Unified View of Portmanteau Tests for General Statistical Models

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(Ⅰ) **Systematic Approach for Portmanteau Tests in View of the Whittle Likelihood Ratio**

by Taniguchi – Amano (2009, JJSS)

(Ⅱ) **A Unified View of Portmanteau Test for Diagnostic Checking**

by Odashima – Taniguchi (2014) (submitted)

(I)

- <u>Box－Pierce (1970)</u> :　For ARMA (p, q),

$$T_{BP} \equiv n \sum_{k=1}^{m} \hat{r}_k^2$$

(Portmanteau Test)

where $\widehat{r_k} = SACF(k)$ of estimated residual process , and n is the sample size .

- Box－Pierce suggested

$$T_{BP} \sim \chi^2_{m-p-q}$$

Asymptotically ,

"if m and n are moderately large ."

- Davies et al（1977） : The approximation

$$T_{BP} \sim \chi^2_{m-p-q}$$ is not adequate !

even for moderately large n & $m = 20$

- Ljung and Box (1978) proposed an improved version of T_{BP} :

$$T_{LB} = n(n+2)\sum_{k=1}^{m}(n-k)^{-1}\hat{r}_k^2$$ (Ljung - Box test)

- However , Ansley and Newbold (1979) reported that the asymptotic significance level by T_{LB} yields a serious understatement .

Various modified versions of T_{BP} & T_{LB} :

- Peña and Rodríguez (2002)
- Lobato et al (2001)
- Li (2004)
- Franq et al (2005)
- Hipel and Mcleod (2005)
- Arranz (2005)
- Katayama (2008)

In this talk , we show that :

(i) T_{BP} is understood to be a special form of Whittle likelihood ratio test T_{PW}

(ii) $For\ any\ \text{``}finite\ \ m\text{''},\ T_{PW}\ \nrightarrow \chi^2\ \ in\ \ distribution.$

(iii) If we assume Bloomfield's exponential spectral density, then

$$T_{PW} \sim \chi^2$$

For general linear models, it is shown that

$$T_{PW} \not\sim \chi^2 \quad \text{in general}$$

Then

(**iv**) we propose a modified version T^*_{PW} of T_{PW}, which satisfies

$$\boxed{T^*_{PW} \sim \chi^2}$$

(**v**) and evaluate the <u>*local power of* T^*_{PW}</u>

- Portmanteau Tests in View of Whittle LR test

Suppose

$$\{X_t\} \sim ARMA(p,q)$$

$$\begin{cases} \sum_{j=0}^{p} \alpha_j X_{t-j} = \sum_{j=0}^{q} \beta_j u_{t-j} & (\alpha_0 = \beta_0 = 1, \alpha_p \neq 0, \beta_q \neq 0) \\ u_t = \sum_{j=-m}^{m} \theta_{2,j} \varepsilon_{t-j} & (\theta_{2,0} = 1, \theta_{2,-j} = \theta_{2,j}) \end{cases}$$

Write

$$\theta_1 = (\theta_{1,1}, \cdots, \theta_{1,p+q})' = (\alpha_1, \cdots, \alpha_p, \beta_1, \cdots, \beta_q)'$$

$$\theta_2 = (\theta_{2,1}, \cdots, \theta_{2,m})'$$

$$\theta = (\theta_1', \theta_2')'$$

- Consider the problem of testing

$$\mathrm{H} : \theta_2 = 0 \qquad \text{vs} \qquad \mathrm{A} : \theta_2 \neq 0$$

- $\boldsymbol{X} = (X_1, \cdots, X_n)'$: observed stretch

- $I_n(\lambda) = \frac{1}{2\pi n} \left| \sum_{t=1}^{n} X_t \, e^{it\lambda} \right|^2$: periodogram

- $f_\theta(\lambda) \equiv f_{(\theta_1, \theta_2)}(\lambda) = \frac{\left| \sum_{j=0}^{q} \beta_j e^{ij\lambda} \right|^2}{\left| \sum_{j=0}^{p} \alpha_j e^{ij\lambda} \right|^2} \times \frac{\sigma_u^2}{2\pi} \left\{ \sum_{j=-m}^{m} \theta_{2,j} \, e^{-ij\lambda} \right\}$

- $D(f_\theta, I_n) = -\frac{1}{4\pi}\int_{-\pi}^{\pi}\left\{log f_\theta(\lambda) + \frac{I_n(\lambda)}{f_\theta(\lambda)}\right\} d\lambda$ （Whittle Likelihood）

Define

$$\widehat{\theta}_1 = arg\ \max_{\theta_1}\ D(f_{(\theta_1,0)}, I_n)$$

$$\widehat{\theta}_2(\widehat{\theta}_1) = arg\ \max_{\theta_2}\ D(f_{(\widehat{\theta}_1,\theta_2)}, I_n)$$

$$\mathrm{T}_{PW} = 2n\left[D\left(f_{(\widehat{\theta}_1,\widehat{\theta}_2(\widehat{\theta}_1))}, I_n\right) - D\left(f_{(\widehat{\theta}_1,0)}, I_n\right)\right]$$

We call T_{PW} a portmanteau test of Whittle type

Theorem 1 Under $H : \theta_2 = 0$,

(i) $T_{PW} - T_{BP} \xrightarrow{P} 0$ & $T_{PW} - T_{LB} \xrightarrow{P} 0 \quad as \quad n \longrightarrow \infty$

