Self-normalized Extreme Eigenvalues of Large Dimensional Covariance Matrices of Heavy-Tailed Multivariate Time Series

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Game Plan

- The Setup
- Background
- Main result
  - Corollaries and applications
  - Elements of the proof I
  - Elements of the proof II
  - The separable case
- Extension to nonlinear models–stochastic volatility and GARCH(1,1)

The Setup

- Data matrix: A $p \times n$ matrix X consisting of n observations of a p-dimensional time series, i.e.,

$$X = \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1n} \\ X_{21} & X_{22} & \cdots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{p1} & X_{p2} & \cdots & X_{pn} \end{bmatrix}.$$

- Sample covariance matrix: the $p \times p$ sample covariance matrix (normalized) is given by

$$XX^T = n\hat{\Gamma}(0) = \left[\sum_{t=1}^{n} X_{it} X_{jt} \right]_{i,j=1}^{p}.$$

- Objective: study the ordered eigenvalues

$$\lambda_{(1)} \geq \lambda_{(2)} \geq \ldots \geq \lambda_{(p)}$$

of the $p \times p$ sample covariance matrix XX^T.

The Setup-continued

Data matrix and sample covariance matrix:

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- Note that if the rows are independent and identically distributed ergodic time series (with mean 0 and variance 1), then for p fixed,

$$\hat{\Gamma}(0) \xrightarrow{P} I_p \,.$$

- Relation to PCA: $\lambda_{(1)}$ is the empirical variance of the first principal component, $\lambda_{(2)}$ of the second, and so on.

Known results for the largest eigenvalue

- Assume the entries of X are iid Gaussian (with mean zero and variance one)
- For $n \to \infty$ and fixed p, Anderson [1963] proved that

$$\sqrt{\frac{n}{2}}\left(\frac{\lambda_{(1)}}{n} - 1\right) \xrightarrow{d} \mathrm{N}(0,1)\,.$$

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- Johnstone [2001] showed that for $p, n \to \infty$ s.t. $p/n \to \gamma \in (0,\infty)$

$$\frac{\sqrt{n} + \sqrt{p}}{\sqrt[3]{\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{p}}}}\left(\frac{\lambda_{(1)}}{\left(\sqrt{n} + \sqrt{p}\right)^2} - 1\right) \xrightarrow{d} \text{Tracy-Widom distribution}$$

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- The assumption of Gaussianity in Johnstone's result can be relaxed to a *moment condition* (c.f. Four Moment Theorem by Tao and Vu [2011]; and work by Erdös, Johansson, Péché, Schlein, Soshnikov, Yau and others).

Setting

- Suppose $X = (X_{it})_{i,t}$, $i = 1, \ldots, p$, $t = 1, \ldots, n$, with

$$X_{it} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} h(k,j) Z_{i-k,t-j} \,.$$

- The noise $(Z_{i,t})$ is iid with regularly varying tails of index $\alpha \in (0,4)$ (infinite fourth moment), i.e.,

$$n\, P(|Z_{11}| > a_n x) \to x^{-\alpha} \text{ as } n \to \infty, \text{ for } x > 0,$$

($a_n = L(n) n^{1/\alpha}$) and

$$\lim_{x\to\infty} \frac{P(Z_{11} > x)}{P(|Z_{11}| > x)} = p_+ \quad \text{and} \quad \lim_{x\to\infty} \frac{P(Z_{11} \le -x)}{P(|Z_{11}| > x)} = 1 - p_+$$

Conditions on h

Summability assumptions on $h(k,l)$:

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}|h(k,j)|^{\delta}<\infty \quad \text{for some } \delta<\min\{1,\alpha\}$$

and

$$\sum_{t=0}^{\infty}\left(\sum_{j=t}^{\infty}|h(k,j)|\right)^{\alpha/2-\epsilon}<\infty, \quad \text{for } k=0,1,2\ldots,.$$

Note: latter condition is implied by

$$\sum_{j=0}^{\infty}j^{2/\alpha+\epsilon'}\,|h(k,j)|<\infty\,, \quad k=0,1,\ldots,,$$

for $\epsilon'>0$ arbitrarily close to zero.

