Asymptotic Distribution of the EPMS Estimator for Financial Derivatives Pricing

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Outline

1. Introduction

- A procedure of financial derivative pricing:

1. Collect data of the underlying assets
2. Model fitting for the prices of the underlying assets under the physical measure (P-model)
3. Transform the fitted model to a risk-neutral counterpart (Q-model)
4. Compute the no-arbitrage price of a contingent claim under the Q-model

Data $\rightarrow$ P-model $\rightarrow$ Q-model $\rightarrow$ Pricing

1. Introduction (BS)

Data $\rightarrow$ P-model $\rightarrow$ Q-model $\rightarrow$ Pricing

- BS model under the P measure:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

or

$$d \log S_t = (\mu - 0.5\sigma^2)dt + \sigma dW_t$$

- Goal: Find a risk-neutral measure Q such that $\{e^{-rt}S_t\}$ is a Q-martingale.

1. Introduction (BS)

Theorem (Girsanov's Theorem)

Let W_t be a Brownian motion on a space $(\Omega, \mathcal{F}, P)$ with information set $\mathcal{F}_t$. Let

$$\Lambda_t = \exp\left\{ \int_0^t \theta(s)dW_s - \frac{1}{2}\int_0^t \theta^2(s)ds \right\}, \ t \leq T,$$

where $\theta(t)$ is a stochastic process satisfying $E^P\left\{ \exp\left(\int_0^t \theta^2(s)ds \right) \right\} < \infty$. Then,

1. Λ_t is a positive P-martingale.
2. If $dQ = \Lambda_T dP$, then $E^Q(X) = E^P(\Lambda_T X)$.
3. $W_t^Q = W_t - \int_0^t \theta(s)ds$ is a Q-Brownian motion.

1. Introduction (BS)

- In particular, if

$$\theta(s) = \frac{\mu - r}{\sigma},$$

then

$$\Lambda_t = \exp\left\{\frac{\mu - r}{\sigma} W_t - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 t\right\},\ t \le T,$$

and

$$W_t^Q = W_t + \left(\frac{\mu - r}{\sigma}\right) t$$

is a Brownian motion under Q.

- The risk-neutral Q model is

$$d \log S_t = (r - 0.5\sigma^2)dt + \sigma dW_t^Q$$

1. Introduction (BS)

- Under the Q model, $\{e^{-rt}S_t\}$ and $\{e^{-rt}f_t\}$ are both Q-martingales, where f_t denotes the value of a contingent claim at time t. Therefore,

$$f_0 = e^{-rT}E^Q(f_T),$$

where f_T is the payoff of the contingent claim.

- European call option: $f_T = (S_T - K)^+$
- Lookback call option: $f_T = S_T - \min_{0 \le t \le T} S_t$
- Up-and-in call option: $f_T = (S_T - K)^+ \delta_{\{\max_{0 \le t \le T} S_t \ge B\}}$

1. Introduction (GARCH Pricing)

Data $\rightarrow$ P-model $\rightarrow$ Q-model $\rightarrow$ Pricing

- Particular features of financial data: non-normality, heavy tail, non-constant volatility, volatility clustering, asymmetry distribution,...
- BS is not suitable to be directly used to depict the dynamics of the prices of the underlying asset.

1. Introduction (GARCH Pricing)

- Two pricing approaches have been adopted in practice: Estimation and Calibration.

  1. Estimation approach: fits the selected model (under P) to historical stock prices, transforms the model to the one defined under the risk-neutral measure Q, and finally performs the numerical evaluation of the option.

  2. Calibration approach: fits the selected model (under Q) to currently observed market prices and performs the numerical evaluation of the option.

- We demonstrate the first approach for GARCH models.

1. Introduction (GARCH Pricing)

Data $\rightarrow$ P-model $\rightarrow$ Q-model $\rightarrow$ Pricing

- Duan (1995)'s P model: $R_t = \log(S_t/S_{t-1})$

$$\begin{cases} R_t = r + \lambda\sigma_t - 0.5\,\sigma_t^2 + \sigma_t\varepsilon_t, & \varepsilon_t \sim N(0,1) \\ \sigma_t^2 = \alpha_0 + \alpha_1\sigma_{t-1}^2\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2 \end{cases}$$

- In finance, the parameter λ is often called the market price of risk or the Sharpe ratio.

- BS model:

$$d\log S_t = (\mu - 0.5\sigma^2)dt + \sigma\,dW_t$$

1. Introduction (GARCH Pricing)

Data $\rightarrow$ P-model $\rightarrow$ Q-model $\rightarrow$ Pricing

- The risk-neutral model is based on the following properties of asset prices:

  1. Martingale property under Q: $E^Q(e^{r(t+1)}S_{t+1} \mid \mathcal{F}_t) = e^{rt}S_t$.
  2. Preserving local higher moments (the type of distribution): As we are considering GARCH models with normal innovations, it suffices to preserve the local variance.

$$\mathrm{var}^Q\left(\log \frac{S_{t+1}}{S_t} \mid \mathcal{F}_{t-1}\right) = \mathrm{var}^P\left(\log \frac{S_{t+1}}{S_t} \mid \mathcal{F}_{t-1}\right).$$

