Global existence of classical solutions of Goursat problem for quasilinear hyperbolic systems with small BV data

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Outline

Consider the following first order quasilinear hyperbolic system:

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0, \tag{1}$$

where $u = (u_1, \dots, u_n)^T$ is the unknown vector function of (t, x) and $A(u)$ is an $n \times n$ matrix with suitably smooth elements $a_{ij}(u)$ $(i, j = 1, \dots, n)$.

The system is assumed to be strictly hyperbolic, i.e., the Jacobian $A(u) = \nabla f(u)$ has n real distinct eigenvalues:

$$\lambda_1(u) < \lambda_2(u) < \dots < \lambda_n(u), \tag{2}$$

and each characteristic field is linearly degenerate in the sense of Lax:

$$\nabla \lambda_i(u) r_i(u) \equiv 0$$

for any given u on the domain under consideration.

We are interested in the Goursat problem for system (1) on an angular domain $D \stackrel{\triangle}{=} \{(t,x) \mid t \geq 0, x_1(t) \leq x \leq x_n(t)\}$, in which the solutions to system (1) are asked to satisfy the following characteristic boundary conditions:

$$\text{on } x = x_1(t): \quad u = \varphi(t) \tag{3}$$

and

$$\text{on } x = x_n(t): \quad u = \psi(t), \tag{4}$$

where $x = x_1(t)$ and $x = x_n(t)$ are the leftmost and the rightmost characteristics passing through the origin $(t,x) = (0,0)$, respectively, such that

$$\begin{cases} \frac{dx_1(t)}{dt} = \lambda_1(\varphi(t)), \\ x_1(0) = 0, \end{cases} \tag{5}$$

$$\begin{cases} \frac{dx_n(t)}{dt} = \lambda_n(\psi(t)), \\ x_n(0) = 0, \end{cases} \tag{6}$$

Moreover,

$$l_1(\varphi(t))\varphi'(t) \equiv 0 \tag{7}$$

and

$$l_n(\psi(t))\psi'(t) \equiv 0, \tag{8}$$

where $\varphi(t) = (\varphi_1(t), \ldots, \varphi_n(t))^T$ and $\psi(t) = (\psi_1(t), \ldots, \psi_n(t))^T$ are any given C^1 vector functions with bounded and possibly large C^1 norm, but of small bounded variation, such that

$$||\varphi(t)||_{C^1}, \ ||\psi(t)||_{C^1} \le M, \tag{9}$$

for some $M > 0$ bounded but possibly large.

Also, we assume that the boundary data $\varphi(t)$, $\psi(t)$ satisfy the conditions of C^1 compatibility at the origin $(0,0)$:

$$\varphi(0)=\psi(0) \tag{10}$$

and

$$\lambda_n(\varphi(0))\varphi'(0)-\lambda_1(\psi(0))\psi'(0)+A(\varphi(0))(\psi'(0)-\varphi'(0))=0. \tag{11}$$

Without loss of generality, we may assume that

$$\varphi(0)=\psi(0)=0.$$

In fact, by the following transformation

$$\widetilde{u}=u-\varphi(0),$$

we can always realize the above assumption.

Previous Works

- For the sufficiently small $\widetilde{t} > 0$, Li and Yu [1985] proved that the Goursat problem always admits a unique continuous, piecewise C^1 solution on the domain $\{(t,x) \mid 0 \le t \le \widetilde{t}, x_1(t) \le x \le x_n(t)\}$.

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- Global existence of the global classical solutions for the Goursat problem of reducible quasilinear hyperbolic system was obtained by Zhou [Chin. Ann. Math. Ser. A 1992].
- If system (1) is linearly degenerate or weakly linearly degenerate, Li, Zhou and Kong [CPDE 1994, Nonlinear Anal. TMA 1997] proved that Cauchy problem admits global classical solutions under the assumption that the C^1 norm of initial data is small and decaying.

However, it is well known that the BV space is a suitable framework for one-dimensional Cauchy problem for the hyperbolic systems of conservation laws (see Bressan [Oxford University Press, 2000]), the result in Bressan [Indiana Univ. Math. J., 1988] suggests that one may achieve global smoothness even if the C^1 norm of the initial data is large.

- Under the assumption that system (1) is strictly hyperbolic and weakly linearly degenerate, Duan and Xu [Nonlinear Anal. 2010] recently generalized Zhou's result on Cauchy problem [Y. Zhou, Chin. Ann. Math. Ser. B 2004] to the Goursat problem. Precisely speaking, they proved that there exists a small constant $\varepsilon > 0$ such that the Goursat problem (1) and (3)-(4) admits a unique global C^1 solution, provided that

$$\int_0^{+\infty} |\varphi'(t)| dt \le \varepsilon, \int_0^{+\infty} |\psi'(t)| dt \le \varepsilon$$

and

$$\int_0^{+\infty} |\varphi_i(t)| dt \le \frac{\varepsilon}{1+M}, \int_0^{+\infty} |\psi_i(t)| dt \le \frac{\varepsilon}{1+M},$$

where

$$M \triangleq \max\{\max_i \sup_{t\ge 0} |\varphi_i'(t)|, \max_i \sup_{t\ge 0} |\psi_i'(t)|\} < +\infty.$$

Under the assumption that system (1) is strictly hyperbolic and linearly degenerate, Dai and Kong [JDE 2007] proved that if the C^1 norm of the initial data is bounded but possibly large, and the BV norm of the initial data is small, then Cauchy problem admits a unique global C^1 solution too.