Setting (cont)

- Let $\lambda_1, \ldots, \lambda_p$ be the eigenvalues of

$$\begin{cases} XX^T, & \text{if } \alpha \in (0,2), \\ XX^T - E(XX^T), & \text{if } \alpha \in (2,4). \end{cases}$$

- Let (D_s) be the iid sequence given by

$$D_s = D_s^{(n)} = \sum_{t=1}^{n} Z_{s,t}^2 \,.$$

Note:

1. The D_s play a key role in determining the asymptotic properties of the ordered eigenvalues $\lambda_{(1)} \geq \cdots \geq \lambda_{(p)}$.
2. Large deviations result implies $pP(D_1 \geq a_{np}^2 x) \to x^{-\alpha/2}$ for $\alpha \in (0,2)$. (Mean correct D_1 for $\alpha \in (2,4)$.)

One more thing!

Set $\mathbf{h}_i = (h_{i0}, h_{i1}, \ldots)^T$ and define the matrix $H = (\mathbf{h}_0, \mathbf{h}_1, \ldots,)$. Let

$$M = H^T H.$$

i.e., the (i, j)th entry of M is

$$M_{ij} = \mathbf{h}_i^T \mathbf{h}_j = \sum_{l=0}^{\infty} h_{il} h_{jl}, \quad i, j = 0, 1, \ldots, .$$

By construction, M is symmetric and non-negative definite and has ordered eigenvalues

$$v_1 \geq v_2 \geq v_3 \geq \cdots$$

Let $r \leq \infty$ be the rank of M so that $v_r > 0$ while $v_{r+1} = 0$ if $r < \infty$.

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Remark: M is the covariance matrix of the vector $\mathbf{X}^* = (X_0^*, X_1^*, \ldots)^T$,

$$X_i^* = \sum_{l=0}^{\infty} h(i, l) Z_l, \quad \{Z_l\} \sim \text{IID}(0, 1)$$

Example

$$X_{i,t} = Z_{i,t} + Z_{i,t-1} - (2Z_{i-1,t} - 2Z_{i-1,t-1})$$

$$H^T = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots \\ -2 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ldots \end{pmatrix} \quad M = H^T H = \begin{pmatrix} 2 & 0 & 0 & \cdots \\ 0 & 8 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ldots \end{pmatrix}$$

which has non-negative eigenvalues $v_1 = 8$ and $v_2 = 2$ ($r = 2$).

Theorem (Main result to the point process convergence)

Let $p = p_n \to \infty$ be a sequence satisfying certain growth conditions (to be specified later) and suppose $k = k_p \to \infty$ is any sequence such that $k^2 = o(p)$.

a) *If $\alpha \in (0,2)$, then*

$$a_{np}^{-2} \max_{i=1,\dots,p} \left| \lambda_{(i)} - \delta_{(i)} \right| \xrightarrow{P} 0\,, \quad n \to \infty\,,$$

where

- $\lambda_{(1)} \ge \cdots \ge \lambda_{(p)}$ *are the ordered eigenvalues of* XX^T.
- $\delta_{(1)} \ge \cdots \ge \delta_{(p)}$ *are the ordered values from the set* $\{D_{(i)} v_j,\ i = 1, \dots, k,\ j = 1, 2, \dots, \}$.

Note: $\delta_{(1)} = v_1 D_{(1)},\ \delta_{(2)} = v_2 D_{(1)} \vee v_1 D_{(2)}$, etc.

Theorem (Main result cont)

b) *If* $\alpha \in (2,4)$*, then*

$$a_{np}^{-2} \max_{i=1,\dots,p} \left| \widetilde{\lambda}_{(i)} - \widetilde{\delta}_i \right| \xrightarrow{P} 0\,, \quad n \to \infty\,,$$

where

- $\tilde{\lambda}_{(1)}, \dots, \tilde{\lambda}_{(p)}$ *are the ordered eigenvalues* (λ_i) *according to their absolute values.*
- $\tilde{\delta}_{(1)} \geq \cdots \geq \tilde{\delta}_{(p)}$ *are the ordered values from the set* $\{(D_{l_i} - ED)v_j,\ i = 1, \dots, k,\ j = 1, 2, \dots, \}$.

