Incomplete Models with Set-Valued Residuals

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Incomplete models with set valued residuals

- A process produces realizations of: outcomes Y, exogenous Z, unobserved continuous U.
- The paper is concerned with models restricting:
  - (1) dependence between U and Z and
  - (2) a function

    $$h(Y, Z, U)$$

    continuous in its first and third arguments

    $$\mathcal{Y}(u, z; h) \equiv \{y : h(y, z, u) = 0\}$$

    which delivers values of Y compatible with values of Z and U.

Incomplete models with set valued residuals

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- If $\mathcal{Y}(u, z, h)$ *and* $\mathcal{U}(y, z, h)$ are non-singleton then these models are generically *partially* identifying.
  - i.e. there may be *sets* of functions h that can deliver observed distributions of (Y, Z).
- We characterize these sets and consider challenges for asymptotic theory.

Examples of complete models

- Function: $h(Y, Z, U)$ has level sets:

$$\mathcal{Y}(u, z; h) \equiv \{y : h(y, z, u) = 0\}$$

$$\mathcal{U}(y, z; h) \equiv \{u : h(y, z, u) = 0\}$$

  - Linear model
$$h(Y, Z, U) = Y - \beta Z - U$$
has singleton $\mathcal{Y}(u, z; h) = \{\beta z + u\}$ and singleton $\mathcal{U}(y, z; h) = \{y - \beta z\}$.
  - Nonlinear model
$$h(Y, Z, U) = Y - g(Z, U)$$
has singleton $\mathcal{Y}(u, z; h) = \{g(z, u)\}$ but non-singleton $\mathcal{U}(y, z; h)$ if Y is discrete or U is non-scalar.

Examples of incomplete models

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$$\mathcal{Y}(u, z; h) \equiv \{y : h(y, z, u) = 0\}$$

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  - Linear model relating two outcomes is incomplete

$$h(Y, Z, U) = Y_1 - \alpha Y_2 - \beta Z - U$$

    It has singleton $\mathcal{U}(y, z; h)$.
  - Non-additive error model

$$h(Y, Z, U) = Y_1 - g(Y_2, Z, U)$$

    has non-singleton $\mathcal{U}(y, z; h)$ if Y discrete or U non-scalar.

  - Dependent censoring has non-singleton $\mathcal{Y}(u, z; h)$ and $\mathcal{U}(y, z; h)$.

$$h(Y, Z, U) = Y_1 - \min\left(g(Z, U), Y_2\right)$$

Contributions

- The paper characterizes identified sets of structures in these models under mean, quantile and full independence restrictions on (U, Z).
  - this builds on Chesher, Rosen and Smolinski (2013, QE), Chesher (2010, Ecta), Chesher and Rosen (2013, AER; 2013 EctJ; recent working papers).
  - identified sets are characterized via systems of moment inequalities and equalities.
  - When outcomes are continuous there may be an uncountable number of inequalities.
  - This poses challenges for inference.
- We use random set theory reviewed in Molchanov (2005, Springer-V) and used for set identification by Beresteanu, Molchanov, and Molinari (2011, Ecta; 2012, JoEcts).
  - unlike BMM we use random sets with support on the space of unobservable U.

Outline of the remainder of the talk

- Restrictions and concepts: structures, models, identified sets.
- Definition of the **identified set** of structures.
- Characterization of the identified set using **selectionability** in the space of unobservables.
- The impact of the independence restriction $U \perp\!\!\!\perp Z$.
- Example: dependent censoring.
- The challenge of implementation - uncountable inequalities.

Restrictions

1. (Y, Z, U) are random vectors defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, endowed with the Borel sets on Ω. The support of (Y, Z, U), denoted $\mathcal{R}_{YZU}$, is a subset of Euclidean space.

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2. A collection of conditional distributions

$$\mathcal{F}_{Y|Z} \equiv \left\{ F_{Y|Z}\left(\cdot|z\right) : z \in \mathcal{R}_Z \right\}$$

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3. There is an $\mathcal{F}$-measurable function $h(\cdot, \cdot, \cdot) : \mathcal{R}_{YZU} \to \mathbb{R}$ such that

$$\mathbb{P}[h(Y, Z, U) = 0] = 1$$

and there is a collection of conditional distributions

$$\mathcal{G}_{U|Z} \equiv \left\{ G_{U|Z}(\cdot|z) : z \in \mathcal{R}_Z \right\}$$

where for all $\mathcal{S} \subseteq \mathcal{R}_{U|Z=z}$

$$G_{U|Z}(\mathcal{S}|z) \equiv \mathbb{P}[U \in \mathcal{S}|z]$$

denotes a conditional distribution of U given $Z = z$.

Identification and What is Observed

- From data we learn $\mathcal{F}_{Y|Z} \equiv \left\{ F_{Y|Z}(\cdot|z) : z \in \mathcal{R}_Z \right\}$ and F_Z.
- These distributions are generated by some **structure** $(h, \mathcal{G}_{U|Z})$ which comprises:
  - A collection of conditional distributions of U given Z,
  $$\mathcal{G}_{U|Z} \equiv \left\{ G_{U|Z}(\cdot|z) : z \in \mathcal{R}_Z \right\}$$
  - A function $h : \mathcal{R}_{YZU} \to \mathbb{R}$ such that
  $$\mathbb{P}[h(Y, Z, U) = 0] = 1.$$
- A **model** $\mathcal{M}$ defines *admissible* structures $(h, \mathcal{G}_{U|Z})$.
- **Identified set**: the structures $(h, \mathcal{G}_{U|Z})$ admitted by a model $\mathcal{M}$ that can deliver $\mathcal{F}_{Y|Z}$.