Problem: If the C^1 norm of the boundary data is bounded but possibly large, and the BV norm of the boundary data is sufficiently small, can we obtain the global existence and uniqueness of C^1 solutions of the Goursat problem for linearly degenerate quasilinear hyperbolic systems ?

Aim of this work

In this paper we exploit to some extent the ideas of Bressan [Indiana Univ. Math. J., 1988], we will develop the method of using continuous Glimm's functional to solve this problem globally and to generalize Dai and Kong's result to the Goursat problem instead of Cauchy problem.

The basic idea here is to combine the techniques employed by Li, Zhou and Kong [CPDE 1994, Nonlinear Anal. TMA 1997], especially the decomposition of waves, with the method of using continuous Glimm's functional. However, we must modify Glimm's functional in order to take care of the presence of the characteristic boundary. This makes our new analysis more complicated than those for the C^1 solutions of the Cauchy problem for linearly degenerate quasilinear hyperbolic systems in Bressan [Indiana Univ. Math. J., 1988], Zhou [Chin. Ann. Math. Ser. B, 2004], Dai and Kong [JDE 2007].

Theorem 1(Shao [Math. Meth. Appl. Sci. 2014])

Suppose that system (1) is strictly hyperbolic and linearly degenerate. Suppose furthermore that $A(u) \in C^2$ in a neighborhood of $u = 0$. Suppose finally that $\varphi(\cdot)$ and $\psi(\cdot)$ are all C^1 functions with respect to their arguments satisfying (5)-(9) and the conditions of C^1 compatibility (10)-(11) at the origin $(t, x) = (0, 0)$. Then for any constant $M > 0$, there exists a small constant $\varepsilon > 0$ such that the Goursat problem (1) and (3)-(4) admits a unique global C^1 solution $u = u(t, x)$ on the angular domain D, provided that

$$\int_0^{+\infty} |\varphi'(t)|dt \leq \varepsilon \ \ and \ \ \int_0^{+\infty} |\psi'(t)|dt \leq \varepsilon.$$

Remark

Suppose that (1) is a non-strictly hyperbolic system with characteristics with constant multiplicity, say, on the domain under consideration,

$$\lambda_1(u) \equiv \cdots \equiv \lambda_p(u) < \lambda_{p+1}(u) < \cdots < \lambda_n(u) \quad (1 \leq p \leq n).$$

Then the conclusion of Theorem 1 still holds.

Remark

An important application is the global existence and uniqueness of classical solutions of the characteristic boundary value problem for the equation of time-like extremal surfaces in Minkowski space $R^{1+(1+n)}$. The extremal surfaces play an important role in the theoretical apparatus of elementary particle physics. A free string is a one-dimensional physical object whose motion is represented by a time-like extremal surface in the Minkowski space. Recall Kong et al.'s work [J. Math. Phys. 2006] at first, By $(x_0, x_1, \cdots, x_{n+1})$ we denote a point in the $1+(1+n)$ dimensional Minkowski space endorsed with the metric

$$ds^2 = -dx_0^2 + dx_1^2 + \cdots + dx_{n+1}^2.$$

Let

$$x_0 = t,\ x_1 = x,\ x_2 = \phi_1(t, x),\ \cdots,\ x_{n+1} = \phi_n(t, x)$$

be a two dimensional surface. It follows that the area element of the surface is

$$dA = \sqrt{1 + |\phi_x|^2 - |\phi_t|^2 - |\phi_t|^2|\phi_x|^2 + (\phi_t \cdot \phi_x)^2}dxdt,$$

where $\phi = (\phi_1 \cdots, \phi_n)^T$, ϕ_t or ϕ_x denote partial differentiation with respect to t or x respectively and $\cdot$ denotes inner product in R^n. The surface is called extremal surface if ϕ is the critical point of the area functional

$$I[\phi] = \int\int \sqrt{1 + |\phi_x|^2 - |\phi_t|^2 - |\phi_t|^2|\phi_x|^2 + (\phi_t \cdot \phi_x)^2}dxdt.$$

The corresponding Euler-Lagrange equation is

$$\Big(\frac{\phi_t+|\phi_x|^2\phi_t-(\phi_t\cdot\phi_x)\phi_x}{\sqrt{1+|\phi_x|^2-|\phi_t|^2-|\phi_t|^2|\phi_x|^2+(\phi_t\cdot\phi_x)^2}}\Big)_t$$
$$-\Big(\frac{\phi_x-|\phi_t|^2\phi_x+(\phi_t\cdot\phi_x)\phi_t}{\sqrt{1+|\phi_x|^2-|\phi_t|^2-|\phi_t|^2|\phi_x|^2+(\phi_t\cdot\phi_x)^2}}\Big)_x=0. \qquad (12)$$