1. Introduction (GARCH Pricing)

Data $\rightarrow$ P-model $\rightarrow$ Q-model $\rightarrow$ Pricing

- Duan (1995)'s P model:

$$\begin{cases} R_t = r + \lambda\sigma_t - 0.5\,\sigma_t^2 + \sigma_t\varepsilon_t, & \varepsilon_t \sim N(0,1) \\ \sigma_t^2 = \alpha_0 + \alpha_1\sigma_{t-1}^2\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2 \end{cases}$$

- The risk-neutral Q model obtained by the locally risk-neutral valuation relationship with an expected utility maximizer: let $\xi_t = \lambda + \varepsilon_t$ and then

$$\begin{cases} R_t = r - 0.5\,\sigma_t^2 + \sigma_t\xi_t, & \xi_t \sim N(0,1) \\ \sigma_t^2 = \alpha_0 + \alpha_1\sigma_{t-1}^2(\xi_{t-1} - \lambda)^2 + \beta\sigma_{t-1}^2 \end{cases}$$

- BS Q-model: $d\log S_t = (r - 0.5\sigma^2)dt + \sigma dW_t^Q$

1. Introduction (con.)

Data $\rightarrow$ P-model $\rightarrow$ Q-model $\rightarrow$ Pricing

- We don't have an explicit representation for the distribution of R_T (or S_T). Thus, the no-arbitrage price is approximated by

$$f_0 = e^{-rT} E^Q(f_T) \approx e^{-rT} \frac{1}{n} \sum_{i=1}^{n} f_T(S_{t,i}, 0 \leq t \leq T),$$

where the random paths of the stock prices $S_{t,i}$ are generated independently from the Q model for $i = 1, \ldots, n$.

- This is called the standard Monte Carlo simulation method.

1. Introduction (con.)

Data → P-model → Q-model → Pricing

- Monte Carlo simulation (MCS) is a natural tool and has been commonly used for solving this problem (Boyle, 1977; Kemna and Vorst, 1990; Duan, 1995; Boyle et al., 1997).
- The computational effort of the standard MCS usually increases dramatically if high precision in option pricing is required.

1. Introduction (con.)

Data $\rightarrow$ P-model $\rightarrow$ Q-model $\rightarrow$ Pricing

- Duan and Simonato (1998) proposed an empirical martingale simulation (EMS) to improve the efficiency of the MCS under a risk-neutral framework.

- The advantage of the EMS is that it can be easily incorporated into the widely known variance reduction procedures, such as antithetic and control-variate simulations, and it is truly simple and practically requires no additional programming efforts.

1. Introduction (con.)

Data $\rightarrow$ P-model $\rightarrow$ Q-model $\rightarrow$ Pricing
Data $\rightarrow$ P-model $\rightarrow \rightarrow \rightarrow \rightarrow \rightarrow$ Pricing

- However, to obtain the explicit expression of a risk-neutral model is not convenient when dealing with a complex model.

- This study proposes a modification of the EMS from the framework under a risk-neutral measure to a dynamic P measure. Thus, we call it EPMS.

2 EPMS

- Let Λ_T denote a Radon-Nikodým derivative of a Q measure with respect to the dynamic P measure.

- For any given payoff f_T of a contingent claim, note that

$$E^Q(f_T) = E^P(f_T \Lambda_T),$$

  where $dQ = \Lambda_T dP$.

- Generate random paths under P measure (i.e., from the physical model)

2 EPMS (con.)

1. Generate n random paths of the stock prices, $S_{t,i}$, $i = 1, 2, \ldots, n$, $t = 0, 1, \ldots, T$, from the dynamic GARCH model.

- Standard Monte Carlo simulation:

$$\begin{array}{c|ccccc} \textit{path } 1 & S_{0,1} & S_{1,1} & S_{2,1} & \cdots & S_{T,1} \\ \textit{path } 2 & S_{0,2} & S_{1,2} & S_{2,2} & \cdots & S_{T,2} \\ \vdots & & & \vdots & & \\ \textit{path } n & S_{0,n} & S_{1,n} & S_{2,n} & \cdots & S_{T,n} \end{array}$$

2 EPMS (con.)

2. The first adjusting step: Assume Λ_t being a change of measure process and denote $\Lambda_{t,i} = \Lambda_t(S_{u,i}\ ,\ 0 \le u \le t)$. Let $\tilde{\Lambda}_{0,i} = \Lambda_{0,i} = 1$ and define $\tilde{\Lambda}_{t,i}$, $i = 1, \dots, n$, iteratively by

$$\tilde{\Lambda}_{t,i} = \frac{W_i(t,n)}{W_0(t,n)},$$

where $W_i(t,n) = \frac{\tilde{\Lambda}_{t-1,i}}{\Lambda_{t-1,i}} \Lambda_{t,i}$ and $W_0(t,n) = \frac{1}{n} \sum_{i=1}^{n} W_i(t,n)$.

2 EPMS (con.)

3. The second adjusting step: Let $\tilde{S}_{0,i} = S_0$ and define the empirical martingale stock prices $\tilde{S}_{t,i}$, $i = 1, \ldots, n$, iteratively by

$$\tilde{S}_{t,i} = S_0 \frac{Z_i(t,n)}{Z_0(t,n)},$$

where $Z_i(t,n) = \frac{\tilde{S}_{t-1,i}}{S_{t-1,i}} S_{t,i}$ and
$Z_0(t,n) = \frac{e^{-rt}}{n} \sum_{i=1}^{n} Z_i(t,n) \tilde{\Lambda}_{t,i}$.

