Alexandrov-Fenchel type inequalities in the hyperbolic space

(Joint work with Guofang Wang and Jie Wu)

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Alexandrov-Fenchel inequality in Euclidean space

- $\Omega \subset \mathbb{R}^n$ convex domain.
- $\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_{n-1})$ set of the principal curvatures of $\Sigma := \partial\Omega$
- $\sigma_k : \mathbb{R}^{n-1} \to \mathbb{R}$ the k-th elementary symmetric function.
- Alexandrov-Fenchel inequality : for $0 \leq j < k \leq n-1$

$$\int_\Sigma \sigma_k \geq C_{n-1}^k \omega_{n-1} \left(\frac{1}{C_{n-1}^j} \frac{1}{\omega_{n-1}} \int_\Sigma \sigma_j \right)^{\frac{n-1-k}{n-1-j}}, \tag{1}$$

where ω_{n-1} is the area of the standard sphere S^{n-1}.

Hyperbolic Alexandrov-Fenchel inequality

Theoreme 1 (G-Wang-Wu)JDG

Let Σ be a horospherical convex hypersurface ($\kappa_i \geq 1$) in the hyperbolic space $\mathbb{H}^n$. We have for $2k \leq n-1$

$$\int_\Sigma \sigma_{2k} \geq C_{n-1}^{2k}\omega_{n-1}\left\{\left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{1}{k}} + \left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{1}{k}\frac{n-1-2k}{n-1}}\right\}^k.$$

Equality holds if and only if Σ is a geodesic sphere.

Weighted hyperbolic Alexandrov-Fenchel inequality

Theoreme 2 (G-Wang-Wu)

Let Σ be a horospherical convex hypersurface in the hyperbolic space $\mathbb{H}^n$. We have

$$\int_\Sigma V\sigma_{2k-1}d\mu \geq C_{n-1}^{2k-1}\omega_{n-1}\left(\left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n}{k(n-1)}}+\left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n-2k}{k(n-1)}}\right)^k.$$

Equality holds if and only if Σ is a geodesic sphere centered at x_0 in $\mathbb{H}^n$.

Asymptotically flat Riemannian manifolds

ADM mass was introduced by Arnowitt-Deser-Misner(61)

Definition 1 : A complete manifold $(\mathcal{M}^n, g)$ is an asymptotically flat (AF) of order τ if there is a compact set K s.t. $\mathcal{M} \setminus K$ is diffeomorphic to $\mathbb{R}^n \setminus B_R(0)$ for some $R > 0$ and

$$g_{ij} = \delta_{ij} + \sigma_{ij}$$

with

$$|\sigma_{ij}| + |x||\partial\sigma_{ij}| + |x|^2|\partial^2\sigma_{ij}| = O(|x|^{-\tau}),$$

Definition 2 : The ADM mass

$$m_{ADM} := \frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{S_r} (\partial_i g_{ij} - \partial_j g_{ii})\nu_j dS, \tag{2}$$

where $\omega_{n-1} = |\mathbb{S}^{n-1}|$, S_r Euclidean sphere, ν unit normal.

Classical results

- Bartnik (86)
  $\tau > (n-2)/2$ and $R_g \in L^1 \Rightarrow m_{ADM}$ is geometric invariant.

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- **Positive mass conjecture(PMT)** : *Any asymptotically flat Riemannian manifold $\mathcal{M}^n$ with a suitable decay order and with nonnegative scalar curvature has nonnegative ADM mass ?*
- **Answer**
  Schoen-Yau $n \leq 7$ (79) LCF $\forall n$ (88)
  Witten Spin manifold $\forall n$ (81)
  Lam Asymptotically flat graph in $\mathbb{R}^{n+1}$ $\forall n$ (10).