Eq. (12) is the equation of time-like extremal surfaces in the Minkowski space $R^{1+(1+n)}$. The extremal surfaces in the Minkowski space are C^2 surfaces with vanishing mean curvature. This is an interesting model in Lorentzian geometry. The Cauchy problem for the equation of time-like extremal surfaces was studied by Kong, Sun and Zhou [J. Math. Phys. 2006]. They give the the necessary and sufficient condition on the global existence of classical solutions of Eq. (12) with the initial data

$$\phi(0,x)=f(x),\quad \phi_t(0,x)=g(x). \qquad (13)$$

Let

$$u = \phi_x, \quad v = \phi_t, \tag{14}$$

where $u = (u_1, \cdots, u_n)^T$ and $v = (v_1, \cdots, v_n)^T$. Then (12) can be equivalently rewritten as

$$\begin{cases} u_t - v_x = 0, \\ \left(\frac{v+|u|^2 v-(u\cdot v)u}{\sqrt{1+|u|^2-|v|^2-|v|^2|u|^2+(u\cdot v)^2}}\right)_t - \left(\frac{u-|v|^2 u+(u\cdot v)v}{\sqrt{1+|u|^2-|v|^2-|v|^2|u|^2+(u\cdot v)^2}}\right)_x = 0. \end{cases} \tag{15}$$

If $\triangle(u, v) = 1 + |u|^2 - |v|^2 - |v|^2|u|^2 + (u \cdot v)^2 > 0$, i.e., the surface is time-like. Then, system (15) is non-strictly hyperbolic, linearly degenerate and has two n-constant multiple eigenvalues

$$\lambda_\pm = \frac{1}{1 + |u|^2}(-(u \cdot v) \pm \sqrt{\triangle(u, v)}\ ). \tag{16}$$

The Riemann invariants for system (15) constructed in Kong, Sun and Zhou [J. Math. Phys. 2006] are

$$\begin{cases} R_i = v_i + \lambda_+ u_i \quad (i = 1, \dots, n), \\ S_i = v_i + \lambda_- u_i \quad (i = 1, \dots, n). \end{cases} \tag{17}$$

By direct computation, they satisfy the following system:

$$\begin{cases} \frac{\partial R_i}{\partial t} + \lambda_- \frac{\partial R_i}{\partial x} = 0 \quad (i = 1, \dots, n), \\ \frac{\partial S_i}{\partial t} + \lambda_+ \frac{\partial S_i}{\partial x} = 0 \quad (i = 1, \dots, n). \end{cases} \tag{18}$$

On an angular domain

$$D = \{(t,x) \mid t \geq 0, x_1(t) \leq x \leq x_2(t)\},$$

where $x = x_1(t)$ and $x = x_2(t)$ are the backward characteristic and forward characteristic passing through the original $O(0,0)$, which are defined by

$$\begin{cases} \frac{dx_1(t)}{dt} = \lambda_-, \\ x_1(0) = 0 \end{cases}$$

and

$$\begin{cases} \frac{dx_2(t)}{dt} = \lambda_+, \\ x_2(0) = 0, \end{cases}$$

respectively.

We consider the characteristic boundary value problem of Eq. (12) with the following characteristic boundary conditions:

$$\begin{cases} x = x_1(t): & \phi(t, x_1(t)) = \phi_1(t), \\ x = x_2(t): & \phi(t, x_2(t)) = \phi_2(t), \end{cases} \tag{19}$$

where $\phi_1(t)$ and $\phi_2(t)$ are any given C^2 functions satisfying the conditions of C^2 compatibility at the origin $(0,0)$. Set $R = (R_1, \cdots, R_n)^T$ and $S = (S_1, \cdots, S_n)^T$, Then, from (18), we know that

$$\begin{cases} R = R(0,0) \overset{\triangle}{=} R_0(0) & on\ x = x_1(t), \\ S = S(0,0) \overset{\triangle}{=} S_0(0) & on\ x = x_2(t). \end{cases} \tag{20}$$

Thus, from (17), we know that the characteristic boundary conditions (19) can be rewritten as

$$\begin{cases} x = x_1(t): & (R, S) = (R_0(0), \phi_1'(t)), \\ x = x_2(t): & (R, S) = (\phi_2'(t), S_0(0)), \end{cases}$$

where

$$S_0(0) = \phi_1'(0) \quad and \quad R_0(0) = \phi_2'(0).$$

We assume that the boundary data are of bounded and possibly large C^2 norm, such that

$$||\phi_1(t)||_{C^2},\ ||\phi_2(t)||_{C^2} \leq M,$$

for some $M > 0$ bounded but possibly large. Also, we assume that the boundary data $\phi_1(t)$, $\phi_2(t)$ satisfy

$$\int_0^{+\infty} |\phi_1''(t)| dt \leq \varepsilon, \quad \int_0^{+\infty} |\phi_2''(t)| dt \leq \varepsilon,$$

where ε is a small parameter.
Therefore Theorem 1 can be applied.