2 EPMS (con.)

MC	*path* 1	$S_{0,1}$	$(S_{1,1}, \Lambda_{1,1})$
	path 2	$S_{0,2}$	$(S_{1,2}, \Lambda_{1,2})$
	$\vdots$	$\vdots$	$\vdots$
	path n	$S_{0,n}$	$(S_{1,n}, \Lambda_{1,n})$
EPMS	*path* 1	$\tilde{S}_{0,1}$	$(\tilde{S}_{1,1}, \tilde{\Lambda}_{1,1})$
	path 2	$\tilde{S}_{0,2}$	$(\tilde{S}_{1,2}, \tilde{\Lambda}_{1,2})$
	$\vdots$	$\vdots$	$\vdots$
	path n	$\tilde{S}_{0,n}$	$(\tilde{S}_{1,n}, \tilde{\Lambda}_{1,n})$

2 EPMS (con.)

MC	*path* 1	$S_{0,1}$	$(S_{1,1}, \Lambda_{1,1})$	$(S_{2,1}, \Lambda_{2,1})$
	path 2	$S_{0,2}$	$(S_{1,2}, \Lambda_{1,2})$	$(S_{2,2}, \Lambda_{2,2})$
	$\vdots$		$\vdots$	$\vdots$
	path n	$S_{0,n}$	$(S_{1,n}, \Lambda_{1,n})$	$(S_{2,n}, \Lambda_{2,n})$
EPMS	*path* 1	$\tilde{S}_{0,1}$	$(\tilde{S}_{1,1}, \tilde{\Lambda}_{1,1})$	$(\tilde{S}_{2,1}, \tilde{\Lambda}_{2,1})$
	path 2	$\tilde{S}_{0,2}$	$(\tilde{S}_{1,2}, \tilde{\Lambda}_{1,2})$	$(\tilde{S}_{2,2}, \tilde{\Lambda}_{2,2})$
	$\vdots$		$\vdots$	$\vdots$
	path n	$\tilde{S}_{0,n}$	$(\tilde{S}_{1,n}, \tilde{\Lambda}_{1,n})$	$(\tilde{S}_{2,n}, \tilde{\Lambda}_{2,n})$

2 EPMS (con.)

MC	*path* 1	$S_{0,1}$	$(S_{1,1}, \Lambda_{1,1})$	$(S_{2,1}, \Lambda_{2,1})$	$(S_{3,1}, \Lambda_{3,1})$
	path 2	$S_{0,2}$	$(S_{1,2}, \Lambda_{1,2})$	$(S_{2,2}, \Lambda_{2,2})$	$(S_{3,2}, \Lambda_{3,2})$
	$\vdots$			$\vdots$	$\vdots$
	path n	$S_{0,n}$	$(S_{1,n}, \Lambda_{1,n})$	$(S_{2,n}, \Lambda_{2,n})$	$(S_{3,n}, \Lambda_{3,n})$
EPMS	*path* 1	$\tilde{S}_{0,1}$	$(\tilde{S}_{1,1}, \tilde{\Lambda}_{1,1})$	$(\tilde{S}_{2,1}, \tilde{\Lambda}_{2,1})$	$(\tilde{S}_{3,1}, \tilde{\Lambda}_{3,1})$
	path 2	$\tilde{S}_{0,2}$	$(\tilde{S}_{1,2}, \tilde{\Lambda}_{1,2})$	$(\tilde{S}_{2,2}, \tilde{\Lambda}_{2,2})$	$(\tilde{S}_{3,2}, \tilde{\Lambda}_{3,2})$
	$\vdots$			$\vdots$	$\vdots$
	path n	$\tilde{S}_{0,n}$	$(\tilde{S}_{1,n}, \tilde{\Lambda}_{1,n})$	$(\tilde{S}_{2,n}, \tilde{\Lambda}_{2,n})$	$(\tilde{S}_{3,n}, \tilde{\Lambda}_{3,n})$

2 EPMS (con.)

MC	$path\ 1$	$S_{0,1}$	$\cdots$	$(S_{T-1,1}, \Lambda_{T-1,1})$	$(S_{T,1}, \Lambda_{T,1})$
	$path\ 2$	$S_{0,2}$	$\cdots$	$(S_{T-1,2}, \Lambda_{T-1,2})$	$(S_{T,2}, \Lambda_{T,2})$
	$\vdots$			$\vdots$	$\vdots$
	$path\ n$	$S_{0,n}$	$\cdots$	$(S_{T-1,n}, \Lambda_{T-1,n})$	$(S_{T,n}, \Lambda_{T,n})$
$EPMS$	$path\ 1$	$\tilde{S}_{0,1}$	$\cdots$	$(\tilde{S}_{T-1,1}, \tilde{\Lambda}_{T-1,1})$	$(\tilde{S}_{T,1}, \tilde{\Lambda}_{T,1})$
	$path\ 2$	$\tilde{S}_{0,2}$	$\cdots$	$(\tilde{S}_{T-1,2}, \tilde{\Lambda}_{T-1,2})$	$(\tilde{S}_{T,2}, \tilde{\Lambda}_{T,2})$
	$\vdots$			$\vdots$	$\vdots$
	$path\ n$	$\tilde{S}_{0,n}$	$\cdots$	$(\tilde{S}_{T-1,n}, \tilde{\Lambda}_{T-1,n})$	$(\tilde{S}_{T,n}, \tilde{\Lambda}_{T,n})$