Theorem (Point process convergence)

Let $p = p_n \to \infty$ be a sequence satisfying certain growth conditions (to be specified later). Then we have the point process convergence,

$$N_p := \sum_{i=1}^{p} \epsilon_{a_{np}^{-2}\lambda_i} \xrightarrow{d} N = \sum_{j=1}^{r} \sum_{i=1}^{\infty} \epsilon_{v_j \Gamma_i^{-2/\alpha}} \,,$$

where $\Gamma_i = E_1 + \ldots + E_i$ is the cumulative sum of iid standard (i.e., mean one) exponentially distributed rv's,

Note: The point process $N^* = \sum_{i=1}^{\infty} \epsilon_{\Gamma_i^{-2/\alpha}}$ is a Poisson process with $E(N^*(dx)) = \alpha/2x^{-\alpha/2-1}dx$.

The largest eigenvalues

Let $d_{(1)} \geq d_{(2)} \geq \cdots$ be the ordered values of the set

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- The theorem implies the joint convergence of the m-largest eigenvalues

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$$\frac{\lambda_{(1)}}{\lambda_{(1)} + \cdots + \lambda_{(m)}} \xrightarrow{d} \frac{v_1 \Gamma_1^{-2/\alpha}}{d_{(1)} + \cdots + d_{(m)}}, \quad n \to \infty.$$

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In fact, we have more!!

Self-normalization

Under the conditions of the theorem, the following limit results hold.

1. If $\alpha \in (0,2)$, then

$$a_{np}^{-2}\Big(\lambda_{(1)}, \sum_{i=1}^{p} \lambda_i\Big) \stackrel{d}{\to} \Big(\Gamma_1^{-2/\alpha}, \sum_{j=1}^{r}\sum_{i=1}^{\infty} v_j \Gamma_i^{-2/\alpha}\Big),$$

   and in particular,

$$\frac{\lambda_{(1)}}{\lambda_1 + \cdots + \lambda_p} \stackrel{d}{\to} \frac{v_1}{v_1 + \ldots + v_r} \frac{\Gamma_1^{-2/\alpha}}{\sum_{i=1}^{\infty} \Gamma_i^{-2/\alpha}}, \quad n \to \infty .$$

2. If $\alpha \in (2,4)$ then

$$\frac{\lambda_{(1)}}{\lambda_1 + \cdots + \lambda_p} \stackrel{d}{\to} \frac{v_1}{v_1 + \ldots + v_r} \frac{\Gamma_1^{-2/\alpha}}{\xi_{\alpha/2}}, \quad n \to \infty ,$$

   where

$$\xi_{\alpha/2} = \lim_{\gamma \downarrow 0} \sum_{i=1}^{\infty} \Big(\Gamma_i^{-2/\alpha} I_{\{\Gamma_i^{-2/\alpha} > \gamma\}} - E\Gamma_i^{-2/\alpha} I_{\{\Gamma_i^{-2/\alpha} > \gamma\}}\Big)$$

Example (cont)

Model: $X_{i,t} = Z_{i,t} + Z_{i,t-1} - (2Z_{i-1,t} - 2Z_{i-1,t-1})$

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Then,

$$N_p := \sum_{i=1}^{p} \epsilon_{a_{np}^{-2}\lambda_i} \xrightarrow{d} N = \sum_{i=1}^{\infty} \left(\epsilon_{8\Gamma_i^{-2/\alpha}} + \epsilon_{2\Gamma_i^{-2/\alpha}} \right).$$

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Results:

- $a_{np}^{-2}\lambda_{(1)} \xrightarrow{d} 8\Gamma_1^{-2/\alpha}$

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- $a_{np}^{-2}(\lambda_{(1)}, \lambda_{(2)}) \xrightarrow{d} (8\Gamma_1^{-2/\alpha}, 2\Gamma_1^{-2/\alpha} \vee 8\Gamma_2^{-2/\alpha})$

