On the rate of convergence in limit theorems for geometric sums of i.i.d. random variables

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Presentation

Title

On the rate of convergence in limit theorems for geometric sums of i.i.d. random variables
(Joint work with Le Truong Giang, UFM, Vietnam)

Contents

Main Points

1. Geometric sums.
2. Limit theorems for geometric sums of i.i.d. random variables (The Renyi's Theorem, 1957).
3. The rate of convergence (o-small rate and O-large rate)

Contents

1. For geometric sums of row-wise triangular arrays of independent identically distributed random variables (The Renyi-type theorems)
2. Main tools: The Trotter-operator method, Taylor's expansion, Geometrically infinitely divisibility, Geometrically strictly stability.

Outline

Outline

- Motivation
- Preliminaries
- Main Results
- Conclusions
- References

Motivation

1. Geometric Sum:
   - Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed (i.i.d.) random variables.
   - Let $\nu \sim Geo(p), p \in (0,1)$ be a geometric random variable with parameter $p \in (0,1)$.
   - Denote $S_\nu = X_1 + X_2 + \ldots + X_\nu$ the geometric sum. By convention $S_0 = 0$.
2. Geometric sum has attracted much attention (both in pure and applied maths).
3. Relationship between Geometric Sums and Ruin Probabilities (in Classical Risk Models) via Pollazeck-Khinchin formula.
4. Renyi's Theorem (1957) is one of well-known limit theorems in Probability Theory and related Problems.

References related to Random sum, Geometric sums and Ruin Probability

- V. M. Kruglov and V. Yu. Korolev, (1990), Limit Theorems for Random Sums, Mosc. St. Univ. Publ., Moscow, (in Russian).
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- V. Kalashnikov, (1997), Geometric Sums: Bounds for Rare Events with Applications, Kuwer Academic Publishers.
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References related to Geometric sums and Ruin Probability

- J.L. Bon, (2002), Geometric Sums in Reliability Evaluation of Regenerative Systems, Information Processes, V. 2, N. 2, 161163.
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Classical Renyi Limit Theorem

Renyi's Theorem, 1957, WLLN

- Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with mean $0 < m = E(X_1)$
- Let $\nu \sim Geo(p), p \in (0,1)$ be a geometric random variable with success probability $P(\nu = k) = p(1-p)^{k-1}, k = 1, 2, \ldots$
- Denote $S_\nu = X_1 + X_2 + \ldots + X_\nu ; S_0 = 0$ the geometric sum.
- Let $Z^{(m)}$ be an exponential random variable with positive mean m, i.e. $Z^{(m)} \sim$ Exp($\frac{1}{m}$).
- Then

$$\boxed{pS_\nu \xrightarrow{d} Z^{(m)} \quad as \quad p \to 0^+}$$

Toda's Theorem, 2012

- Let $X_1, X_2, \ldots$ be a sequence of independent non-identically distributed random variables with mean $0 = E(X_n), 0 < \sigma_n^2 = D(X_n) < \infty; n = 1, 2, \ldots$.
- Let a_j be a real sequence such that $a := \lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{j=1}^{n} a_j$ exists.
- Let $\lim\limits_{n\to\infty} n^{-\alpha}\sigma_n^2 = 0$ for some $0 < \alpha < 1$ and $\sigma^2 := \frac{1}{n}\sum\limits_{j=1}^{n} \sigma_j^2 > 0$.
- for all $\epsilon > 0$, we have an analog of Lindeberg's condition:

$$\lim_{p\to 0}\sum_{j=1}^{\infty}(1-p)^{j-1} pE\left[X_j^2\{\mid X_j \mid \geq \epsilon p^{-\frac{1}{2}}\}\right] = 0.$$

Toda's Theorem, 2012

Theorem

Then

$$\boxed{p^{\frac{1}{2}} \sum_{j=1}^{\nu} (X_j + p^{\frac{1}{2}} a_j) \xrightarrow{d} W_{0,a,\sigma} \quad as \quad p \to 0.}$$

where $W_{0,a,\sigma}$ is an asymmetric Laplace distributed random variable with parameters $(0, a, \sigma)$.