(ii) $For\ any\ fixed \quad m \equiv dim\theta_2,$

$$T_{PW} \nrightarrow \chi^2_{m-p-q} \qquad \text{in distribution}$$

- ARMA $\longrightarrow$ General Linear Process

Suppose

$$X_t = \sum_{j=0}^{\infty} a_j(\theta_1) u_{t-j} \ , \qquad \sum_{j=0}^{\infty} a_j(\theta_1)^2 < \infty, \quad \theta_1 = (\theta_{1,1}, \cdots, \theta_{1,r})'.$$

- *Suppose that* $\{X_t\}$ has the spectral density

$$f_\theta(\lambda) \equiv f_{(\theta_1,\theta_2)}(\lambda) = \left|\sum_{j=0}^{\infty} a_j(\theta_1)e^{ij\lambda}\right|^2 \times \frac{\sigma_n^2}{2\pi}\left\{\sum_{j=-m}^{m}\theta_{2,j}e^{-ij\lambda}\right\}$$

$$= \quad g_{\theta_1}(\lambda) \times h_{\theta_2}(\lambda) \qquad (say)$$

where $\theta_2 = (\theta_{2,1}, \cdots, \theta_{2,m})'.$

- $\mathrm{H_G} : \theta_2 = 0 \quad \text{vs} \quad \mathrm{A_G} : \theta_2 \neq 0$

- $\mathrm{F} \equiv \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{\partial}{\partial\theta} \, log \; f_\theta(\lambda) \cdot \frac{\partial}{\partial\theta'} \, log \; f_\theta(\lambda) = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \quad ; \; nonsingular$

Theorem 2

$Assume \quad m = dim\ \theta_2 > r = dim\ \theta_1. \quad Then,\ under\ \ \mathrm{H_G},$

$if\ \ \mathrm{F_{21}F_{11}^{-1}F_{12}}\ \ is\ idempotent\ with\ \ rank = r,$

$$\mathrm{T}_{PW} \xrightarrow{d} \chi^2_{m-r} \quad as\ \ n \to \infty$$

$Corollary, \quad If$

$$g_{\theta_1}(\lambda) = \frac{\sigma^2}{2\pi}\ exp\left[\sum_{j=0}^{r} \theta_{1,j}\ cos\ j\lambda\right], (\theta_{1,0} = 1) \qquad \underline{exponential\ model}$$

$$\Rightarrow \quad \boxed{\mathrm{T}_{PW} \xrightarrow{d} \chi^2_{m-r} \qquad under\ \ \mathrm{H}_G}$$

- However, in general, $\mathrm{T}_{PW} \nrightarrow \chi^2_{m-r}$

- In the case of $ARMA(p,q)$, Katayama(2008) proposed a $modified\ statistic\ T^*_{PW}$ of T_{PW} such that $T^*_{PW} \to \chi^2_{m-p-q}$.

- For the general linear process, such a modification is possible.

- Let $\widetilde{F_{ij}} \equiv F_{ij}\Big((\widehat{\theta}_1, \widehat{\theta}_2(\widehat{\theta}_1))\Big)$ & $\widetilde{W} \equiv \widetilde{F_{21}}\ \Big(\widetilde{F_{12}}\widetilde{F_{21}}\Big)^{-1}\ \widetilde{F_{12}}$, and

$$\mathrm{T}^+_{PW} \equiv T_{PW} - n\widehat{\theta}_2(\widehat{\theta}_1)'\widetilde{W}\widehat{\theta}_2(\widehat{\theta}_1)\ .$$

Theorem 3 *Assume* $m > r$. *Then, under* $\mathrm{H_G}$,

$$\mathrm{T}^+_{PW} \xrightarrow{d} \chi^2_{m-r} \quad as \quad n \to \infty$$

- Local Power Properties

Testing Problem $\mathrm{H}_G : \theta_2 = 0 \quad vs \quad A_G : \theta_2 \neq 0$

- Local Alternative

$$\mathrm{A}_G^{(n)} : \theta_2 = \frac{1}{\sqrt{n}} h$$

Theorem 4 $\quad Under \quad \mathrm{A}_{\mathrm{G}}^{(\mathrm{n})},$

$$\mathrm{T}_{PW}^{+} \xrightarrow{d} \chi_{m-r}^{2}\{C\} \qquad (\text{noncentral } \chi^2)$$

$$where \quad \mathrm{C} = \mathbf{I}_{\mathrm{m}} - \mathrm{F}_{21}(\mathrm{F}_{12}\mathrm{F}_{21})^{-1}\mathrm{F}_{12}\ ,$$

- Another Tests

Let $$(\widetilde{\theta}_1, \widetilde{\theta}_2) = arg \max_{(\theta_1,\theta_2)} D(f_{(\theta_1,\theta_2)}, I_n)$$

- $$\mathrm{T}_{WLR} \equiv 2n \left[D(f_{(\widetilde{\theta}_1,\widetilde{\theta}_2)}, I_n) - D(f_{(\widehat{\theta}_1,0)}, I_n) \right]$$ *(Whittle likelihood ratio)*

- $$LM = n \left\{ \frac{\partial}{\partial\theta} l(\widehat{\theta}_1, 0) \right\}' \left[E\left\{ -\frac{\partial^2}{\partial\theta\partial\theta'} l(\widehat{\theta}_1, 0) \right\} \right]^{-1} \left\{ \frac{\partial}{\partial\theta} l(\widehat{\theta}_1, 0) \right\}$$ *(Lagrange multiplier test)*

l : log - likelihood