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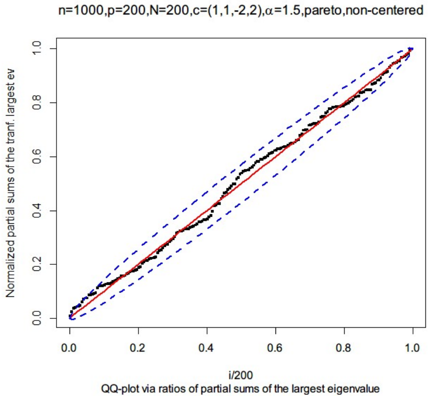
- $$\frac{\lambda_{(1)}}{\lambda_1 + \cdots + \lambda_p} \xrightarrow{d} \frac{8}{10} \frac{\Gamma_1^{-2/\alpha}}{\sum_{i=1}^{\infty} \Gamma_i^{-2/\alpha}}, \quad n \to \infty.$$

QQ-Plot via ratio of partial sums to $\lambda_{(1)}$

Model: $X_{i,t} = Z_{i,t} + Z_{i,t-1} - (2Z_{i-1,t} - 2Z_{i-1,t-1})$, Pareto noise wih $\alpha = 1.5$ and $\alpha = 3.0$, replications = 200



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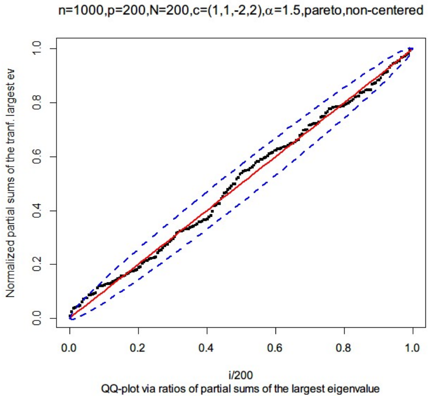

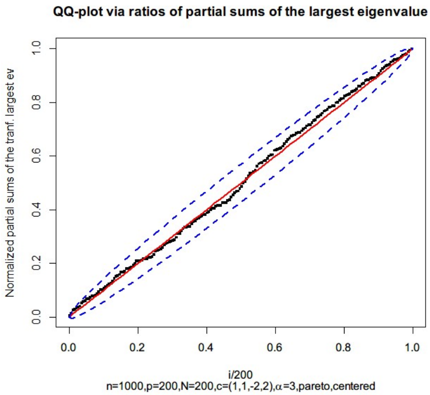

Example: Ratio of largest to second largest, $\lambda_{(1)}/\lambda_{(2)}$:

Recall:

$$\frac{\lambda_{(1)}}{\lambda_{(2)}} \xrightarrow{d} \begin{cases} 4, & \text{if } 8\Gamma_2^{-2/\alpha} < 2\Gamma_1^{-2/\alpha}, \\ \dfrac{\Gamma_1^{-2/\alpha}}{\Gamma_2^{-2/\alpha}}, & \text{otherwise} \end{cases}$$

It follows that

$$\begin{aligned} \lim_{n\to\infty} P(\lambda_{(1)} = 4\lambda_{(2)}) &= P(2\Gamma_1^{-2/\alpha} > 8\Gamma_2^{-2/\alpha}) \\ &= P(\frac{E_1}{E_1+E_2} < 2^{-\alpha}) = 2^{-\alpha} = .354(\alpha = 1.5) \end{aligned}$$

and

$$\lim_{n\to\infty} P(\lambda_{(1)} = 4\lambda_{(2)} | \lambda_{(1)} > a_{np}^2 x) = P(\frac{E_1}{E_1+E_2} < 2^{-\alpha} | E_1 < 8x^{-\alpha/2}).$$

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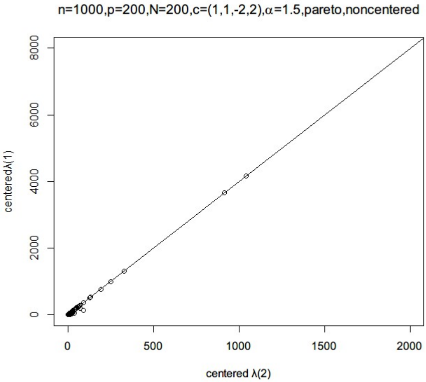