Goal

- The rate of convergence in Renyi-type limit theorems
- The rate of convergence in generalized Renyi-type limit theorems (for negative-binomial random sums).

References related to the Renyi-type limit theorems

- V. M. Kruglov and V. Yu. Korolev, (1990), Limit Theorems for Random Sums, Mosc. St. Univ. Publ., Moscow, (in Russian).
- V. Kalashnikov, (1997), Geometric Sums: Bounds for Rare Events with Applications, Kuwer Academic Publishers.
- E. V. Sugakova, (1995), Estimates in the Renyi theorem for differently distributed terms, Ukrainian Mathematical Journal, Vol. 47, No. 7, 1128–1134.

References related to the Renyi-type limit theorems

- Samuel Kotz, Tomasz J. Kozubowski and Krzysztof Podgorski, (2001), The Laplace Distribution and Generalizations, Indiana UniversityPurdue University, Indianapolis.
- Erol A. Pekoz and Adrian Rollin, (2011), New rates for exponential approximation and the theorems of Renyi and Yaglom, The Annals of Probability, Vol. 39, No. 2, 587–608.
- Alexis Akira Toda, (2012), Weak Limit of the Geometric Sum of Independent But Not Identically Distributed Random Variables.
- John Pike and Haining Ren, Stein's method and the Laplace distribution.

The Trotter's Operator

Definition, H. F. Trotter, 1959

Let $C_B(\mathcal{R})$ be the set of all real-valued bounded and uniformly continuous functions f on $\mathcal{R}$ and $\|f\| = \sup\limits_{x\in\mathcal{R}} |f(x)|$ Let X be a random variable. A linear operator $A_X : C_B(\mathbb{R}) \to C_B(\mathbb{R})$, is said to be Trotter operator and it is defined by

$$\boxed{A_X f(t) := Ef(X+t) = \int_{\mathbb{R}} f(x+t)dF_X(x)}$$

where F_X is the distribution function of X, $t \in \mathbb{R}, f \in C_B(\mathbb{R})$.

The Trotter's Operator

Properties

- The operator A_X is a linear positive "contraction" operator, i.e.,

$$\| A_X f \| \leq \| f \|,$$

for each $f \in C_B(\mathbb{R})$.

- The operators A_{X_1} and A_{X_2} commute.

The Trotter's Operator

Properties

- The equation $A_X f(t) = A_Y f(t)$ for $f \in C_B(\mathbb{R}), t \in \mathbb{R}$, provided that X and Y are identically distributed random variables.
- If $X_1, X_2, \ldots, X_n$ are independent random variables, then for $f \in C_B(\mathbb{R})$

$$A_{X_1+\ldots+X_n}(f) = A_{X_1} \ldots A_{X_n}(f).$$

Trotter Operator

Properties

- Suppose that $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ are independent random variables (in each group) and they are independent. Then for each $f \in C_B(\mathbb{R})$

$$\|A_{X_1+\ldots+X_n}(f) - A_{Y_1+\ldots+Y_n}(f)\| \leq \sum_{i=1}^{n} \| A_{X_i}(f) - A_{Y_i}(f) \| .$$

- For two independent random variables X and Y, for each $f \in C_B(\mathbb{R})$ and $n = 1, 2, \ldots$

$$\| A_X^n(f) - A_Y^n(f) \| \leq n \| A_X(f) - A_Y(t) \| .$$

Trotter Operator

Properties

- Suppose that $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$, are i.i.d. random variables (in each group) and they are independent.
- Assume that $\nu \in \text{Geo}(p), p \in (0,1)$ is geometric distributed random variable, independent of all X_j and $Y_j, j = 1, 2, \ldots$. Then, for each $f \in C_B(\mathbb{R})$

$$\parallel A_{X_1+\ldots+X_\nu}(f) - A_{Y_1+\ldots+Y_\nu}(f) \parallel \leq \frac{1}{p} \parallel A_{X_1}(f) - A_{Y_1}(f) \parallel .$$