2 EPMS (con.)

4. Approximate f_0 by

$$\tilde{f}_0 = e^{-rT}\frac{1}{n}\sum_{i=1}^{n}\tilde{f}_{T,i}\tilde{\Lambda}_{T,i}, \qquad (vs. \quad f_0 = E^P(e^{-rT}f_T\Lambda_T))$$

where $\tilde{f}_{T,i} = f_T(\tilde{S}_{t,i};\ t = 0, 1, \ldots, T)$.

2 EPMS (con.)

- Both $\tilde{\Lambda}_{t,i}$ and $\tilde{S}_{t,i}$ satisfy the "empirical P-martingale property":

$$\tilde{\Lambda}_{0,i} = \frac{1}{n}\sum_{i=1}^{n} \tilde{\Lambda}_{t,i} \quad (vs. \quad \Lambda_0 = E^P(\Lambda_t)),$$

and

$$S_0 = \frac{1}{n}\sum_{i=1}^{n} e^{-rt}\tilde{S}_{t,i}\tilde{\Lambda}_{t,i} \quad (vs. \quad S_0 = E^P(e^{-rt}S_t\Lambda_t)),$$

for any integer n and $t = 1, \ldots, T$.

3 Asymptotic Results

3.1 Strong consistency of the EPMS

Theorem 1

Let $\{e^{-rt}S_t\Lambda_t\}$ be a positive P-martingale process over the time index set $\{t : t = 0, 1, \ldots, T\}$. Suppose that the payoff function, $f(S_1, \ldots, S_T)$, satisfies $\mathrm{E}^Q(|f(S_1, \ldots, S_T)|) < \infty$ and is Lipschitz continuous. Then we have

$$n^{-1}\sum_{i=1}^{n}\{f(\tilde{S}_{1,i}, ..., \tilde{S}_{T,i})\,\tilde{\Lambda}_{T,i}\} \to \mathrm{E}_0\{f(S_1, ..., S_T)\,\Lambda_T\},$$

almost surely, as $n \to \infty$, where $\tilde{\Lambda}_{t,i}$ and $\tilde{S}_{t,i}$ are generated by the EPMS.

3.1 Strong consistency of the EPMS (con.)

- For example, the payoff functions of European calls, $(S_T - K)^+$, are Lipschitz continuous.
- However, some payoff functions of contingent claims do not satisfy the Lipschitz continuity, like the digital, $f(S_T) = \delta_{\{S_T > K\}}$, and barrier $f(S_t, 0 \leq t \leq T) = (S_T - K)^+ \delta_{\{S_{\max} \geq B\}}$, options.

3.1 Strong consistency of the EPMS (con.)

- In order to accommodate the case of discontinuous payoff functions, the Lipschitz continuity is replaced by the following generic Lipschitz condition.

Definition 1

A function $f(x)$, mapping from R_+^m to R, is said to satisfy the generic Lipschitz condition if there exists $q < \infty$ such that

$$|f(x)| < q(1 + ||x||)$$

for any $x \in R_+^m$, where $||\cdot||$ stands for the Euclidean norm, and there exists a finite partition, A_ℓ, $\ell = 1, \ldots, k$, of its domain such that each A_ℓ is a connected set and $f(\cdot)$ is Lipschitz continuous over any A_ℓ.

3.1 Strong consistency of the EPMS (con.)

Theorem 2

Let $\{e^{-rt}S_t\Lambda_t\}$ be a positive P-martingale process over the time index set $\{t : t = 0, 1, \ldots, T\}$. If the payoff function, $f(S_1, \ldots, S_T)$, satisfies $E^Q(|f(S_1, \ldots, S_T)|) < \infty$ and the generic Lipschitz condition, and the multivariate distribution of $(S_1, \ldots, S_T)$ under Q has a bounded density function, then as $n \to \infty$,

$$n^{-1}\sum_{i=1}^{n}\{f(\tilde{S}_{1,i}, \ldots, \tilde{S}_{T,i})\,\tilde{\Lambda}_{T,i}\} \to \mathrm{E}_0\{f(S_1, \ldots, S_T)\,\Lambda_T\},$$

almost surely, where $\tilde{\Lambda}_{t,i}$ and $\tilde{S}_{t,i}$ are generated by the EPMS.

3.2 Asymp. dist. of the EPMS with piecewise smooth and continuous payoffs

- Let $f : \Re \to \Re$ be a piecewise smooth and continuous function:

$$f(x) = \sum_{j=1}^{m+1} f_j(x)\delta_{A_j}(x), \tag{1}$$

  where A_j's form a partition of $\Re$.