For the sake of completeness, in this section we briefly recall John's formula on the decomposition of waves with some supplements, which will play an important role in our proof.

Let

$$v_i = l_i(u)u \quad (i = 1, \ldots, n) \tag{21}$$

and

$$w_i = l_i(u)u_x \quad (i = 1, \ldots, n), \tag{22}$$

where $l_i(u) = (l_{i1}(u), \ldots, l_{in}(u))$ denotes the i-th left eigenvector.

By

$$l_i(u)r_j(u) \equiv \delta_{ij} \quad (i, j = 1, \ldots, n),$$

where $r_i(u) = (r_{i1}(u), \ldots, r_{in}(u))^T$ denotes the i-th right eigenvector corresponding to $\lambda_i(u)$, it is easy to see that

$$u = \sum_{k=1}^{n} v_k r_k(u) \tag{23}$$

and

$$u_x = \sum_{k=1}^{n} w_k r_k(u). \tag{24}$$

Let

$$\frac{d}{d_i t} = \frac{\partial}{\partial t} + \lambda_i(u)\frac{\partial}{\partial x} \tag{25}$$

be the directional derivative along the i-th characteristic. We have (cf. [John, CPAM 1974])

$$\frac{dv_i}{d_i t} = \sum_{j,k=1}^{n} \beta_{ijk}(u) v_j w_k \ \ (i = 1, \dots, n), \tag{26}$$

where

$$\beta_{ijk}(u) = (\lambda_k(u) - \lambda_i(u)) l_i(u) \nabla r_j(u) r_k(u). \tag{27}$$

Hence, we have

$$\beta_{iji}(u) \equiv 0, \ \ \forall\ i, j. \tag{28}$$

On the other hand, we have (cf. [John, CPAM 1974])

$$\frac{dw_i}{d_i t} = \sum_{j,k=1}^{n} \gamma_{ijk}(u) w_j w_k \quad (i = 1, \dots, n), \tag{29}$$

where

$$\gamma_{ijk}(u) = \frac{1}{2}\{(\lambda_j(u) - \lambda_k(u)) l_i(u) \nabla r_k(u) r_j(u) - \nabla \lambda_i(u) r_j(u) \delta_{ik} + (j|k)\}, \tag{30}$$

in which $(j|k)$ denotes all the terms obtained by changing j and k in the previous terms. We have

$$\gamma_{ijj}(u) \equiv 0, \quad \forall\ j \neq i \qquad (i, j = 1, \dots, n). \tag{31}$$

Moreover, if the i-th characteristic $\lambda_i(u)$ is linearly degenerate in the sense of Lax, we have

$$\gamma_{iii}(u) \equiv 0. \tag{32}$$

Noting (24), by (29) we have

$$\frac{\partial w_i}{\partial t} + \frac{\partial(\lambda_i(u)w_i)}{\partial x} = \sum_{j,k=1}^{n} \Gamma_{ijk}(u)w_j w_k \stackrel{def}{=} G_i(t,x), \tag{33}$$

equivalently,

$$d[w_i(dx - \lambda_i(u)dt)] = \sum_{j,k=1}^{n} \Gamma_{ijk}(u)w_j w_k dt \wedge dx = G_i(t,x)dt \wedge dx, \tag{34}$$

where

$$\Gamma_{ijk}(u) = \frac{1}{2}(\lambda_j(u) - \lambda_k(u))l_i(u)[\nabla r_k(u)r_j(u) - \nabla r_j(u)r_k(u)]. \tag{35}$$

Hence, we have

$$\Gamma_{ijj}(u) \equiv 0, \qquad \forall\ i,j. \tag{36}$$

By the existence and uniqueness of local classical solutions of quasilinear hyperbolic systems (see [Li and Yu, 1985]), in order to prove Theorem 1, it suffices to establish a uniform a priori estimate for the C^0 norm of u and u_x on any given domain of existence of the C^1 solution $u = u(t,x)$.

By (2), there exist sufficiently small positive constants δ and δ_0 such that

$$\lambda_{i+1}(u) - \lambda_i(v) \geq \delta_0, \qquad \forall\ |u|,\ |v| \leq \delta\ \ (i = 1, \dots, n-1). \quad (3.1)$$

For the time being it is supposed that on the domain of existence of the C^1 solution $u = u(t,x)$ to the Goursat problem (1) and (3)-(4), we have

$$|u(t,x)| \leq \delta. \quad (3.2)$$

At the end of the proof of Lemma 5, we will explain that this hypothesis is reasonable.