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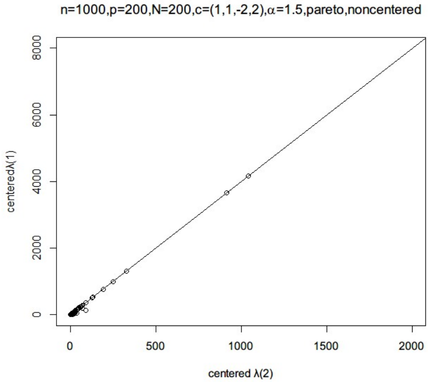

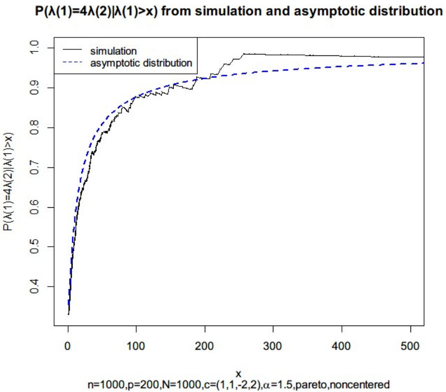

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Growth conditions on p_n

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$$P(|Z_1| > x) = L_1(x)x^{-\alpha} \quad \text{and} \quad P(|Z_1 Z_2| > x) = L_2(x)x^{-\alpha}$$

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General conditions:

- For $\alpha \in (0,1)$, $\limsup_{n\to\infty} p\,[n\,p\,P(|Z_1 Z_2| > a_{np}^2)] = 0$.
- For $\alpha \in (1,2)$, there exists $\gamma \in (\alpha, 2)$ but arbitrarily close to α such that $\limsup_{n\to\infty} p^\gamma\,[n\,p\,P(|Z_1 Z_2| > a_{np}^2)] = 0$.
- For $\alpha \in (2,4)$, there exists $\gamma \in (\alpha, 4)$ arbitrarily close to α such that $\limsup_{n\to\infty} n^{\gamma/2-1} p^\gamma [n\,p\,P(|Z_1 Z_2| > a_{np}^2)] = \infty$.

Growth conditions on p_n

Case $P(Z_1 > x) \sim cx^{-\alpha}$: Here $L_2(x) = C\log(x)$.

- For $\alpha \in (0, 2)$,

$$p_n = O(n^\beta), \quad \text{for any } \beta > 0\,.$$

  Can allow for a touch faster growth rate ($p_n = O(\exp\{c_n\})$, where $c_n^2/n \to 0$ in the $\alpha \in (0, 1)$ case.

- For $\alpha \in (2, 4)$,

$$p_n = O(n^\beta), \quad \beta \in (0, (4-\alpha)/[2(\alpha-1)])\,.$$

  This excludes the case $p_n \sim cn$.

Elements of the proof I:

Special case: $X_{i,t} = \theta_0 Z_{i,t} + \theta_1 Z_{i-1,t}$

$$\begin{aligned}\sum_{t=1}^{n} X_{it}^2 &= \sum_{t=1}^{n} \underbrace{\theta_0^2 Z_{i,t}^2 + \theta_1^2 Z_{i-1,t}^2}_{\text{tail index } \alpha/2} + 2\theta_0\theta_1 \sum_{t=1}^{n} \underbrace{Z_{i,t} Z_{i-1,t}}_{\text{tail index } \alpha} \\ &= \theta_0^2 D_i + \theta_1^2 D_{i-1} + o_p(a_{np}^2)\end{aligned}$$

and

$$\begin{aligned}\sum_{t=1}^{n} X_{it} X_{i+1,t} &= \theta_0\theta_1 \sum_{t=1}^{n} Z_{i,t}^2 + o_p(a_{np}^2) \\ &= \theta_0\theta_1 D_i + o_p(a_{np}^2)\end{aligned}$$

Elements of the proof I:

Special case: $X_{i,t} = \theta_0 Z_{i,t} + \theta_1 Z_{i-1,t}$

$$
\begin{aligned}
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&= \theta_0^2 D_i + \theta_1^2 D_{i-1} + o_p(a_{np}^2)
\end{aligned}
$$

and

$$
\begin{aligned}
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&= \theta_0\theta_1 D_i + o_p(a_{np}^2)
\end{aligned}
$$

$$
\begin{aligned}
\begin{pmatrix} \mathbf{X}_i^T \mathbf{X}_i & \mathbf{X}_{i+1}^T \mathbf{X}_i \\ \mathbf{X}_{i+1}^T \mathbf{X}_i & \mathbf{X}_{i+1}^T \mathbf{X}_{i+1} \end{pmatrix} &\approx \begin{pmatrix} \theta_0^2 & \theta_0\theta_1 \\ \theta_0\theta_1 & \theta_1^2 \end{pmatrix} D_i + \begin{pmatrix} \theta_1^2 & 0 \\ 0 & 0 \end{pmatrix} D_{i-1} \\
&\quad + \begin{pmatrix} 0 & 0 \\ 0 & \theta_0^2 \end{pmatrix} D_{i+1}
\end{aligned}
$$

The covariance matrix can be *approximated* by

$$XX^T = \sum_{i=1}^{p} D_i M_i + o_p(a_{np}^2)\,,$$

where M_i is the $p \times p$ matrix consisting of all zeros except for a 2×2 matrix,

$$M = \begin{pmatrix} \theta_0^2 & \theta_0\theta_1 \\ \theta_0\theta_1 & \theta_1^2 \end{pmatrix},$$

whose NW corner is pinned to the i^{th} position on the diagonal. For example,

$$M_1 = \begin{pmatrix} \theta_0^2 & \theta_0\theta_1 & 0 & \cdots & 0 \\ \theta_0\theta_1 & \theta_1^2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad M_2 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \theta_0^2 & \theta_0\theta_1 & \cdots & 0 \\ 0 & \theta_0\theta_1 & \theta_1^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Denote the order statistics of the D_i's by $D_{(1)} \geq D_{(2)} \geq \cdots \geq D_{(p)}$ and write $D_{L_i} = D_{(i)}$.

Then,

- $XX^T = \sum_{i=1}^{p} D_{L_i} M_{L_i} + o_p(a_{np}^2)$ in the sense that

$$a_{np}^{-2} \| XX^T - \sum_{i=1}^{p} D_{L_i} M_{L_i} \|_2 \stackrel{P}{\to} 0 \,,$$

  where

$$\|A\|_2 = \sqrt{\text{largest eigenvalue of } AA^T} \ \text{(operator 2-norm).}$$

Elements of the proof II

- For $k \to \infty$ sufficiently slow,

$$a_{np}^{-2} \left\| XX^T - \sum_{i=1}^{k} D_{L_i} M_{L_i} \right\|_2 \stackrel{P}{\to} 0 \,.$$

- Since the D_s are iid, $(L_1, \ldots, L_p)$ is a random permutation of $(1, \ldots, p)$ and hence the set $A_k = \{|L_i - L_j| > 1,\ i, j = 1, \ldots, k, i \neq j\}$ has probability converging to 1 provided $k^2 = o(p)$.

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- On the set A_k, the matrix $\sum_{i=1}^{k} D_{L_i} M_{L_i}$ is block diagonal with nonzero eigenvalues $D_{L_i} v_1,\ i = 1, \dots, k$. Here we used the fact that M_{L_i} is a rank 1 matrix with nonzero ev equal to $v_1 = \theta_0^2 + \theta_1^2$.

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- By Weyl's inequality

$$a_{np}^{-2} \max_{i=1,\ldots,k} |\lambda_{(i)} - D_{L_i} v_1| \leq a_{np}^{-2} \left\| XX^T - \sum_{i=1}^{k} D_{L_i} M_{L_i} \right\|_2 \xrightarrow{P} 0 .$$

Elements of the proof II (cont)

- Large deviations: $D_s^{(n)} = \sum_{t=1}^{n} Z_{s,t}^2$.

$$\sup_{x>b_n}\left|\frac{P(D_1 > x)}{nP(Z_1^2 > x)} - 1\right| \to 0\,,$$

  where $b_n/a_n^2 \to \infty$.