- $A_1 = (-\infty, k_1)$, $A_j = [k_{j-1}, k_j)$, for $j = 2, 3, \ldots, m+1$ and $k_{m+1} = \infty$, and $\delta_A(\cdot)$ is an indicator function.

- f_j's have continuous first order derivatives and $f_j(k_j) = f_{j+1}(k_j)$, $j = 1, \ldots, m$, to ensure the continuity of f.

3.2 Asymp. dist. of the EPMS with continuous payoffs (con.)

- We use $f'(x)$ to denote the right first derivative and write

$$f'(x) = \sum_{j=1}^{m+1} f_j'(x)\delta_{A_j}(x). \tag{2}$$

Definition 2

A function $f : \Re \to \Re$ is said to have a polynomial growth rate q if there exist a constant $C > 0$ and a positive integer q such that for any real number $x \in \Re$, $|f(x)| \leq C(1 + |x|^q)$.

3.2 Asymp. dist. of the EPMS with continuous payoffs (con.)

Theorem 3

Let the asset price S_T be a positive random variable with a continuous distribution, Λ_T be a Radon-Nikodým derivative, and the payoff function $f(S_T)$ be piecewise smooth and continuous as defined in (1). If $f'(\cdot)$ exists and has a polynomial growth rate q, $\mathrm{E}(S_T^{2(q+1)}\Lambda_T^2) < \infty$ and $\mathrm{E}(\Lambda_T^2) < \infty$, then

$$\sqrt{n}(C_{EPMS}^{(n)} - C) \xrightarrow{\mathcal{L}} \mathcal{N}(0, V), \quad \text{as } n \to \infty,$$

where C is the true derivative price, $\xrightarrow{\mathcal{L}}$ denotes convergence in distribution, and

3.2 Asymp. dist. of the EPMS with continuous payoffs (con.)

Theorem 3 (con.)

$$\begin{aligned} V &= e^{-2rT}\Big(\mathrm{Var}[f(S_T)\Lambda_T] + \mathrm{Var}[S_T\Lambda_T]\Phi^2 + \mathrm{Var}[\Lambda_T]\Psi^2 \\ &\quad -2\{\Phi\mathrm{Cov}[f(S_T)\Lambda_T, S_T\Lambda_T] + \Psi\mathrm{Cov}[f(S_T)\Lambda_T, \Lambda_T] \\ &\quad -\Phi\Psi\mathrm{Cov}[S_T\Lambda_T, \Lambda_T]\}\Big), \qquad (3) \end{aligned}$$

in which $\Phi = e^{-rT}\mathrm{E}[f'(S_T)S_T\Lambda_T]/S_0$, and
$\Psi = \mathrm{E}[f(S_T)\Lambda_T] - S_0 e^{rT}\Phi$.

3.3 Asymp. dist. of the EPMS with piecewise smooth and discontinuous payoffs

- In financial markets, there are derivative contracts with discontinuous payoffs such as the binary (digital) options.
- Yuan and Chen (2009): a conjecture for the asymptotic distribution of the EMS estimator when the payoffs are discontinuous.
- We derive the asymptotic distribution of the EPMS estimator when f is discontinuous.

3.3 Asymp. dist. of the EPMS with discontinuous payoffs (con.)

Theorem 4

Let the asset price S_T be a positive continuous random variable with density function $p(\cdot, T)$, Λ_T be a Radon-Nikodým derivative, and $f(S_T)$ be a piecewise smooth and discontinuous payoff function that jumps at k_i with jump height J_i, $i = 1, \ldots, m$. If $f'(\cdot)$ exists and has a polynomial growth rate q, $\mathrm{E}(S_T^{2(q+1)}\Lambda_T^2) < \infty$ and $\mathrm{E}(\Lambda_T^2) < \infty$, then

$$\sqrt{n}(C_{EPMS}^{(n)} - C) \xrightarrow{\mathcal{L}} \mathcal{N}(0, V^*), \quad \text{as } n \to \infty,$$

where C is the true derivative price and

3.3 Asymp. dist. of the EPMS with discontinuous payoffs (con.)

Theorem 4 (con.)

$$
\begin{aligned}
V^* \;=\; & e^{-2rT}\Big(\mathrm{Var}[f(S_T)\Lambda_T] + \mathrm{Var}[S_T\Lambda_T]\Phi^{*2} + \mathrm{Var}[\Lambda_T]\Psi^{*2} \\
& -2\{\Phi^*\mathrm{Cov}[f(S_T)\Lambda_T, S_T\Lambda_T] + \Psi^*\mathrm{Cov}[f(S_T)\Lambda_T, \Lambda_T] \\
& -\Phi^*\Psi^*\mathrm{Cov}[S_T\Lambda_T, \Lambda_T]\}\Big), \qquad (4)
\end{aligned}
$$

in which $\Phi^* = \Phi + \frac{1}{S_0 e^{rT}} \sum_{i=1}^{m} J_i\, k_i\, \Lambda_T(k_i) \times p(k_i, T)$ and $\Psi^* = \mathrm{E}[f(S_T)\Lambda_T] - S_0 e^{rT}\Phi^*$.

3.3 Asymp. dist. of the EPMS with discontinuous payoffs (con.)

- If there is no jump, then J_i's are zero and the V^* in Theorem 4 reduces to V in Theorem 3.
- If the measures P and Q coincide, i.e., $\Lambda_T = 1$, then the EPMS reduces to the EMS.