For any fixed $T > 0$, let

$$U_\infty(T) = \sup_{0 \le t \le T} \sup_{x_1(t) \le x \le x_n(t)} |u(t,x)|, \tag{3.3}$$

$$V_\infty(T) = \sup_{0 \le t \le T} \sup_{x_1(t) \le x \le x_n(t)} |v(t,x)|, \tag{3.4}$$

$$W_\infty(T) = \sup_{0 \le t \le T} \sup_{x_1(t) \le x \le x_n(t)} |w(t,x)|, \tag{3.5}$$

$$\widetilde{W}_1(T) = \max_{i=1,\dots,n} \max_{j \ne i} \sup_{\widetilde{C}_j} \int_{\widetilde{C}_j} |w_i(t,x)| dt, \tag{3.6}$$

where $|\cdot|$ stands for the Euclidean norm in $\mathbf{R}^n$, $v = (v_1, \dots, v_n)^T$ and $w = (w_1, \dots, w_n)^T$ in which v_i and w_i are defined by (21) and (22) respectively, while $\widetilde{C}_j$ stands for any given j-th characteristic on the domain $D(T) = \{(t,x) \mid 0 \le t \le T, x_1(t) \le x \le x_n(t)\}$. Clearly, $V_\infty(T)$ is equivalent to $U_\infty(T)$.

First we recall some basic L^1 estimates. They are essentially due to Schatzman [Indiana Univ. Math. J., 34, 1985; Lectures in Applied Mathematics, Vol. 23, 1986] and Zhou [Chin. Ann. Math. Ser. B, 25, 2004].

Lemma 1.

Let $\phi = \phi(t, x) \in C^1$ satisfy

$$\phi_t + (\lambda(t,x)\phi)_x = F(t,x),\ 0 \le t \le T, x \in R,$$

$$\phi(0,x) = g(x),$$

where $\lambda \in C^1$. Then

$$\int_{-\infty}^{+\infty} |\phi(t,x)| dx \le \int_{-\infty}^{+\infty} |g(x)| dx$$

$$+ \int_0^T \int_{-\infty}^{+\infty} |F(t,x)| dx dt,\ \ \forall\ t \le T, \tag{3.7}$$

provided that the right-hand side of the inequality is bounded.

Lemma 2

Let $\phi = \phi(t,x)$ and $\psi = \psi(t,x)$ be C^1 functions satisfying

$$\phi_t + (\lambda(t,x)\phi)_x = F_1(t,x),\ 0 \le t \le T, x \in R,$$

$$\phi(0,x) = g_1(x)$$

and

$$\psi_t + (\mu(t,x)\psi)_x = F_2(t,x),\ 0 \le t \le T, x \in R,$$

$$\psi(0,x) = g_2(x),$$

respectively, where $\lambda, \mu \in C^1$ such that there exists a positive constants δ_0 independent of T verifying

$$\mu(t,x) - \lambda(t,x) \ge \delta_0,\ 0 \le t \le T, x \in R.$$

Then

$$\int_0^T \int_{-\infty}^{+\infty} |\phi(t,x)||\psi(t,x)| dxdt \le C\Big(\int_{-\infty}^{+\infty} |g_1(x)| dx + \int_0^T \int_{-\infty}^{+\infty} |F_1(t,x)| dxdt \Big) \times \Big(\int_{-\infty}^{+\infty} |g_2(x)| dx + \int_0^T \int_{-\infty}^{+\infty} |F_2(t,x)| dxdt \Big), \tag{3.8}$$

provided that the two factors on the right-hand side of the inequality is bounded.

In the present situation, similar to the above basic L^1 estimates (3.7)-(3.8), we have

Lemma 3

Under the assumptions of Theorem 1, on any given domain of existence $D(T) = \{(t,x) \mid 0 \leq t \leq T, x_1(t) \leq x \leq x_n(t)\}$ of the C^1 solution $u = u(t,x)$ to the Goursat problem (1) and (3)-(4), there exists a positive constant k_1 independent of ε, T and M such that

$$\int_{x_1(t)}^{x_n(t)} |w_i(t,x)|dx \leq k_1 \Big\{ \int_0^{+\infty} (|\varphi'(t)| + |\psi'(t)|)dt + \int_0^T \int_{x_1(t)}^{x_n(t)} |G_i(t,x)|dxdt \Big\}, \qquad \forall\ t \leq T, \tag{3.9}$$

provided that the right-hand side of the inequality is bounded.

Proof. For any given t with $0 \leq t \leq T$, we denote the point $(t, x_n(t))$ (resp. $(t, x_1(t))$) by A (resp. B). We rewrite (34) as

$$d(|w_i(t,x)|(dx - \lambda_i(u)dt)) = \ sgn(w_i)G_i dxdt, \qquad (3.10)$$

and we integrating it in the region AOB to get
(i) For $i = 2, \ldots, n-1$, then we have

$$\int_{BA} |w_i(t,x)|dx \leq \int_0^t |w_i(t, x_n(t))(x_n'(t) - \lambda_i(u(t, x_n(t))))|dt$$

$$+ \int_0^t |w_i(t, x_1(t))(x_1'(t) - \lambda_i(u(t, x_1(t))))|dt + \int\int_{AOB} |G_i|dxdt. \qquad (3.11)$$

On the other hand, by (22), we have

$$w_i(t, x_n(t)) = l_i(u(t, x_n(t)))u_x(t, x_n(t)) \quad (i = 1, \ldots, n). \qquad (3.12)$$