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  with distributions determined by h and observed $\mathcal{F}_{Y|Z}$.

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- Our analysis exploits a dual feature of the level sets:

$$u^* \in \mathcal{U}\left(y^*, z; h\right) \Leftrightarrow y^* \in \mathcal{Y}\left(u^*, z; h\right)$$

true because

$$\begin{aligned} u^* &\in \mathcal{U}\left(y^*, z; h\right) \Leftrightarrow h\left(y^*, z, u^*\right) = 0 \\ y^* &\in \mathcal{Y}\left(u^*, z; h\right) \Leftrightarrow h\left(y^*, z, u^*\right) = 0 \end{aligned}$$

Definitions: Selections and Selectionability

- The random outcome set

$$\mathcal{Y}(U, Z; h) \equiv \{y : h(y, Z, U) = 0\}$$

  can be regarded as a collection of random variables.

- *Definition*: Random variable A is a **selection** of random set $\mathcal{A}$ if $\mathbb{P}[A \in \mathcal{A}] = 1$.

- *Definition*: Distribution F is **selectionable** w.r.t. the distribution of a random set $\mathcal{A}$ if F is the distribution of a selection of $\mathcal{A}$.

Sets of Outcomes: Selections and Selectionability

- For each structure $(h, \mathcal{G}_{U|Z})$, and any z, stochastic variation in

$$U \sim G_{U|Z=z} \in \mathcal{G}_{U|Z}$$

delivers a random set

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- *Definition*. A structure $\left(h, \mathcal{G}_{U|Z}\right)$ delivers a conditional distribution $F_{Y|Z=z}$ if and only if $F_{Y|Z=z}$ is **selectionable** w.r.t. the distribution of $\mathcal{Y}(U, z; h)$ when $U \sim G_{U|Z=z}$

Observational equivalence

- Structures $\left(h, \mathcal{G}_{U|Z}\right)$ and $\left(h', \mathcal{G}'_{U|Z}\right)$ are observationally equivalent relative to $\mathcal{F}_{Y|Z}$ if a.e. $z \in \mathcal{R}_Z$

$$F_{Y|Z=z} \text{ is selectionable w.r.t. } \mathcal{Y}\left(U, z; h\right) \text{ with } U \sim G_{U|Z=z}$$

and

$$F_{Y|Z=z} \text{ is selectionable w.r.t.} \mathcal{Y}\left(U, z; h'\right) \text{ with } U \sim G'_{U|Z=z}$$

The Identified Set

- The **identified set** of structures comprises admissible $\left(h, \mathcal{G}_{U|Z}\right)$ such that the conditional distributions $F_{Y|Z=z} \in \mathcal{F}_{Y|Z}$ are selectionable with respect to the conditional distributions of random sets $\mathcal{Y}(U, z; h)$ obtained with $U \sim G_{U|Z=z}$, a.e. $z \in \mathcal{R}_Z$.

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- Because of the dual relationship between level sets:

$$F_{Y|Z=z} \text{ is selectionable w.r.t. } \mathcal{Y}(U, z; h) \text{ when } U \sim G_{U|Z=z}$$

$$\Longleftrightarrow$$

$$G_{U|Z=z} \text{ is selectionable w.r.t. } \mathcal{U}(Y, z; h) \text{ when } Y \sim F_{Y|Z=z}$$

The Identified Set

- *Theorem.* The identified set of structures $\left(h, \mathcal{G}_{U|Z}\right)$ are those such that $G_{U|Z=z}$ is selectionable with respect to the conditional (on $Z = z$) distribution of random set $\mathcal{U}(Y, Z; h)$ induced by $F_{Y|Z=z}$, a.e. $z \in \mathcal{R}_Z$.

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- One characterization comes on using Artstein's (1983, IJM) inequality.
  - Distribution F_A is selectionable w.r.t. the distribution of random set $\mathcal{A}$ if and only if for all closed sets $\mathcal{S}$

$$F_A(\mathcal{S}) \geq \mathbb{P}[\mathcal{A} \subseteq \mathcal{S}]$$

The Identified Set

- *Theorem*. The identified set of structures $\left(h, \mathcal{G}_{U|Z}\right)$ are those such that

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  - $\mathbb{P}[\mathcal{A} \subseteq \mathcal{S}]$ is the containment function of random set $\mathcal{A}$.

Characterization via Conditional Moment Inequalities

- The conditional containment functional of $\mathcal{U}(Y, Z; h)$ applied to set $\mathcal{S} \subseteq \mathcal{R}_U$ is

$$C_h(\mathcal{S}|z) \equiv \mathbb{P}\left[\mathcal{U}(Y, Z; h) \subseteq \mathcal{S} | Z = z\right].$$

- The identified set of structures $(h, \mathcal{G}_{U|Z})$ are the admissible structures that satisfy the **moment inequalities**:

$$G_{U|Z}(\mathcal{S}|z) \geq C_h(\mathcal{S}|z)$$

for all closed sets $\mathcal{S} \subseteq \mathcal{R}_U$ and a.e. $z \in \mathcal{R}_Z$.