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- Penrose inequality : $m_{ADM} \geq \frac{1}{2}\left(\frac{|\Sigma|}{w_{n-1}}\right)^{(n-2)/(n-1)}$
  Huisken-Ilmanen and Bray $n = 3$ and Bray-Lee $n \leq 7$

Asymptotically hyperbolic manifolds

Hyperbolic mass is introduced by Wang and Cruściel-Herzlich

Hyperbolic metric $(\mathbb{H}^n, b = dr^2 + \sinh^2 r d\Theta^2)$, where $d\Theta^2$ standard metric on the sphere

Definition 3 : A Riemannian manifold $(\mathcal{M}^n, g)$ is *asymptotically hyperbolic of decay order* τ if $\exists$ a compact subset K and a diffeomorphism at infinity $\Phi : \mathcal{M}^n \setminus K \to \mathbb{H}^n \setminus B$ (B is a closed ball), s.t. $(\Phi^{-1})^* g$ and b are equivalent on $\mathbb{H}^n \setminus B$ and

$$\|(\Phi^{-1})^* g - b\|_b + \|\bar{\nabla}\left((\Phi^{-1})^* g\right)\|_b + \|\bar{\nabla}^2\left((\Phi^{-1})^* g\right)\|_b = O(e^{-\tau r}),$$

where $\bar{\nabla}$ is covariant derivative w.r.t. hyperbolic metric b.

- **Definition 4 :**

$$\mathbb{N}_b := \{V \in C^\infty(\mathbb{H}^n) | \mathrm{Hess}^b V = Vb\}.$$

- **Remark :** $\mathbb{N}_b$ is an $(n+1)$-dimensional vector space spanned

$$V_{(0)} = \cosh r,\ V_{(1)} = x^1 \sinh r,\ \cdots,\ V_{(n)} = x^n \sinh r,$$

where $x^1, x^2, \cdots, x^n$ are the coordinate functions on $\mathbb{S}^{n-1}$. $\mathbb{N}_b$ is equiped with a Lorentz inner product η with signature $(+,-,\cdots,-)$ s.t.

$$\eta(V_{(0)}, V_{(0)}) = 1, \quad \text{and} \quad \eta(V_{(i)}, V_{(i)}) = -1 \quad \text{for} \quad i = 1, \cdots, n.$$

- **Definition 5 :**

$$\mathbb{N}_b^1 = \{V = \cosh \mathrm{dist}_b(x_0, \cdot), x_0 \in \mathbb{H}^n\}$$

- **Condition :**

$$\int_{\mathcal{M}} \cosh r \, |R + n(n-1)| d\mu < \infty, \quad \tau > n/2$$

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- **Definition 6 :** Mass functional of $(\mathcal{M}^n, g)$ with respect to Φ

$$H_\Phi(V) = \lim_{r \to \infty} \int_{S_r} (V(\mathrm{div}^b e - d\mathrm{tr}^b e) + (\mathrm{tr}^b e) dV - e(\nabla^b V, \cdot)) \nu d\mu,$$

where $e := \Phi_* g - b$, S_r is a geodesic sphere with radius r, ν is the outer normal.

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- If A is an isometry of the hyperbolic metric b

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- **Definition 7 :** $m_{ADM}^{\mathbb{H}} := \dfrac{1}{(n-1)\,\omega_{n-1}} \inf_{V\in\mathbb{N}_b^1} H^\Phi(V)$

Gauss-Bonnet curvature

- **Definition 8 :** *k-th Gauss-Bonnet curvature* :
$L_k := \frac{1}{2^k}\delta^{i_1 i_2 \cdots i_{2k-1} i_{2k}}_{j_1 j_2 \cdots j_{2k-1} j_{2k}} R_{i_1 i_2}{}^{j_1 j_2} \cdots R_{i_{2k-1} i_{2k}}{}^{j_{2k-1} j_{2k}}$,
where

$$\delta^{j_1 j_2 \cdots j_r}_{i_1 i_2 \cdots i_r} = \det \begin{pmatrix} \delta^{j_1}_{i_1} & \delta^{j_2}_{i_1} & \cdots & \delta^{j_r}_{i_1} \\ \delta^{j_1}_{i_2} & \delta^{j_2}_{i_2} & \cdots & \delta^{j_r}_{i_2} \\ \vdots & \vdots & \vdots & \vdots \\ \delta^{j_1}_{i_r} & \delta^{j_2}_{i_r} & \cdots & \delta^{j_r}_{i_r} \end{pmatrix}.$$