4 Simulation Study

- GARCH models with normal, shifted gamma and double exponential innovations are considered.
- The change of measure process: the Esscher transform
- The coverage rates of the EPMS price estimator are investigated when the payoff are continuous or discontinuous.

4 Simulation Study (con.)

Esscher transform

- Let the density of the log returns under the P measure be $g(x)$ and the density of the log returns under the Q measure be
$$g(x;\theta) \propto e^{\theta x} g(x)$$
- The parameter θ is chosen such that $e^{-rt}S_t$ is a Q martingale.

4.1 GARCH-N model

- Duan (1995):

$$\begin{cases} R_t = r + \lambda\sigma_t - 0.5\,\sigma_t^2 + \sigma_t\varepsilon_t, \quad \varepsilon_t \sim N(0,1) \\ \sigma_t^2 = \alpha_0 + \alpha_1\sigma_{t-1}^2\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2 \end{cases} \tag{5}$$

Proposition 1

For Model (5), the change of measure process Λ_t^{ess} derived by the Esscher transform is

$$\Lambda_t^{ess} = \prod_{k=1}^{t} \exp\{-[\lambda^2\sigma_k^2 + 2\lambda\sigma_k(R_k - \mu_k)]/(2\sigma_k^2)\},$$

where $\mu_k = r + \lambda\sigma_k - 0.5\,\sigma_k^2$.

4.1 GARCH-N model (con.)

- The risk-neutral GARCH-N counterpart of Model (5) under the Q^{ess} measure is written as follows,

$$\begin{cases} R_t = r - 0.5\,\sigma_t^2 + \sigma_t \xi_t, & \xi_t \sim N(0,1) \\ \sigma_t^2 = \alpha_0 + \alpha_1 \sigma_{t-1}^2 (\xi_{t-1} - \lambda)^2 + \beta \sigma_{t-1}^2 \end{cases} \quad (6)$$

  which is the same as Duan (1995)'s result derived from the locally risk-neutral valuation relationship with an expected utility maximizer.

4.1 GARCH-N model (con.)

- Parameter setting (Duan and Simonato, 1998): S_0=100, r=0.10 (annualized and 1 year = 365 days), $\alpha_0 = 0.00001$, $\alpha_1 = 0.20$, $\beta_1 = 0.70$, $\lambda = 0.01$, $T = 1, 3, 9$ months, $S_0/K = 0.9, 1, 1.1$, and $\sigma_1^2 = \alpha_0/(1 - \alpha_1 - \beta_1)$.

- Denote the ratios of the standard deviations obtained by the MCS, EMS and EPMS with 10,000 sample paths and 100 replications as

$$\mathrm{RS}_1 \equiv \mathrm{std.(MCS)}/\,\mathrm{std.(EPMS)}$$

and

$$\mathrm{RS}_2 \equiv \mathrm{std.(EMS)}/\,\mathrm{std.(EPMS)}.$$

4.1 GARCH-N model (con.)

Table: European call option prices and the ratios of the standard deviations, RS_1 and RS_2, for a GARCH-N model.

		Maturity = 1 Month			Maturity = 3 Months			Maturity = 9 Months		
	S_0/K	1.1	1.0	0.9	1.1	1.0	0.9	1.1	1.0	0.9
European call										
MCS	mean	9.9235	2.5352	0.1164	11.7523	5.0073	1.0616	16.7597	10.4681	5.1226
	std.	0.0536	0.0338	0.0087	0.0898	0.0665	0.0307	0.1335	0.1132	0.0837
EMS	mean	9.9208	2.5335	0.1164	11.7573	5.0108	1.0626	16.7600	10.4683	5.1230
	std.	0.0061	0.0172	0.0088	0.0162	0.0301	0.0253	0.0322	0.0470	0.0524
EPMS	mean	9.9208	2.5335	0.1163	11.7570	5.0103	1.0626	16.7596	10.4679	5.1224
	std.	0.0066	0.0169	0.0082	0.0175	0.0305	0.0238	0.0363	0.0507	0.0529
RS_1		8.1385	1.9933	1.0585	5.1406	2.1786	1.2889	3.6774	2.2334	1.5814
RS_2		0.9295	1.0133	1.0709	0.9244	0.9858	1.0619	0.8872	0.9270	0.9904

4.1 GARCH-N model (con.)

- The EPMS is comparable to the EMS and the relative efficiency of the EPMS against the MCS increases as the time to maturity increases in most cases.

4.2 GARCH-SG model

- Siu, Tong and Yang (2004):

$$\begin{cases} R_t = r + \lambda\sigma_t - 0.5\,\sigma_t^2 + \sigma_t\varepsilon_t, \;\; \varepsilon_t \sim SG(0,1,a) \\ \sigma_t^2 = \alpha_0 + \alpha_1\sigma_{t-1}^2\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2 \end{cases} \quad (7)$$

Proposition 2

For Model (7), the change of measure process Λ_t^{ess} derived by the Esscher transform is

$$\Lambda_t^{ess} = \prod_{k=1}^{t} (1 - \delta_k^* \sigma_k/\sqrt{a})^a \exp\{\delta_k^*(R_k - \mu_k + \sqrt{a}\sigma_k)\},$$

where $\mu_k = r + \lambda\sigma_k - 0.5\,\sigma_k^2$, $\delta_k^* = \sqrt{a}/\sigma_k - b_k^q$ and $b_k^q = [1 - \exp\{(\mu_k - r - \sqrt{a}\sigma_k)/a\}]^{-1}$.