  - Proof draws on Artstein (1983, IJM), Molchanov (2005, Springer-V) and Norberg (1992, IJM).

Test sets

- We find a collection, $\mathrm{Q}(h, z)$, of sets such that if

$$G_{U|Z}(\mathcal{S}|z) \geq C_h(\mathcal{S}|z)$$

holds for all $\mathcal{S} \in \mathrm{Q}(h, z)$ then it holds for all closed $\mathcal{S} \subseteq \mathcal{R}_U$.

- We show that $\mathrm{Q}(h, z)$ contains only certain unions of the sets comprising the conditional (on z) support of $\mathcal{U}(Y, Z; h)$.

- We determine a class of members of $\mathrm{Q}(h, z)$ for which moment inequalities reduce to moment **equalities**.

- We show that two types of model **always** deliver moment **equalities**.
  - *complete* models - these have singleton outcome sets, $\mathcal{Y}(u, z; h)$
  - models with *singleton* residual sets, $\mathcal{U}(y, z; h)$.

Distributional Restrictions

- So far the analysis has proceeded without restrictions on the distribution of unobserved heterogeneity.

- Now consider the impact of a **stochastic independence condition**:

$$\forall z \in \mathcal{R}_Z : \quad G_{U|Z}(\cdot|z) = G_U(\cdot)$$

Stochastic Independence

- Let $\mathcal{S}^c$ denote the complement of $\mathcal{S}$. We show that the identified set comprises admissible $(h, \mathcal{G}_{U|Z})$ such that for all sets $\mathcal{S} \in \mathsf{Q}(h, z)$ and a.e. $z \in \mathcal{R}_Z$

$$1 - C_h(\mathcal{S}^c|z) \geq G_{U|Z}(\mathcal{S}|z) \geq C_h(\mathcal{S}|z)$$

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$$1 - C_h(\mathcal{S}^c|z) \geq G_{U|Z}(\mathcal{S}|z) \geq C_h(\mathcal{S}|z)$$

- Under the restriction $U \perp\!\!\!\perp Z$ the identified set comprises admissible $(h, \mathcal{G}_{U|Z})$ such that for all $\mathcal{S} \in \mathrm{Q}(h,z)$:

$$\inf_{z \in \mathcal{R}_Z} (1 - C_h(\mathcal{S}^c|z)) \geq G_U(\mathcal{S}) \geq \sup_{z \in \mathcal{R}_Z} (C_h(\mathcal{S}|z))$$

Endogenous censoring

- Outcomes: Y_1 and Y_2 are observed.

$$Y_1 = \min(g(Z,U),\ Y_2)$$

with U and Z independent and $g(Z,U)$ monotone increasing in scalar $U \sim Unif[0,1]$.

- Example: Demand with fixed supply.

$$\underbrace{Y_1}_{\text{amount sold}} = \min(\underbrace{g(Z,U)}_{\text{amount demanded}}, \underbrace{Y_2}_{\text{amount supplied}})$$

Level sets and test sets

- Structural function

$$h(y, z, u) = Y_1 - \min(g(Z, U),\ Y_2)$$

has level sets

$$\mathcal{U}(y, z; h) = \begin{cases} \text{interval:} \quad (g^{-1}(z, y_2), 1] & \text{if} \quad y_1 = y_2 \\ \text{singleton:} \quad \{g^{-1}(z, y_1)\} & \text{if} \quad y_1 < y_2 \end{cases}$$

- Support of $\mathcal{U}(y, z; h)$ comprises intervals.$(t, 1]$ and singletons $\{t\}$, $t \in [0, 1]$.
- $\mathrm{Q}(h, z)$ is *all* closed intervals: $[t_1, t_2] \subset [0, 1]$.
- We consider a selection of intervals: with $m = 1/M \in (0, 1)$:

$$\begin{bmatrix} [0, m] & [0, 2m] & [0, 3m] & \cdots & \cdots & [0, 1] \\ & [m, 2m] & [m, 3m] & \cdots & \cdots & [m, 1] \\ & & [2m, 3m] & \cdots & \cdots & [2m, 1] \\ & & & \ddots & & \vdots \end{bmatrix}$$

Inequalities

- For each interval $[t_1, t_2] \subset [0,1]$ there is an inequality:

$$\inf_{z \in \mathcal{R}_Z} \left(\mathbb{P}_F[(Y_1 = Y_2) \wedge (Y_2 \leq g(z, t_2))|z] + \mathbb{P}_F[(Y_1 < Y_2) \wedge (g(z, t_1) \leq Y_1 \leq g(z, t_2))|z] \right)$$

$$\geq (t_2 - t_1) \geq$$

$$\sup_{z \in \mathcal{R}_Z} \left(1[t_2 = 1] \times \mathbb{P}_F[(Y_1 = Y_2) \wedge (g(z, t_1) \leq Y_1 \leq g(z, t_2))\,|z] + \mathbb{P}_F[(Y_1 < Y_2) \wedge (g(z, t_1) \leq Y_1 \leq g(z, t_2))|z] \right)$$