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where
$$\delta^{j_1 j_2\cdots j_r}_{i_1 i_2\cdots i_r} = \det\begin{pmatrix} \delta^{j_1}_{i_1} & \delta^{j_2}_{i_1} & \cdots & \delta^{j_r}_{i_1} \\ \delta^{j_1}_{i_2} & \delta^{j_2}_{i_2} & \cdots & \delta^{j_r}_{i_2} \\ \vdots & \vdots & \vdots & \vdots \\ \delta^{j_1}_{i_r} & \delta^{j_2}_{i_r} & \cdots & \delta^{j_r}_{i_r} \end{pmatrix}.$$
- **Decomposition of the Gauss-Bonnet curvature**
$$L_k = P^{stjl}_{(k)} R_{stjl},$$
where
$$2^k P^{stlj}_{(k)} =$$
$$\delta^{i_1 i_2\cdots i_{2k-3} i_{2k-2} st}_{j_1 j_2\cdots j_{2k-3} j_{2k-2} j_{2k-1} j_{2k}} R_{i_1 i_2}{}^{j_1 j_2}\cdots R_{i_{2k-3} i_{2k-2}}{}^{j_{2k-3} j_{2k-2}} g^{j_{2k-1} l} g^{j_{2k} j}$$

- $k=1$,

$$L_1 = R = R_{stjl} P_{(1)}^{stjl},$$

and

$$P_{(1)}^{stjl} = \frac{1}{2}(g^{sj}g^{tl} - g^{sl}g^{tj}).$$

- $k=2$,

$$L_2 = \frac{1}{4}\delta^{i_1i_2i_3i_4}_{j_1j_2j_3j_4} R^{j_1j_2}{}_{i_1i_2} R^{j_3j_4}{}_{i_3i_4} = \|Rm\|^2 - 4\|Ric\|^2 + R^2$$

and

$$P_{(2)}^{stjl} = R^{stjl} + R^{tj}g^{sl} - R^{tl}g^{sj} - R^{sj}g^{tl} + R^{sl}g^{tj} + \frac{1}{2}R(g^{sj}g^{tl} - g^{sl}g^{tj}$$

Properties of tensor P

- $P_{(k)}$ shares the same symmetry and antisymmetry with the Riemann curvature tensor that

$$P_{(k)}^{stjl} = -P_{(k)}^{tsjl} = -P_{(k)}^{stlj} = P_{(k)}^{jlst}.$$

- $P_{(k)}$ satisfies the first Bianchi identity, i.e., $P^{stjl} + P^{tjsl} + P^{jstl} = 0$.
- $P_{(k)}$ is divergence-free,

$$\nabla_s P_{(k)}^{stjl} = 0.$$

Gauss-Bonnet-Chern Mass on AH manifolds

- A new four-tensor $\widetilde{Riem}$

$$\widetilde{\mathrm{Riem}}_{ijsl}(g) = \tilde{R}_{ijsl}(g) := R_{ijsl}(g) + g_{is}g_{jl} - g_{il}g_{js}.$$

-

$$\tilde{L}_k := \frac{1}{2^k}\delta^{i_1 i_2 \cdots i_{2k-1} i_{2k}}_{j_1 j_2 \cdots j_{2k-1} j_{2k}} \tilde{R}_{i_1 i_2}{}^{j_1 j_2} \cdots \tilde{R}_{i_{2k-1} i_{2k}}{}^{j_{2k-1} j_{2k}} = \tilde{R}_{stjl}\tilde{P}^{stjl}_{(k)},$$

-

$$2^k \tilde{P}^{stjl}_{(k)} := \delta^{i_1 i_2 \cdots i_{2k-3} i_{2k-2} st}_{j_1 j_2 \cdots j_{2k-3} j_{2k-2} j_{2k-1} j_{2k}} \tilde{R}_{i_1 i_2}{}^{j_1 j_2} \cdots \tilde{R}_{i_{2k-3} i_{2k-2}}{}^{j_{2k-3} j_{2k-2}} g^{j_{2k-1} j} g^{j_{2k} l}.$$