4.2 GARCH-SG model (con.)

- The risk-neutral GARCH-SG counterpart of Model (7) under the Q^{ess} measure is written as follows,

$$\begin{cases} R_t = r + \lambda\sigma_t - 0.5\,\sigma_t^2 - \sqrt{a}(\sigma_t - \sigma_t^*) + \sigma_t^*\xi_t^*, \;\; \xi_t^* \sim SG(0,1,a) \\ \sigma_t^2 = \alpha_0 + \alpha_1\{\sigma_{t-1}^*\xi_{t-1}^* + \sqrt{a}(\sigma_{t-1}^* - \sigma_{t-1})\}^2 + \beta\sigma_{t-1}^2 \end{cases} \tag{8}$$

where $\sigma_t^* = \sqrt{a}/b_t^q$ and b_t^q are defined as in Proposition 2.

4.2 GARCH-SG model (con.)

Table: European call option prices and the ratios of the standard deviations, RS_1 and RS_2, for a GARCH-SG model.

		Maturity = 1 Month			Maturity = 3 Months			Maturity = 9 Months		
	S_0/K	1.1	1.0	0.9	1.1	1.0	0.9	1.1	1.0	0.9
European call										
MCS	mean	9.9075	2.5273	0.1367	11.7330	5.0012	1.0905	16.7166	10.4327	5.1232
	std.	0.0497	0.0359	0.0103	0.0832	0.0627	0.0322	0.1395	0.1185	0.0856
EMS	mean	9.9098	2.5310	0.1332	11.7287	4.9953	1.0911	16.7200	10.4380	5.1268
	std.	0.0061	0.0152	0.0102	0.0140	0.0314	0.0293	0.0301	0.0433	0.0490
EPMS	mean	9.9094	2.5283	0.1328	11.7286	4.9981	1.0894	16.7193	10.4348	5.1244
	std.	0.0066	0.0178	0.0096	0.0183	0.0311	0.0254	0.0335	0.0455	0.0501
RS_1		7.5612	2.0152	1.0812	4.5442	2.0115	1.2660	4.1642	2.6046	1.7068
RS_2		0.9255	0.8528	1.0719	0.7643	1.0091	1.1523	0.8996	0.9515	0.9780

4.2 GARCH-DE model

- Consider the following GARCH model:

$$\begin{cases} R_t = r + \lambda\sigma_t - 0.5\,\sigma_t^2 + \sigma_t\varepsilon_t, \ \ \varepsilon_t \sim DE(0,1) \\ \sigma_t^2 = \alpha_0 + \alpha_1\sigma_{t-1}^2\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2 \end{cases} \quad (9)$$

Proposition 2

For Model (9), the change of measure process Λ_t^{ess} derived by the Esscher transform is

$$\Lambda_t^{ess} = \prod_{k=1}^{t} [1 - 0.5\,(\delta_k^*\sigma_k)^2] \exp\{\delta_k^*(R_k - \mu_k)\},$$

where $\mu_k = r + \lambda\sigma_k - 0.5\,\sigma_k^2$,
$\delta_k^* = \{-\sigma_k + [a_k\sigma_k^2 + 2(a_k - 1)^2]^{0.5}\}/[\sigma_k(1 - a_k)]$ and
$a_k = \exp(\lambda\sigma_k - 0.5\,\sigma_k^2)$.

4.2 GARCH-DE (con.)

Table: European call option prices and the ratios of the standard deviations, RS_1, for a GARCH-DE model.

		Maturity = 1 Month			Maturity = 3 Months			Maturity = 9 Months		
	S_0/K	1.1	1.0	0.9	1.1	1.0	0.9	1.1	1.0	0.9
European call										
MCS	mean	9.9344	2.4539	0.1355	11.7512	4.9143	1.0202	16.7253	10.3705	4.9977
	std.	0.0509	0.0356	0.0114	0.0810	0.0607	0.0309	0.1369	0.1153	0.0844
EPMS	mean	9.9418	2.4583	0.1357	11.7615	4.9214	1.0222	16.7407	10.3828	5.0054
	std.	0.0097	0.0176	0.0107	0.0185	0.0275	0.0240	0.0331	0.0442	0.0476
RS_1		5.2542	2.0188	1.0679	4.3832	2.2113	1.2865	4.1406	2.6098	1.7726

4.3 Coverage rates of the EPMS

- Self-quanto options with payoff $f(S_T) = S_T \max(S_T - K, 0)$ under the GARCH-N model: investigate the finite sample performance of Theorem 3
- The asymptotic confidence interval of confidence level $1 - \alpha$ for the EPMS price estimator:

$$\left[C_{EPMS}^{(n)} - z_{\alpha/2}\sqrt{V},\ C_{EPMS}^{(n)} + z_{\alpha/2}\sqrt{V} \right].$$