Example - a Gaussian process

- Probabilities are generated as follows.

$$
\begin{aligned}
Y^* &= V_1 \\
Y_2 &= \gamma_{21} z + V_2
\end{aligned}
$$

$$Y_1 = \min(Y^*, Y_2)$$

$$V \equiv \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \perp\!\!\!\perp Z, \qquad V \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \right)$$

$$Z \in \{-1, 0, +1\} \qquad \gamma_{21} \in \{0.5, 1.0\}$$

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- The **model** is:

$$Y_1 = \min(\beta_0 + \sigma\Phi^{-1}(U), Y_2) \qquad U \perp\!\!\!\perp Z \qquad U \sim Unif(0,1)$$

and in the "data generating process" $(\beta_0, \sigma) = (0, 1)$.

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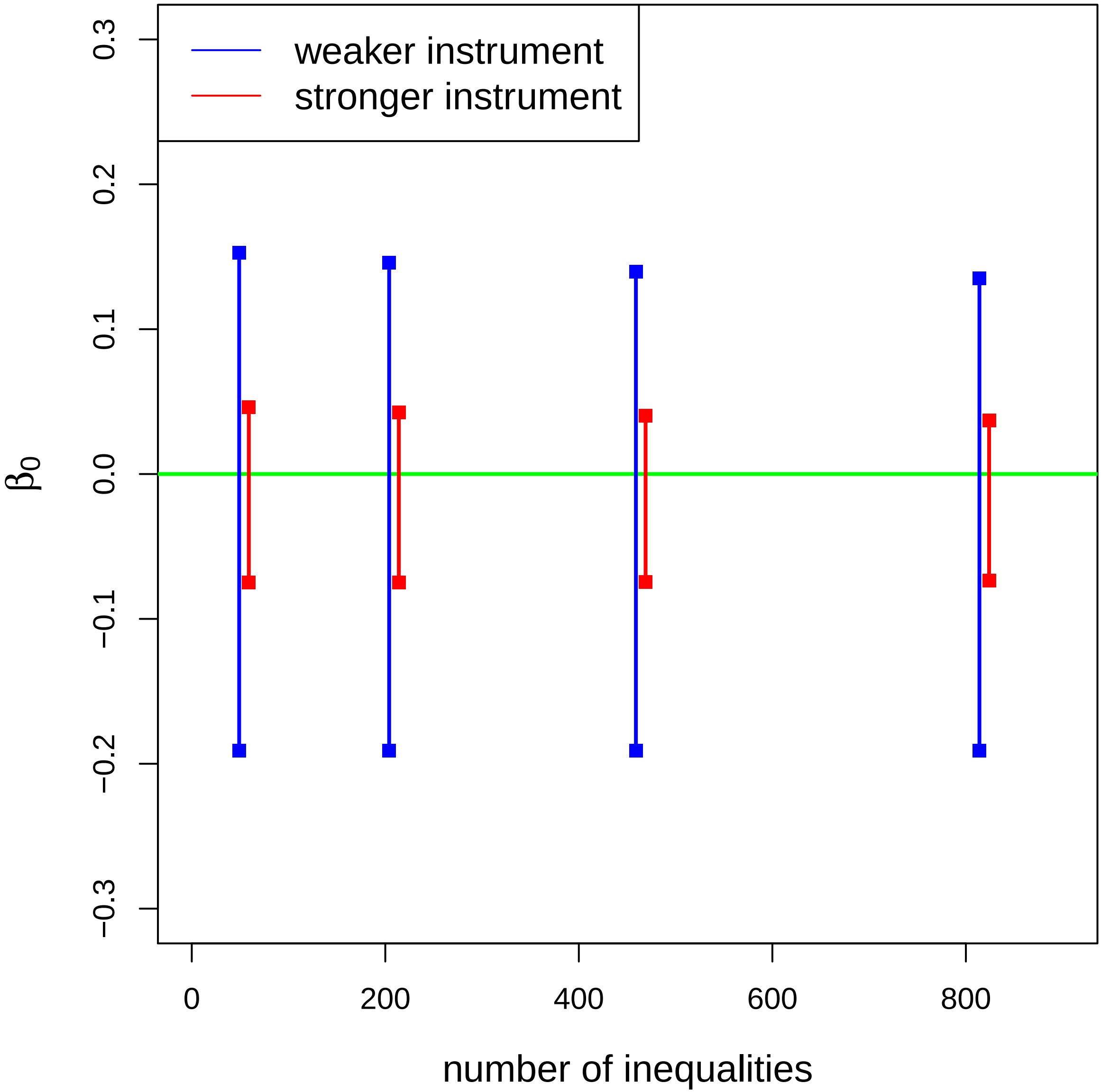
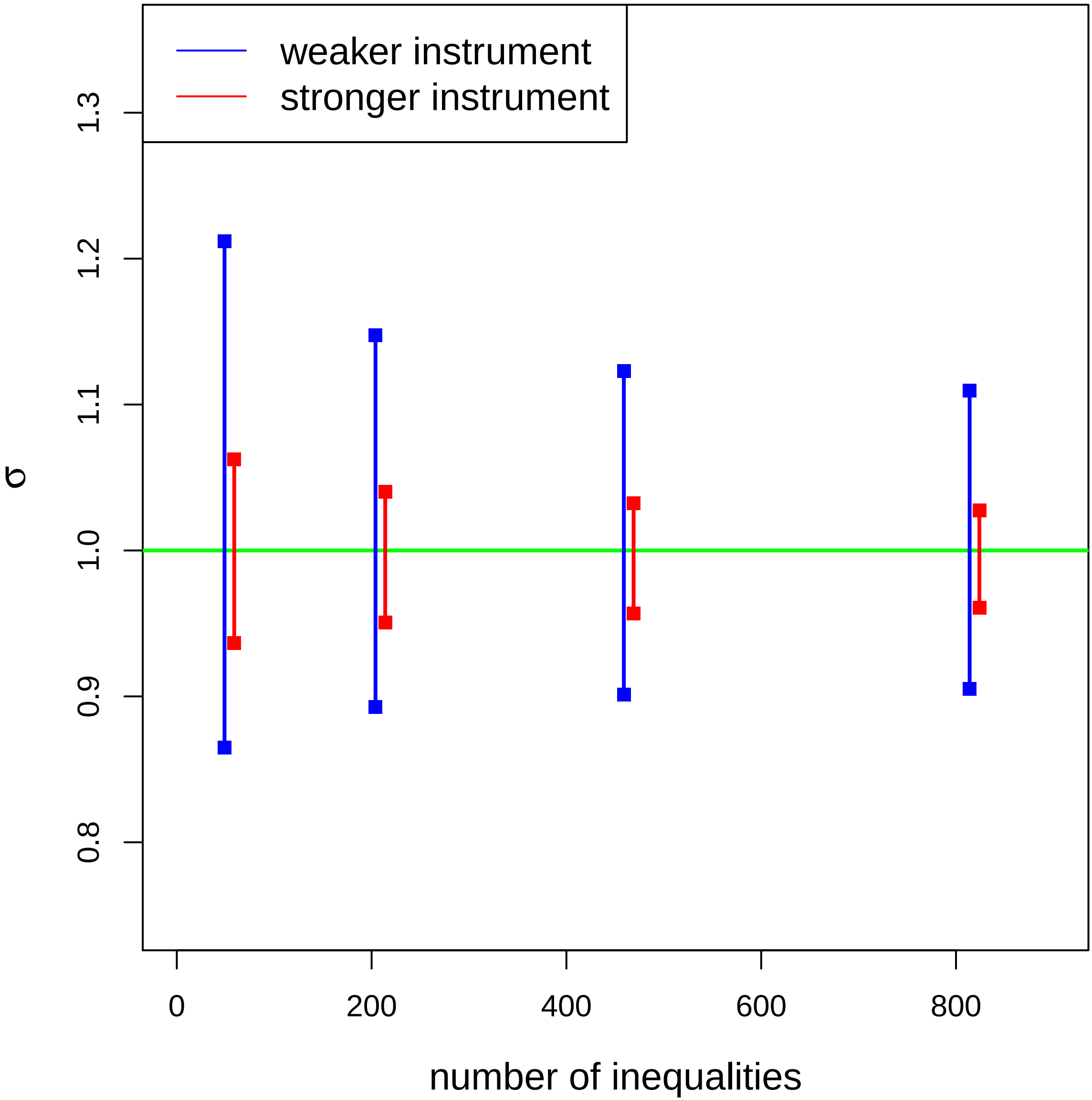
  and in the "data generating process" $(\beta_0, \sigma) = (0, 1)$.