- Classical EVT plus large deviations implies:

$$\sum_{i=1}^{p} \epsilon_{a_{np}^{-2}\lambda_i} \sim \sum_{i=1}^{p} \epsilon_{a_{np}^{-2} v_1 D_i} \stackrel{d}{\to} N = \sum_{i=1}^{\infty} \epsilon_{v_1 \Gamma_i^{-2/\alpha}}\,.$$

Elements of the proof II

- Important tool: $\|A\|_2 = \sqrt{\text{largest eigenvalue of } AA^T}$ (operator 2-norm).
- Define $D \in \mathbb{R}^{p \times p}$ by $D_{ii} = (XX^T)_{ii}$ and $D_{ij} = 0$ for $i \neq j$. Then

$$a_{np}^{-2} \left\| XX^T - D \right\|_2 \xrightarrow{P} 0 \text{ as } p, n \to \infty.$$

- By Weyl's inequality

$$a_{np}^{-2} \left| \lambda_{(1)} - \max_{1 \le i \le p} \sum_{t=1}^{n} X_{it}^2 \right| \le a_{np}^{-2} \left\| XX^T - D \right\|_2 \xrightarrow{P} 0 \text{ as } p, n \to \infty$$

and likewise for $\lambda_{(2)}, \lambda_{(3)}, \ldots$

Hence, we "only" have to derive the extremal behavior of the diagonal elements $(\sum_{t=1}^{n} X_{it}^2)_i$ of XX^T.

The separable case

Suppose $h(k,l)$ is separable, i.e., $h(k,l) = \theta_k c_l$ and

$$X_{it} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \theta_k c_l Z_{i-k,t-j}\,.$$

In this case,

$$\mathbf{h}_i^T \mathbf{h}_j = \theta_i \theta_j C, \quad C = \sum_{l=0}^{\infty} c_l^2.$$

The matrix $M = H^T H$ is then rank 1 with eigenvalue $v_1 = \Theta C$ ($\Theta = \sum_{i=0}^{\infty} \theta_i^2$). Limits same as IID case, namely

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$$\frac{\lambda_{(1)}}{\lambda_1 + \cdots + \lambda_p} \xrightarrow{d} \frac{\Gamma_1^{-2/\alpha}}{\sum_{i=1}^{\infty} \Gamma_i^{-2/\alpha}} , \quad n \to \infty .$$

Stochastic volatility models—special case

Suppose the rows are independent copies of the SV process given by

$$X_t = \sigma_t Z_t$$

where (Z_t) is iid RV(α) and $(\ln \sigma_t^2)$ is a purely nondeterministic stationary Gaussian process (this can be weakened), independent of (Z_t).

Theorem *Suppose $p_n, n \to \infty$ such that*

$$\limsup_{n\to\infty} \frac{p_n}{n^\beta} < \infty\,, \quad \text{for some } \beta > 0 \text{ satisfying}$$

1. $\beta < \infty$ *if* $\alpha \in (0,1)$, *and*
2. $\beta < \frac{2-\alpha}{\alpha-1}$ *if* $\alpha \in (1,2)$.

Then, we have the point process convergence,

$$N_p := \sum_{i=1}^{p} \epsilon_{a_{np}^{-2}\lambda_{(i)}} \xrightarrow{d} N = \sum_{i=1}^{\infty} \epsilon_{\Gamma_i^{-2/\alpha}}\,.$$

Stochastic volatility models—special case

Point process convergence:

$$N_p := \sum_{i=1}^{p} \epsilon_{a_{np}^{-2}\lambda_{(i)}} \xrightarrow{d} N = \sum_{i=1}^{\infty} \epsilon_{\Gamma_i^{-2/\alpha}} .$$

Remarks:

- Proof uses a large deviation result of Davis and Hsing (1995); see also Mikosch and Wintenberger (2012).
- Likely that we can weaken the restriction on β
- Similar results hold for GARCH processes if X_t is RV(α) with $\alpha \in (0, 2)$.

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