Trotter Operator

Properties

Let

$$\boxed{\lim_{n\to\infty} \| A_{X_n}(f) - A_X(f) \| = 0, \forall f \in C_B^r(\mathbb{R}), r \in \mathbb{N}}$$

Then,

$$\boxed{X_n \xrightarrow{d} X \text{ as } n \to \infty}$$

References related to the Trotter's Operator

- P. L. Butzer, L. Hahn, and U. Westphal, (1975), On the rate of approximation in the central limit theorem, Journal of Approximation Theory, vol. 13, 327–340.
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References related to the Trotter's Operator

- A. Renyi, (1970), Probability theory, Akademiai Kiado, Budapest.
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- H. F. Trotter, (1959), An elementary proof of the central limit theorem, Arch. Math (Basel), 10, 226–234.

Geometrically infinitely divisibility (G.I.D.)

Definition, Klebanov, 1984

A real-valued random variable X is said to be a geometrically infinitely divisible (g.i.d.) if for any $p \in (0,1)$, there exists a sequence of real-valued independent identically distributed random variables $X_j(p)$, such that

$$X \stackrel{d}{=} \sum_{j}^{\nu} X_j(p),$$

where $\nu \sim Geo(p), p \in (0,1)$, and ν and all $X_j(q), j = 1, 2, \ldots$ are independent.

Geometrically infinitely divisibility (G.I.D.)

Theorem, Klebanov, 1984

A characteristic function of X is geometrically infinitely divisible if, and only if, it has the form

$$f_X(t) = \frac{1}{1 - \ln \Psi(t)},$$

where $\Psi(t)$ is an infinitely divisibility characteristic function.

Geometrically infinitely divisibility (G.I.D.)

Theorem, Klebanov, 1984

Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables. Suppose that ν is a positive integer-valued random variable having geometric distribution with success probability $P(\nu = k) = p(1-p)^{k-1}, k = 1, 2, \ldots; p \in (0, 1)$. Assume that ν and $X_1, X_2, \ldots$ are independent. Let us denote by $S_\nu = X_1 + X_2 + \ldots + X_\nu$ the geometric random sum. Suppose that

$$p \sum_{j=1}^{\nu} X_j(p) \xrightarrow{d} X \quad \text{as} \quad q \to 0.$$

Then, X is geometrically infinitely divisible random variable.

Geometrically strictly stability (G.S.S.)

Definition, Klebanov, 1984

A real-valued random variable Y is said to be a geometrically strictly stability (g.s.s.) if for any $q \in (0,1)$, there exists a positive constant $c = c(p) > 0$ such that

$$Y \stackrel{d}{=} c(p) \sum_{j}^{\nu} Y_j(p),$$

where $\nu \sim Geo(p), p \in (0,1)$, independent of all $Y_j(p), j = 1, 2, \ldots$.

Geometrically strictly stable (G.S.S.)

Theorem, Klebanov, 1984

Let Y be a non-degenerate distributed random variable. The characteristic function of Y is geometrically strictly stable if, and only if, it has the form

$$f_Y(t) = \frac{1}{1 + \lambda \mid t \mid^{\alpha} \exp\left(-\frac{i}{2}\theta\alpha \mathsf{sgn}(t)\right)},$$

where λ, α, θ are parameters such that
$0 < \alpha \leq 2, \mid \theta \mid \leq \theta_{\alpha} = \min(1, \frac{2}{\alpha} - 1), \lambda > 0.$