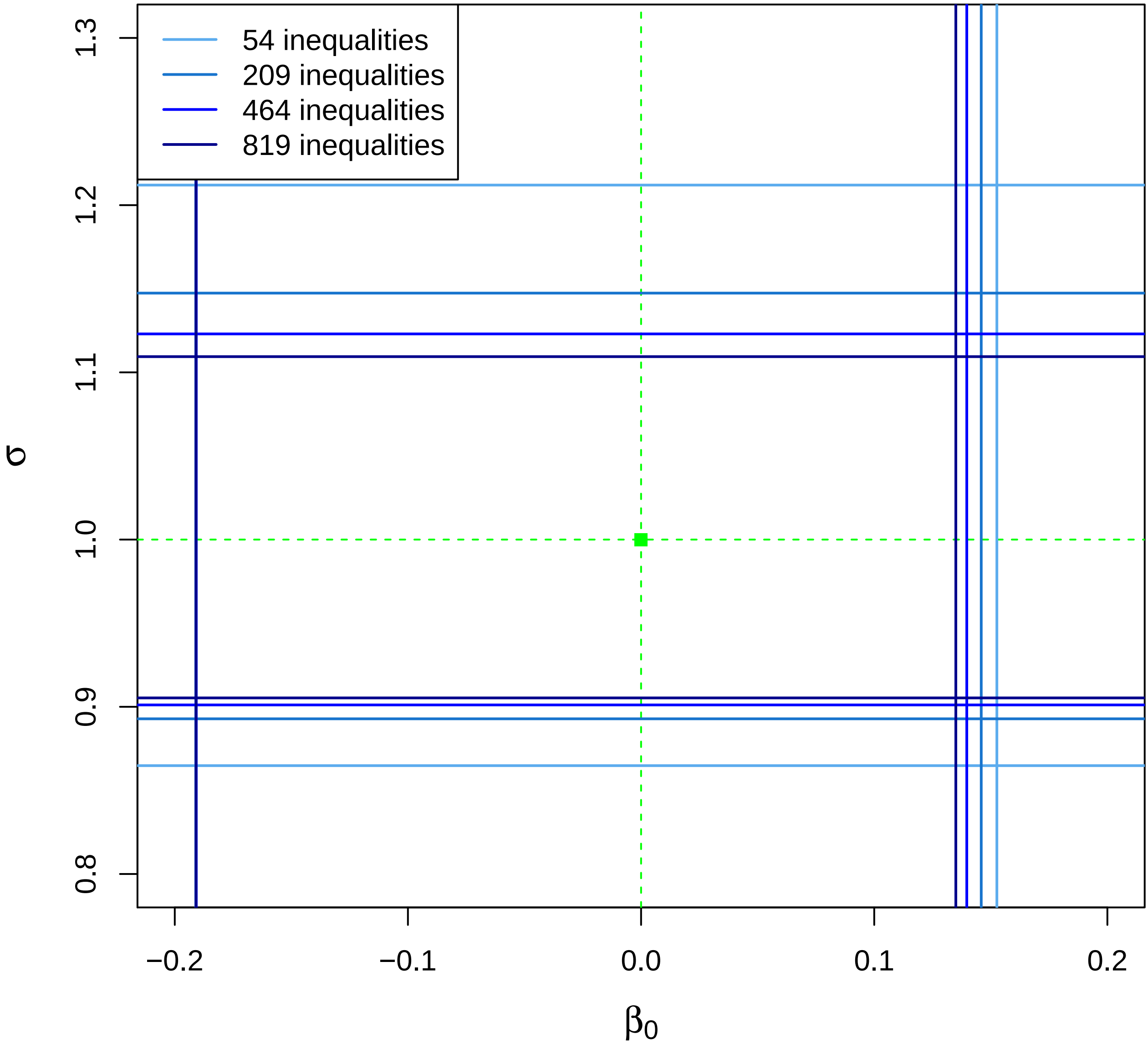
- We calculate outer regions for (β_0, σ) using collections of intervals with $M \in \{10, 20, 30, 40\}$.