Consequently, we conclude from (1), (3) and (3.12) that

$$w_i(t, x_n(t))(x_n'(t) - \lambda_i(u(t, x_n(t)))) = l_i(\psi(t))\psi'(t) \quad (i = 1, \dots, n). \tag{3.13}$$

Likewise, we have

$$w_i(t, x_1(t))(x_1'(t) - \lambda_i(u(t, x_1(t)))) = l_i(\varphi(t))\varphi'(t) \quad (i = 1, \dots, n). \tag{3.14}$$

Substituting (3.13)-(3.14) into (3.11), we obtain

$$\int_{x_1(t)}^{x_n(t)} |w_i(t,x)|dx \leq c_1 \Big\{ \int_0^{+\infty} (|\varphi'(t)| + |\psi'(t)|)dt + \int_0^T \int_{x_1(t)}^{x_n(t)} |G_i(t,x)|dxdt \Big\}, \tag{3.15}$$

where here and henceforth, $c_i (i = 1, 2, \dots)$ will denote positive constants independent of ε, T and M.

(ii) For $i = n$, then we have

$$\int_{BA} |w_n(t,x)|dx \le \int_0^t |w_n(t,x_n(t))(x_n'(t) - \lambda_n(u(t,x_n(t))))|dt$$

$$+ \int_0^t |w_n(t,x_1(t))(x_1'(t) - \lambda_n(u(t,x_1(t))))|dt + \int\int_{AOB} |G_n|dxdt. \tag{3.16}$$

Noting (8) and (3.13), we get

$$w_n(t,x_n(t))(x_n'(t) - \lambda_n(u(t,x_n(t)))) = l_n(\psi(t))\psi'(t) \equiv 0. \tag{3.17}$$

It thus follows from (3.16) that

$$\int_{x_1(t)}^{x_n(t)} |w_n(t,x)|dx \le c_2 \Big\{ \int_0^{+\infty} |\varphi'(t)|dt$$

$$+ \int_0^T \int_{x_1(t)}^{x_n(t)} |G_n(t,x)|dxdt \Big\}. \tag{3.18}$$

(iii) For $i = 1$, the argument in step (iii) is similar to the one in step (ii), instead of formula (3.18) we have

$$\int_{x_1(t)}^{x_n(t)} |w_1(t,x)|dx \leq c_3 \Big\{ \int_0^{+\infty} |\psi'(t)|dt + \int_0^T \int_{x_1(t)}^{x_n(t)} |G_1(t,x)|dxdt \Big\}. \tag{3.19}$$

The proof of Lemma 3 is finished. □

Lemma 4

Under the assumptions of Theorem 1, on any given domain of existence $D(T) = \{(t,x) \mid 0 \le t \le T, x_1(t) \le x \le x_n(t)\}$ of the C^1 solution $u = u(t,x)$ to the Goursat problem (1) and (3)-(4), there exists a positive constant k_2 independent of ε, T and M such that

$$\int_0^T \int_{x_1(t)}^{x_n(t)} |w_i(t,x)||w_j(t,x)|dxdt$$

$$\le k_2\Big(\int_0^{+\infty}(|\varphi'(t)|+|\psi'(t)|)dt + \int_0^T\int_{x_1(t)}^{x_n(t)}|G_i(t,x)|dxdt\Big)$$

$$\times\Big(\int_0^{+\infty}(|\varphi'(t)|+|\psi'(t)|)dt+\int_0^T\int_{x_1(t)}^{x_n(t)}|G_j(t,x)|dxdt\Big), \qquad \forall\ i \ne j \tag{3.20}$$

provided that the right-hand side of the inequality is bounded.

Proof. For $i, j \in \{1, \ldots, n\}$ and $i \neq j$, without loss of generality, we suppose that $i < j$. We introduce the "continuous Glimm's functional"

$$Q(t) = \int\int_{x_1(t)<x<y<x_n(t)} |w_j(t,x)||w_i(t,y)|dxdy. \qquad (3.21)$$

Then, it is easy to see that

$$\frac{dQ(t)}{dt} = x_n'(t)|w_i(t, x_n(t))| \int_{x_1(t)}^{x_n(t)} |w_j(t,x)|dx$$

$$-x_1'(t)|w_j(t, x_1(t))| \int_{x_1(t)}^{x_n(t)} |w_i(t,x)|dx$$

$$+\int\int_{x_1(t)<x<y<x_n(t)} \frac{\partial}{\partial t}(|w_j(t,x)|)|w_i(t,y)|dxdy$$

