, the dimension of $\mathcal{S}_k(P)$ is at most n.

Sketch of Proof

We first prove the following five properties of $\mathcal{S}_k(P)$:

(1) $0 \in \mathcal{S}_k(P)$.

(2) $(z,t) \in \mathcal{S}_k(P) \Rightarrow \delta_\lambda((z,t)) \in \mathcal{S}_k(P)$, $\forall \lambda > 0$.

(3) $(z_1,t_1),(z_2,t_2) \in \mathcal{S}_k(P) \Rightarrow (z_1,t_1)\circ(z_2,t_2) \in \mathcal{S}_k(P)$.

(4)$(z,t) \in \mathcal{S}_k(P) \Rightarrow (-z,t) \in \mathcal{S}_k(P)$.

(5) If $(z,t) \in \mathcal{S}_k(P)$ for some $t > 0 (t < 0)$, then all points $(0,t)$ satisfying $t > 0 (t < 0)$ are in $\mathcal{S}_k(P)$. Moreover, the polynomial P is independent of t in this case.

Then by using this five properties we can get the desired result.

Geometric Structure

Theorem (Tian-Y)

Let u be a nontrivial H-harmonic function in $B_d(0,1)$. *Then the horizontal singular set in* $B_d(0,\frac{1}{2})$ *is a countable union of* C^1 *sub-manifolds in* $\mathbb{R}^{2n+1}$ *with dimension at most* $2n-1$. *Thus the horizontal singular set of u is at most* $(2n-1)$*-countably rectifiable.*

Sketch of Proof

Step 1. We first write $\mathcal{S}(u)$ as

$$\mathcal{S}(u) = \bigcup_{k\geq 2} \mathcal{S}_k(u).$$

Because u is a non-trivial $\mathbb{H}$-harmonic function on $\mathbb{H}^n$ and has the strong unique continuity property, that is a finite union.

Step 2. Do the Taylor extension of u at point z for $z \in \mathcal{S}_k(u)$. Then

$$u(z \circ p) = P_z(p) + \mathcal{O}(d^{k+1}(z^{-1} \circ p)).$$

Let

$$S_k(u) = \bigcup_{j=0}^{2n-1} S_k^j, \mathcal{S}_k^j(u) = \overline{\mathcal{S}}_k^j(u) \cup \widetilde{\mathcal{S}}_k^j(u),$$

where

$$\overline{\mathcal{S}}_k^j(u) = \{z \in \mathcal{S}_k(u) : dim\mathcal{S}_k(P_z) = j, P_z \text{ is independent of } t\},$$

$$\widetilde{\mathcal{S}}_k^j(u) = \{z \in \mathcal{S}_k(u) : dim\mathcal{S}_k(P_z) = j, P_z \text{ depends on } t\},$$

Step 3. show that $\overline{\mathcal{S}}^j_k(u)$ and $\widetilde{\mathcal{S}}^j_k(u)$ both are countable union of j–dimensional C^1–manifolds respectively. That is the result we need.

A Measure Estimate Result for Horizontal Singular Sets

Corollary

Suppose that u is an H-harmonic function in $B_d(0,1) \subseteq H^1$, then the horizontal singular set S of u in $B_d(0,1)$ has the following measure estimate:

$$H^2(S) \leq C < \infty,$$

where C is an absolutely positive constant.

Thank you!