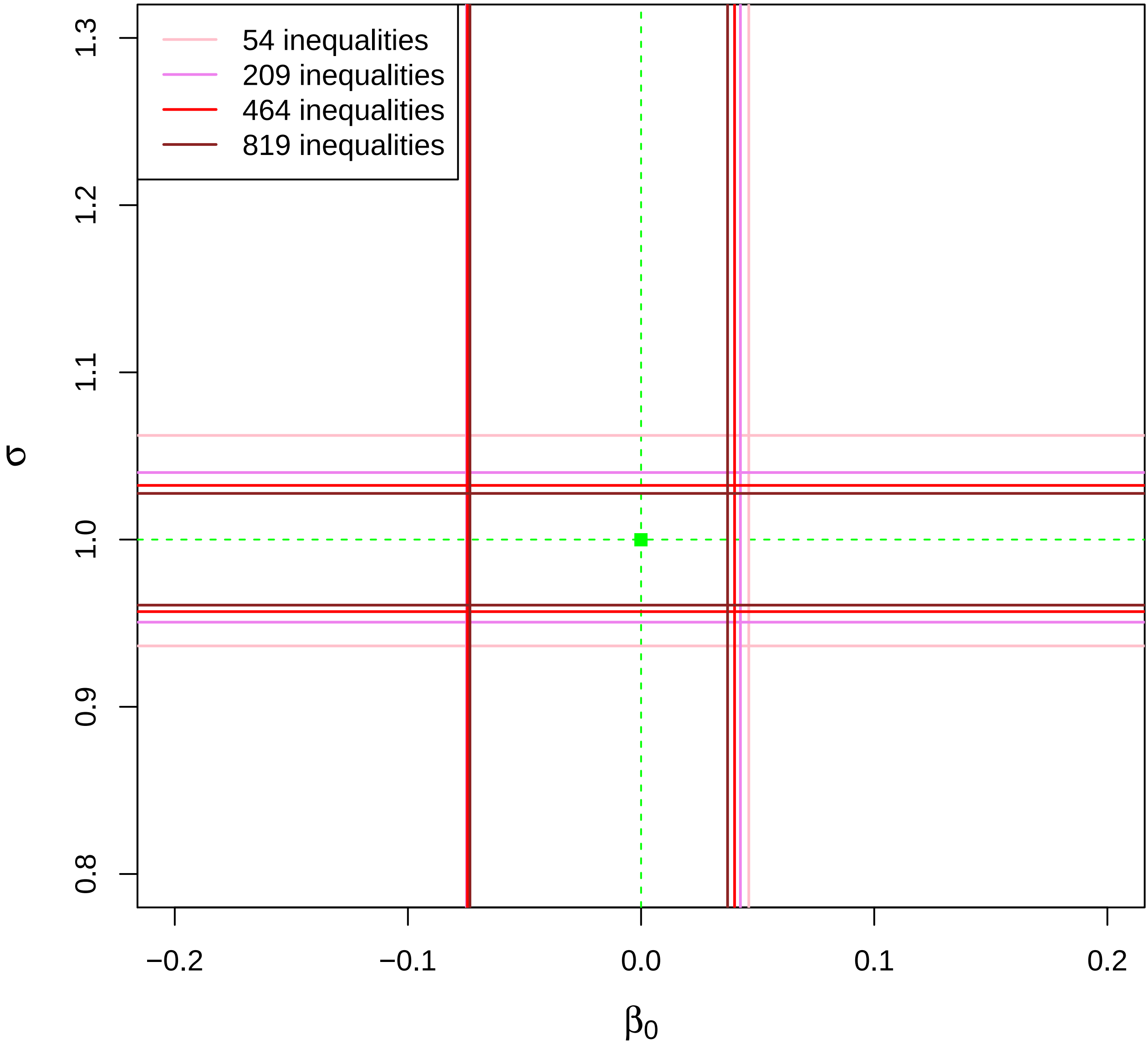
weaker instrument
stronger instrument
0.3
0.2
0.1
0.0
−0.1
−0.2
−0.3
β_0
0
200
400
600
800
number of inequalities