$$
\begin{aligned}
&+\int\int_{x_1(t)<x<y<x_n(t)} |w_j(t,x)|\frac{\partial}{\partial t}(|w_i(t,y)|)dxdy\\
&= x_n'(t)|w_i(t,x_n(t))|\int_{x_1(t)}^{x_n(t)}|w_j(t,x)|dx\\
&-x_1'(t)|w_j(t,x_1(t))|\int_{x_1(t)}^{x_n(t)}|w_i(t,x)|dx\\
&-\int\int_{x_1(t)<x<y<x_n(t)}\frac{\partial}{\partial x}(\lambda_j(u)|w_j(t,x)|)|w_i(t,y)|dxdy\\
&-\int\int_{x_1(t)<x<y<x_n(t)}|w_j(t,x)|\frac{\partial}{\partial y}(\lambda_i(u)|w_i(t,y)|)dxdy\\
&+\int\int_{x_1(t)<x<y<x_n(t)} sgn(w_j)G_j(t,x)|w_i(t,y)|dxdy\\
&+\int\int_{x_1(t)<x<y<x_n(t)}|w_j(t,x)|\ sgn(w_i)G_i(t,y)dxdy
\end{aligned}
$$

$$
\begin{aligned}
&= -\int_{x_1(t)}^{x_n(t)} (\lambda_j(u(t,x)) - \lambda_i(u(t,x)))|w_i(t,x)||w_j(t,x)|dx \\
&+(x_n'(t) - \lambda_i(u(t,x_n(t))))|w_i(t,x_n(t))| \int_{x_1(t)}^{x_n(t)} |w_j(t,x)|dx \\
&+(\lambda_j(u(t,x_1(t))) - x_1'(t))|w_j(t,x_1(t))| \int_{x_1(t)}^{x_n(t)} |w_i(t,x)|dx \\
&\quad + \int\int_{x_1(t)<x<y<x_n(t)} sgn(w_j)G_j(t,x)|w_i(t,y)|dxdy \\
&+ \int\int_{x_1(t)<x<y<x_n(t)} |w_j(t,x)|\ sgn(w_i)G_i(t,y)dxdy. \qquad (3.22)
\end{aligned}
$$

Noting (3.1) and using (3.13)-(3.14), we get from (3.22) that

$$\frac{dQ(t)}{dt} \le -\delta_0 \int_{x_1(t)}^{x_n(t)} |w_i(t,x)||w_j(t,x)|dx$$

$$+|w_i(t,x_n(t))(x_n'(t)-\lambda_i(u(t,x_n(t))))| \int_{x_1(t)}^{x_n(t)} |w_j(t,x)|dx$$

$$+|w_j(t,x_1(t))(x_1'(t)-\lambda_j(u(t,x_1(t))))| \int_{x_1(t)}^{x_n(t)} |w_i(t,x)|dx$$

$$+\int_{x_1(t)}^{x_n(t)} |G_j(t,x)|dx \int_{x_1(t)}^{x_n(t)} |w_i(t,x)|dx$$

$$+\int_{x_1(t)}^{x_n(t)} |G_i(t,x)|dx \int_{x_1(t)}^{x_n(t)} |w_j(t,x)|dx$$

$$
\begin{aligned}
&\leq -\delta_0 \int_{x_1(t)}^{x_n(t)} |w_i(t,x)||w_j(t,x)|dx \\
&+|l_i(\psi(t))\psi'(t)| \int_{x_1(t)}^{x_n(t)} |w_j(t,x)|dx + |l_j(\varphi(t))\varphi'(t)| \int_{x_1(t)}^{x_n(t)} |w_i(t,x)|dx \\
&+ \int_{x_1(t)}^{x_n(t)} |G_j(t,x)|dx \int_{x_1(t)}^{x_n(t)} |w_i(t,x)|dx \\
&+ \int_{x_1(t)}^{x_n(t)} |G_i(t,x)|dx \int_{x_1(t)}^{x_n(t)} |w_j(t,x)|dx. \qquad (3.23)
\end{aligned}
$$

It then follows from Lemma 3 that

$$
\frac{dQ(t)}{dt} + \delta_0 \int_{x_1(t)}^{x_n(t)} |w_i(t,x)||w_j(t,x)|dx
$$

$$\leq k_1\left(|l_j(\varphi(t))\varphi'(t)| + \int_{x_1(t)}^{x_n(t)} |G_j(t,x)|dx\right)$$
$$\times\left(\int_0^{+\infty}(|\varphi'(t)| + |\psi'(t)|)dt + \int_0^T\int_{x_1(t)}^{x_n(t)} |G_i(t,x)|dxdt\right)$$
$$+k_1\left(|l_i(\psi(t))\psi'(t)| + \int_{x_1(t)}^{x_n(t)} |G_i(t,x)|dx\right)$$
$$\times\left(\int_0^{+\infty}(|\varphi'(t)|+|\psi'(t)|)dt+\int_0^T\int_{x_1(t)}^{x_n(t)} |G_j(t,x)|dxdt\right). \quad (3.24)$$