Examples

1. the exponential random variable $Z^m \sim \text{Exp}(\frac{1}{m})$ is GID

$$Z^{(m)} \stackrel{d}{=} p \sum_{j}^{\nu} Z_j$$

where $Z_j \sim \text{Exp}(\frac{1}{m})$

2. the Laplace random variable is GSS

$$W_{0,\sigma} \stackrel{d}{=} p^{\frac{1}{2}} \sum_{j}^{\nu} W_j,$$

Exponential random variable with mean m

1. Density function:

$$p_Z(x) = \frac{1}{m} e^{-\frac{1}{m}x}, x \geq 0.$$

2. Characteristic function:

$$f_Z(t) = \frac{1}{1 - imt}$$

Symmetric Laplace random variable $W_{0,\sigma}$

1. Density function:

$$p_W(x) = \frac{\sigma}{2} exp^{-\sigma|x|}.$$

2. Characteristic function:

$$f_W(t) = \frac{1}{1 + i\frac{1}{\sigma^2}t^2}$$

Note: W is a double exponential random variable.

References related to the G.I.D. and D.S.S.

- Emad-Eldin A. A. Aly and Nadjib Bouzar, (2000), *On geometric infinitely divisibility and Stability*, Ann. Inst. Statist. Math. Vol. 52, No.4, pp. 790-799.
- V. Kalashnikov, (1997), *Geometric sums: bounds for rare events with applications. Risk analysis, reliability, queuing*. Mathematics and its Applications, 413. Kluwer Academic Publishers Group, Dordrecht, xviii+265 pp.
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Lipschitz class

The modulus of continuity of function f is defined for $f \in C_B(\mathbb{R}), \delta \geq 0$ by

$$\omega(f;\delta) := \sup_{|h| \leq \delta} \parallel f(x+h) - f(x) \parallel . \tag{1}$$

- The modulus of continuity $\omega(f;\delta)$ is a monotonely decreasing function of δ with $\omega(f;\delta) \to 0$ for $\delta \to 0^+$, and

$$\omega(f;\lambda\delta) \leq (1+\lambda)\omega(f;\delta) \quad \text{for} \quad \lambda > 0.$$

- A function $f \in C_B(\mathbb{R})$ is said to satisfy a Lipschitz condition of order $\alpha, 0 < \alpha \leq 1$, in symbol $f \in Lip(\alpha)$, if $\omega(f;\delta) = O(\delta^\alpha.)$ It is easily seen that $f \in Lip(1)$, if $f^{'} \in C_B(\mathbb{R})$.

Main Results

Theorem 1

- Let $(X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots)$ be a row-wise triangular array of non-negative independent identically distributed random variables with $E(\mid X_{n1} \mid^k) < +\infty, n = 1, 2, \ldots, k = 1, 2, \ldots, r; r = 1, 2, \ldots.$
- Let ν be a geometric random variable with parameter $p, p \in (0, 1)$ and for every $n = 1, 2, \ldots X_{n1}, X_{n2}, \ldots, \nu$ are independent.
- Moreover, assume that

$$\boxed{E\mid X_{n1} \mid^k = E\mid Z_1^{(m)} \mid^k, k = 1, 2, \ldots, r; r = 1, 2, \ldots.}$$

Continuous

Theorem 1

Then, for $f \in C_B^r(\mathbb{R})$

$$\boxed{\| A_{pS_\nu} f - A_{Z^{(m)}} \| = o(p^{r-1}), \quad \text{as} \quad p \to 0.}$$

where $Z^{(m)}$ is a exponential distributed random variable with positive mean $E(Z^{(m)}) = m$.

An analog of Renyi Theorem

Corollary 1

- $(X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots)$ be a row-wise triangular array of non-negative independent identically distributed random variables with mean $0 < E(X_{nj}) = m, j = 1, 2, \ldots, n; n = 1, 2, \ldots$.
- ν be a geometric random variable with parameter $p, p \in (0, 1)$ and for every $n = 1, 2, \ldots X_{n1}, X_{n2}, \ldots, \nu$ are independent.