weaker instrument
stronger instrument
σ
1.3
1.2
1.1
1.0
0.9
0.8
0
200
400
600
800
number of inequalities



54 inequalities
209 inequalities
464 inequalities
819 inequalities
σ
β_0
1.3
1.2
1.1
1.0
0.9
0.8
−0.2
−0.1
0.0
0.1
0.2



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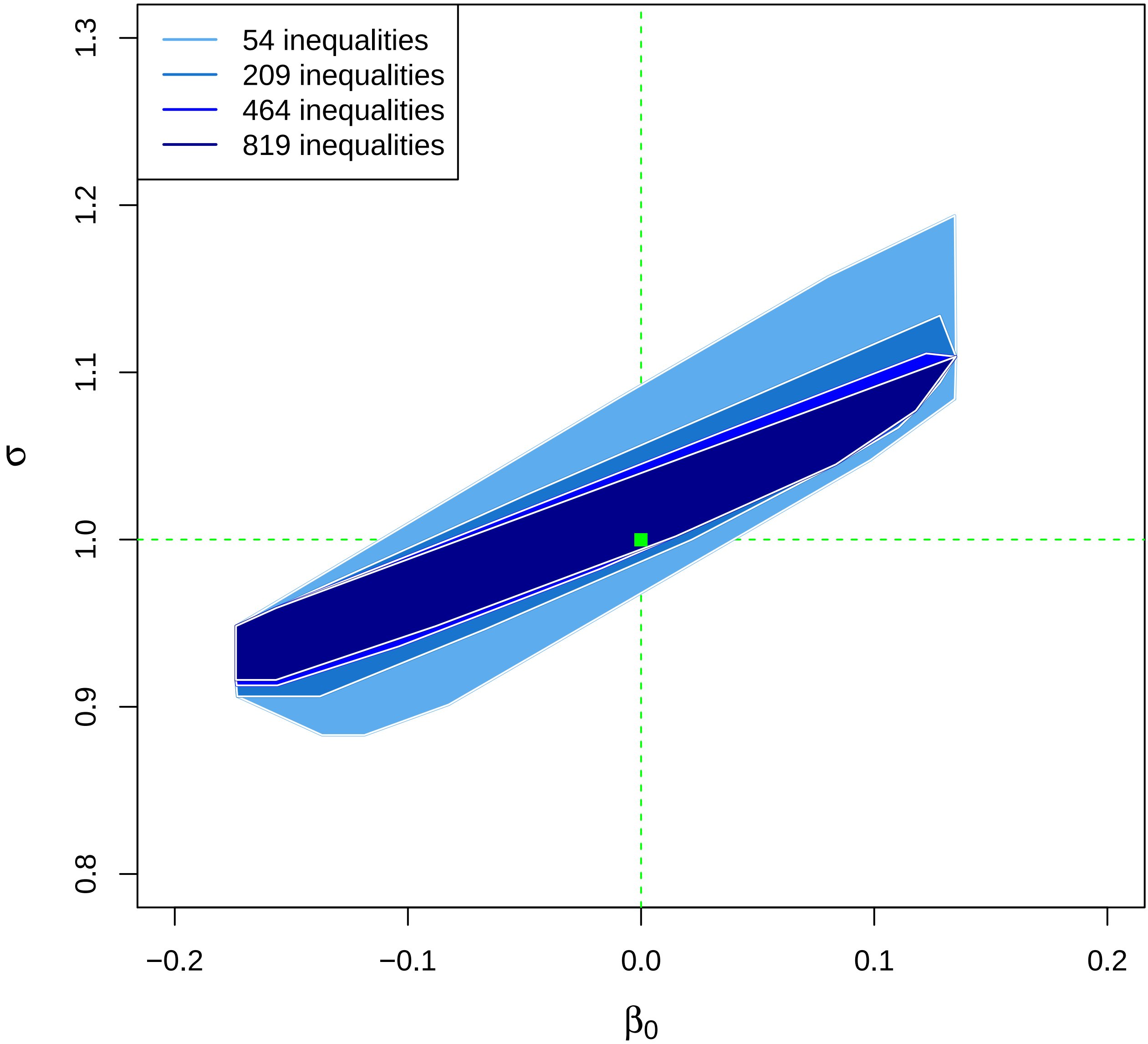