Therefore

$$\delta_0 \int_0^T \int_{x_1(t)}^{x_n(t)} |w_i(t,x)||w_j(t,x)|dxdt$$

$$\leq Q(0) + c_4\left(\int_0^{+\infty} |\varphi'(t)|dt + \int_0^T \int_{x_1(t)}^{x_n(t)} |G_j(t,x)|dxdt\right)$$

$$\times\left(\int_0^{+\infty} (|\varphi'(t)| + |\psi'(t)|)dt + \int_0^T \int_{x_1(t)}^{x_n(t)} |G_i(t,x)|dxdt\right)$$

$$+c_5\left(\int_0^{+\infty} |\psi'(t)|dt + \int_0^T \int_{x_1(t)}^{x_n(t)} |G_i(t,x)|dxdt\right)$$

$$\times\left(\int_0^{+\infty} (|\varphi'(t)| + |\psi'(t)|)dt + \int_0^T \int_{x_1(t)}^{x_n(t)} |G_j(t,x)|dxdt\right)$$

$$\leq Q(0)+c_6\left(\int_0^{+\infty}(|\varphi'(t)|+|\psi'(t)|)dt+\int_0^T\int_{x_1(t)}^{x_n(t)}|G_i(t,x)|dxdt\right)$$

$$\times\left(\int_0^{+\infty}(|\varphi'(t)|+|\psi'(t)|)dt+\int_0^T\int_{x_1(t)}^{x_n(t)}|G_j(t,x)|dxdt\right). \quad (3.25)$$

In view of $Q(0)=0$, we immediately get the desired conclusion. The proof of Lemma 4 is finished. □

Lemma 5

Under the assumptions of Theorem 1, there exists a constant $\varepsilon > 0$ so small that on any given domain of existence $D(T) = \{(t,x) \mid 0 \le t \le T, x_1(t) \le x \le x_n(t)\}$ of the C^1 solution $u = u(t,x)$ to the Goursat problem (1) and (3)-(4), there exists a positive constant k_3 independent of ε, T and M, such that the following uniform a priori estimates hold:

$$\widetilde{W}_1(T) \le k_3\varepsilon, \tag{3.26}$$

$$U_\infty(T), V_\infty(T) \le k_3\varepsilon \tag{3.27}$$

and

$$W_\infty(T) \le k_3 M. \tag{3.28}$$

Proof of Theorem 1

Under the assumptions of Theorem 1, from (3.27) and (3.28), we know that there exists $\varepsilon > 0$ suitably small such that on any given domain of existence $D(T) = \{(t,x) \mid 0 \le t \le T, x_1(t) \le x \le x_n(t)\}$ of the C^1 solution $u = u(t,x)$ to the Goursat problem (1) and (3)-(4), the C^1 norm of the solution possesses a uniform a priori estimate independent of T. This leads to the conclusion of Theorem 1 immediately. The proof of Theorem 1 is finished. □

1-Dimensional compressible isentropic Euler equations for Chaplygin gases

$$\begin{cases} \partial_t\rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho v) + \partial_x(\rho v^2 + p) = 0, \end{cases} \tag{4.1}$$

where ρ, v and p are the density, velocity and pressure of the gas, respectively, the state equation is

$$p(\rho) = -\frac{1}{\rho}, \qquad for \ \rho > 0. \tag{4.2}$$

This model is known as the Chaplygin gas, it was introduced by Chaplygin, Tsien and von Karman as a suitable mathematical approximation for calculating the lifting force on a wing of an airplane in aerodynamics. In this model, the sound speed is given by $c = \frac{1}{\rho}$.

The relativistic Euler equations for Chaplygin gases

The Euler system of conservation laws of energy and momentum for a Chaplygin gas in special relativity reads:

$$\begin{cases} \partial_t\Big((p+\rho c^2)\frac{v^2}{c^2(c^2-v^2)}+\rho\Big)+\partial_x\Big((p+\rho c^2)\frac{v}{c^2-v^2}\Big)=0, \\ \partial_t\Big((p+\rho c^2)\frac{v}{c^2-v^2}\Big)+\partial_x\Big((p+\rho c^2)\frac{v^2}{c^2-v^2}+p\Big)=0, \end{cases} \tag{4.3}$$

where ρ, p and v represent the proper energy density, the pressure and the particle speed respectively, and the constant c is the speed of light. The equations of state is

$$p(\rho)=-\frac{1}{\rho}, \qquad for\,\rho>0. \tag{4.4}$$

System (4.3) models the dynamics of plane waves in special relativistic fluids in a two dimensional Minkowski time-space (x^0, x^1):

$$\mathrm{d}iv\ T = 0, \tag{4.5}$$

where T is the stress-energy tensor for the fluid:

$$T^{ij} = (p + \rho c^2)u^i u^j + p\eta^{ij}, \tag{4.6}$$

with all indices running from 0 to 1 with $x^0 = ct$. In (4.6),

$$\eta^{ij} = \eta_{ij} = \mathrm{d}iag(-1, 1)$$

denotes the flat Minkowski metric, u the 2-velocity of the fluid particle (the velocity of the frame of isotropy of the perfect fluid), ρ the mass-energy density of the fluid as measured in units of mass in a reference frame moving with the fluid particle.

A gas is called a Chaplygin gas if it satisfies the exotic equation of state (4.4). A Chaplygin gas owns a negative pressure and occurs in certain theories of cosmology. Such a gas has been advertised as a possible model for dark energy.

It is well-known that these systems are strictly hyperbolic and that their fields are linearly degenerate, so Theorem 1 is obviously applicable.

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