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- $$\tilde{L}_k := \frac{1}{2^k}\delta^{i_1 i_2 \cdots i_{2k-1} i_{2k}}_{j_1 j_2 \cdots j_{2k-1} j_{2k}} \tilde{R}_{i_1 i_2}{}^{j_1 j_2} \cdots \tilde{R}_{i_{2k-1} i_{2k}}{}^{j_{2k-1} j_{2k}} = \tilde{R}_{stjl} \tilde{P}^{stjl}_{(k)},$$

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- $\tilde{P}_{(k)}$ satisfies the same properties as $P_{(k)}$

Examples

- $\tilde{L}_1 = L_1 + n(n-1) = R + n(n-1)$
- $\tilde{L}_2 = L_2 + 2(n-2)(n-3)R + n(n-1)(n-2)(n-3)$
- $\tilde{P}^{stjl}_{(1)} = P^{stjl}_{(1)} = \frac{1}{2}(g^{sj}g^{tl} - g^{sl}g^{tj})$
- $\tilde{P}^{stjl}_{(2)} =$
  $\tilde{R}^{stjl} + \tilde{R}^{tj}g^{sl} - \tilde{R}^{sj}g^{tl} - \tilde{R}^{tl}g^{sj} + \tilde{R}^{sl}g^{tj} + \frac{1}{2}\tilde{R}(g^{sj}g^{tl} - g^{sl}g^{tj})$
  where $\tilde{R}^{sj} = g_{tl}\tilde{R}^{stjl}, \quad \tilde{R} = g_{sj}\tilde{R}^{sj}$.
- $\tilde{P}^{stjl}_{(2)} = P^{stjl}_{(2)} + (n-2)(n-3)P^{stjl}_{(1)}$.

- **Conditions :**
  $\tau > \frac{n}{k+1}$, $V \in \mathbb{N}_b$ and $V\tilde{L}_k$ is integrable in $(\mathcal{M}^n, g)$

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- **Definition 9 :** Gauss-Bonnet-Chern Mass functional

$$H_k^{\Phi}(V) = \lim_{r\to\infty} \int_{S_r} \left(\left(V\bar{\nabla}_l e_{js} - e_{js}\bar{\nabla}_l V \right) \tilde{P}_{(k)}^{ijsl} \right) \nu_i d\mu$$

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- **Definition 10 : Gauss-Bonnet-Chern mass**

$$m_k^{\mathbb{H}} := \frac{(n-2k)!}{2^{k-1}(n-1)!\,\omega_{n-1}} \inf_{V\in\mathbb{N}_b^1} H_k^\Phi(V). \tag{3}$$

provided H_k^Φ is timeline future directed on $\mathbb{N}_b^1$.

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  provided H_k^Φ is timeline future directed on $\mathbb{N}_b^1$.
- **Remark** $m_k^{\mathbb{H}}$ vanishes if $\tau > \frac{n}{k}$, and well-defined and non-trivial range for GBC mass $m_k^{\mathbb{H}}$ is $\tau \in (\frac{n}{k+1}, \frac{n}{k}]$. The decay order of the anti-de Sitter Schwarzschild type metric is $\frac{n}{k}$.

Anti-de Sitter Schwarzschild manifolds

- **Definition 11 :** anti-de Sitter Schwarzschild metric $(m > 0)$

$$g_{\mathrm{adS-Sch}} = \left(1 + \rho^2 - \frac{2m}{\rho^{\frac{n}{k}-2}}\right)^{-1} d\rho^2 + \rho^2 d\Theta^2;$$

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- ADS metric can be realized as a graph over the hyperbolic metric $\mathbb{H}^n$. ;
- $\tilde{L}_k(g) = 0$;
- $m_k^{\mathbb{H}} = m^k$.