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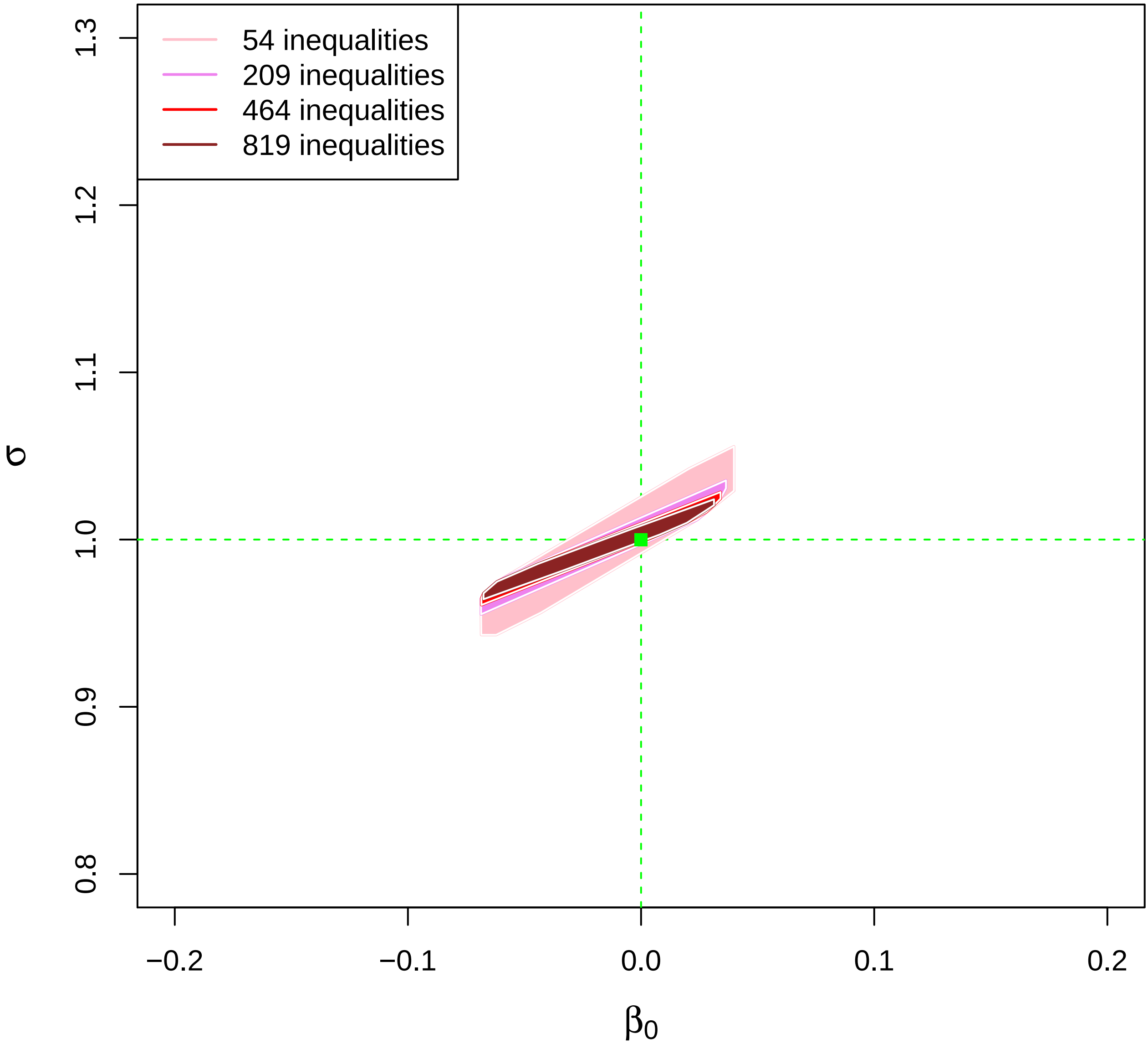

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Remarks

- We provide characterizations of sharp identified sets for a broad class of incomplete models with set-valued residuals.
- Opens the door to application of *weakly restrictive* models with discrete outcomes, high dimensional heterogeneity, random coefficients, structural relationships defined by inequalities.
- When outcomes have finite support estimation and inference as in Andrews and Shi (Ecta, 2013), Chernozhukov, Lee and Rosen (Ecta, 2013), Lee, Song, Whang (2013, JoEct), Armstrong (2012, working paper), Chetverikov (2012, working paper) is applicable.
- With continuous outcomes identified sets are characterized by an *uncountable* number of moment inequalities.
- Challenges for implementation and asymptotic theory:
  - can data be informative about how many and which inequalities to use in practice?
  - too few or bad choices deliver excessively large outer regions
  - too many and in finite samples there may be low quality inference.