4.3 Coverage rates of the EPMS (con.)

Table: Coverage rates of the EPMS for computing self-quanto calls with the GARCH-N model.

	n = 500 sample paths								
	Maturity= 30 days			Maturity= 90 days			Maturity= 270 days		
S_0/K	1.10	1.00	0.90	1.10	1.00	0.90	1.10	1.00	0.90
25% cov. rate	0.232	0.234	0.257	0.257	0.249	0.244	0.236	0.239	0.264
50% cov. rate	0.512	0.483	0.520	0.497	0.499	0.504	0.499	0.495	0.497
75% cov. rate	0.742	0.743	0.776	0.746	0.746	0.758	0.743	0.741	0.745
95% cov. rate	0.947	0.959	0.946	0.952	0.957	0.951	0.947	0.945	0.950
	n = 10, 000 sample paths								
	Maturity= 30 days			Maturity= 90 days			Maturity= 270 days		
S_0/K	1.10	1.00	0.90	1.10	1.00	0.90	1.10	1.00	0.90
25% cov. rate	0.283	0.251	0.254	0.261	0.246	0.246	0.250	0.260	0.260
50% cov. rate	0.525	0.494	0.531	0.514	0.513	0.492	0.529	0.526	0.522
75% cov. rate	0.755	0.766	0.768	0.758	0.766	0.734	0.784	0.771	0.762
95% cov. rate	0.952	0.954	0.942	0.956	0.949	0.952	0.959	0.952	0.947

4.3 Coverage rates of the EPMS (con.)

- Examine the validity of Theorem 4: digital option pricing, $f(S_T) = \delta_{\{S_T > K\}}$, is considered under the Black-Scholes model since the density $p(\cdot, T)$ of S_T is analytically available.

- The asymptotic confidence interval of confidence level $1 - \alpha$ for the EPMS price estimator:

$$\left[C_{EPMS}^{(n)} - z_{\alpha/2}\sqrt{V^*},\ C_{EPMS}^{(n)} + z_{\alpha/2}\sqrt{V^*} \right].$$

4.3 Coverage rates of the EPMS (con.)

Table: Coverage rates of the EPMS for digital option pricing with the Black-Scholes model.

	n = 500 sample paths								
	Maturity= 30 days			Maturity= 90 days			Maturity= 270 days		
S_0 / K	1.10	1.00	0.90	1.10	1.00	0.90	1.10	1.00	0.90
Theorem 3									
25% cov. rate	0.121	0.048	0.213	0.062	0.035	0.109	0.014	0.019	0.039
50% cov. rate	0.317	0.237	0.442	0.292	0.237	0.369	0.155	0.184	0.229
75% cov. rate	0.659	0.555	0.738	0.612	0.569	0.683	0.520	0.546	0.571
95% cov. rate	0.880	0.863	0.917	0.923	0.877	0.952	0.902	0.947	0.956
Theorem 4									
25% cov. rate	0.200	0.237	0.258	0.214	0.237	0.267	0.180	0.239	0.229
50% cov. rate	0.483	0.507	0.514	0.503	0.515	0.534	0.450	0.495	0.478
75% cov. rate	0.659	0.740	0.738	0.772	0.761	0.783	0.708	0.764	0.721
95% cov. rate	0.880	0.927	0.917	0.923	0.940	0.952	0.902	0.947	0.956
	n = 10, 000 sample paths								
	Maturity= 30 days			Maturity= 90 days			Maturity= 270 days		
S_0 / K	1.10	1.00	0.90	1.10	1.00	0.90	1.10	1.00	0.90
Theorem 3									
25% cov. rate	0.122	0.049	0.206	0.053	0.049	0.106	0.013	0.022	0.040
50% cov. rate	0.338	0.197	0.457	0.252	0.221	0.337	0.173	0.166	0.221
75% cov. rate	0.641	0.559	0.738	0.583	0.583	0.649	0.529	0.525	0.578
95% cov. rate	0.948	0.901	0.956	0.919	0.896	0.933	0.896	0.882	0.913
Theorem 4									
25% cov. rate	0.210	0.205	0.272	0.196	0.248	0.246	0.178	0.210	0.224
50% cov. rate	0.445	0.451	0.525	0.445	0.524	0.485	0.426	0.479	0.493
75% cov. rate	0.741	0.740	0.778	0.726	0.758	0.740	0.698	0.728	0.740
95% cov. rate	0.948	0.935	0.956	0.938	0.944	0.942	0.935	0.940	0.950

5 Conclusion

- An EPMS is proposed to improve the efficiency of computing the no-arbitrage prices under the dynamic P measure.

- The proposed method can be applied to compute the no-arbitrage prices even when the risk-neutral model can not be expressed explicitly.

- The strong consistency and the asymptotic normality of the EPMS estimator are established.

5 Conclusion (con.)

- Simulation results show that the EPMS is comparable to the EMS and the relative efficiency of the EPMS against the MCS increases as the time to maturity increases.
- Simulation results also show that the asymptotically normal distribution serves as a persuasive approximation for samples consisting of as few as 500 simulation paths.
- The extensions to path-dependent contingent claims or high-dimensional payoffs are interesting topics, and we refer these extensions to our future study.

Thank You for Your Attention!