Theoreme 3 (G-Wang-Wu)

Suppose that $(\mathcal{M}^n, g)$ is an asymptotically hyperbolic manifold of decay order $\tau > \frac{n}{k+1}$ and for $V \in \mathbb{N}_b$, $V\tilde{L}_k$ is integrable on $(\mathcal{M}^n, g)$, then the mass functional $H_k^{\Phi}(V)$ is well-defined and does not depend on the choice of the coordinates at infinity used in the definition in sense that $H_k^{\Phi_1}(V) = H_k^{\Phi_2}(V \circ A)$ with some isometry A of b.

Positive Mass Theorem

Theoreme 4 (G-Wang-Wu)

Let $(\mathcal{M}^n, g) = (\mathbb{H}^n, b + V^2 df \otimes df)$ be the graph of a smooth function $f : \mathbb{H}^n \to \mathbb{R}$ which satisfies $V\tilde{L}_k$ is integrable and $(\mathcal{M}^n, g)$ is asymptotically hyperbolic of decay order $\tau > \frac{n}{k+1}$. Then we have

$$m_k^{\mathbb{H}} = c(n,k) \int_{\mathcal{M}^n} \frac{1}{2} \frac{V\tilde{L}_k}{\sqrt{1 + V^2 |\bar{\nabla} f|^2}} dV_g.$$

where $V = V_{(0)} = \cosh r$. In particular, $\tilde{L}_k \geq 0$ implies $m_k^{\mathbb{H}} \geq 0$.

Penrose Inequality by Weighted hyperbolic Alexandrov-Fenchel inequality

Theoreme 5 (G-Wang-Wu)

Assume conditions given in Theorem 4 hold and $f : \mathbb{H}^n \setminus \Omega \to \mathbb{R}$ with $\partial\Omega = \Sigma$. If each connected component of Σ is horospherical convex, then

$$m_k^{\mathbb{H}} \geq \frac{1}{2^k}\left(\left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n}{k(n-1)}} + \left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n-2k}{k(n-1)}}\right)^k,$$

provided that

$$\tilde{L}_k \geq 0.$$

Equality $\Leftrightarrow$ anti-de Sitter Schwarzschild type metric.

Remark : $k = 1$ Lima-Girão,Dahl-Gicquaud-Sakovich(12).

Idea of proof of Theorem 1 : case $k = 2$

- set $p_k := \sigma_k / C^k_{n-1}$.
- **Energy functional**
  $$Q(\Sigma) := |\Sigma|^{-\frac{n-5}{n-1}} \int_\Sigma (p_4 - 2p_2 + 1) d\mu.$$
  for any convex hypersurface.
- **Inverse curvature flow**
  $$\frac{d}{dt} X = \frac{p_3}{p_4} \nu.$$
- **Variational inequality**
  $$\frac{d}{dt} \log Q(\Sigma_t) \leq$$
  $$(n-5) \int_{\Sigma_t} (p_5 - 2p_3 + p_1) \frac{p_3}{p_4} - (p_4 - 2p_2 + 1) d\mu.$$

Properties of flow :

- Flow preserves the horospherical convexity ;
- Q is non-increasing under flow ;
- Flow converges asymptotically to a conformal sphere at infinity (**Gerhardt**)
  H-convexity $\Leftrightarrow$ positive Schouten tensor at ∞
  $\Rightarrow \lim_{t\to\infty} Q(\Sigma_t) \geq w_{n-1}^{\frac{4}{n-1}}$

Monotonicity of Q

Refined Newton-MacLaurin inequality

Lemma

Let $n > 5$ and $\kappa_i \geq 1$ for all i. Then

$$\frac{p_3}{p_4}(p_5 - 2p_3 + p_1) \leq p_4 - 2p_2 + 1. \tag{4}$$

Equality holds if and only if

$$(i)\, \kappa_i = \kappa_j \,\forall\, i, j, \text{ or } (ii)\, \exists\, i \text{ with } \kappa_i > 1 \,\&\, \kappa_j = 1 \,\forall j \neq i. \tag{5}$$

Remark. Classical Newton-MacLaurin inequalities :

$$\frac{p_3 p_5}{p_4} \leq p_4, \quad -2\frac{p_3^2}{p_4} \leq -2p_2, \quad \frac{p_3 p_1}{p_4} \geq 1$$

Two inequalities

- **Claim 1.** $3(p_2p_4 - p_3^2) + (p_3p_1 - p_4) \leq 0$. Equality holds if and only if κ satisfies (5).
- **Claim 2.** $3(p_5p_3 - p_4^2) + (p_3p_1 - p_4) \leq 0$. Equality holds if and only if κ satisfies (5).

For $\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_n)$, define auxiliary function

$$F_n(x) := \sum_{i=0}^{n} C_n^i p_i x^{n-i} = \Pi_{i=1}^{n}(x + \kappa_i)$$

$$F_n'(x) = \Pi_{i=1}^{n-1}(x + \kappa_i').$$

Facts. $\forall 1 \leq i \leq n-1$, $p_k(\kappa) = p_k(\kappa')$.

Results in low dimensions

- **Claim 1 in case $n = 4$.**

$$\begin{aligned} & 3(p_2p_4 - p_3^2) + p_1p_3 - p_4 \\ = \; & \frac{1}{16}\sum_{cyc} \kappa_1\kappa_2(1-\kappa_1\kappa_2)(\kappa_3-\kappa_4)^2 \leq 0. \end{aligned}$$

- **Claim 2 in case $n = 5$.**

$$\begin{aligned} & 3(p_5p_3 - p_4^2) + (p_3p_1 - p_4) \\ = \; & \frac{1}{100}\sum_{cyc} \kappa_1\kappa_2(\kappa_3-\kappa_4)^2 \\ & + \frac{3}{100}\Big(-\sum_{cyc} (\kappa_1\kappa_2\kappa_3)^2(\kappa_4-\kappa_5)^2\Big) \\ = \; & \frac{1}{100}\sum_{cyc} \left[\kappa_1\kappa_2+\kappa_2\kappa_3+\kappa_1\kappa_3 - 3(\kappa_1\kappa_2\kappa_3)^2\right](\kappa_4-\kappa_5)^2 \\ \leq \; & 0. \end{aligned}$$

Idea of proof of Theorem 2

Step 1 Set $V = \cosh(r)$, $u = \langle \bar{\nabla} V, \nu \rangle > 0$ support function and $p_j = \frac{1}{C_{n-1}^j}\sigma_j$ normalized j-th mean curvature.

Two Minkowski integral formulas

- $$\int_\Sigma uVp_k d\mu \geq \int_\Sigma V^2 p_{k-1} d\mu,$$
  for any convex hypersurface. Equality holds iff Σ is a centered geodesic sphere.
- $$\int_\Sigma u^2 p_k d\mu \geq \int_\Sigma Vup_{k-1} d\mu,$$
  for any convex hypersurface. Equality holds iff Σ is a centered geodesic sphere.

Step 2 : A key lemma

Lemma

Let $1 \leq k < n-1$. Any horospherical convex hypersurface Σ in the hyperbolic space $\mathbb{H}^n$ satisfies

$$\int_\Sigma V p_{k+1} \geq \int_\Sigma \left(V p_{k-1} + \frac{p_{k+1}}{V} \right) d\mu.$$

- **Energy functional**
$$E := \int_\Sigma \left(V p_{k+1} - V p_{k-1} - \frac{p_{k+1}}{V} \right) d\mu.$$
- **Conformal flow :**
$$\frac{\partial X}{\partial t} = -V\nu$$

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- **Conformal flow :**
  $$\frac{\partial X}{\partial t} = -V\nu$$
- **Properties of flow :**
  - E is non-increasing under flow ;
  - limit of E under flow is 0 ;
  - Flow preserves the horospherical convexity

Step 3 :

- Induction argument ;
- Hyperbolic Alexandrov-Fenchel inequality without weight ;

Thank you for your attention !