Then,

$$\boxed{pS_\nu \xrightarrow{d} Z^{(m)}, \quad \text{as} \quad p \to 0}$$

where $S_\nu = X_{n1} + X_{n2} + \ldots + X_{n\nu}, \quad$ for $\quad n = 1, 2, \ldots$ and $Z^{(m)}$ is a exponential distributed random variable with positive mean $E(Z^{(m)}) = m$. Note that $S_0 = 0$ by convention.

Main Results

Theorem 3

- $(X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots)$ be a row-wise triangular array of non-negative valued, independent and identically distributed random variables with finite r-th absolute moment $E(| X_{nj} |^r) < +\infty, j = 1, 2, \ldots; r \geq 1$.
- ν be a geometric variable with parameter $p, p \in (0, 1)$ and ν is independent of all $X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots$.
- $$\boxed{E(| X_{nj} |^k) = E(| Z_j^{(m)} |^k); \quad k = 1, 2, \ldots, r-1; r \geq 1.}$$

Continuous

Theorem 3

Then, for every $f \in C_B^{r-1}(\mathbb{R})$,

$$\boxed{\| A_{pS_\nu} f - A_{Z^{(m)}} f \| \leq \frac{2p^{r-1}}{(r-1)!} \omega(f^{(r-1)}; p) \left(m^{r-1}(r-1)! + m^r r! \right)}$$

where $Z_j^{(m)}$ are independent exponential distributed random variables with common mean m, i.e. $Z_j^{(m)} \sim Exp(\frac{1}{m}), j = 1, 2, \ldots$

Main Results

Corollary 2

- $(X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots)$ be a row-wise triangular array of non-negative valued, independent and identically distributed random variables with mean $E(X_{n1}) = m < +\infty$ and finite variance $0 < D(X_{n1}) = \sigma^2 < +\infty, j = 1, 2, \ldots, n; n = 1, 2, \ldots$.
- ν be a geometric variable with parameter $p, p \in (0, 1)$,
- ν is independent of all $X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots$.

Then, for every $f \in C_B^1(\mathbb{R})$,

$$\boxed{\| A_{pS_\nu} f - A_{Z(m)} f \| \leq 2\omega(f'; p) \left(m + \frac{1}{2}\sigma^2 + m^2 \right)}$$

Main Results

Continuous

In particular, suppose that
$f^{'} \in Lip(\alpha, M), 0 < \alpha \leq 1, 0 < M < +\infty$. Then

$$\| A_{pS_\nu} f - A_{Z(m)} f \| \leq 2 \left(m + \frac{1}{2}\sigma^2 + m^2 \right) M p^\alpha.$$

Main Results

Corollary 3

- $(X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots)$ be a row-wise triangular array of non-negative valued, independent and standard normal distributed random variables.
- ν be a geometric variable with parameter p, $p \in (0, 1)$, and ν is independent of all $X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots$.
- Denote by $S_\nu^2 := X_{n1}^2 + \ldots + X_{n\nu}^2$ by the geometric sum of squared standard normal random variables (another word we call is by chi-squared random variable with geometric degree of freedom)

Main Results

Corollary 3 (continuous)

Then, for every $f \in C_B^2(\mathbb{R})$,

$$\boxed{\| A_{pS_\nu^2} f - A_{Z^{(1)}} f \| \leq \frac{p}{2} \| f^{''} \| \left(1 + 24\omega(f^{''}; p)\right),}$$

where $Z^{(1)} \sim Exp(1)$.

Main Results

Theorem 4

- Let $(X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots)$ be a row-wise triangular array of independent identically distributed random variables with $E(| X_{n1} |^k) < +\infty, n = 1, 2, \ldots, k = 1, 2, \ldots, r; r = 1, 2, \ldots.$
- Let ν be a geometric random variable with parameter $p, p \in (0, 1)$ and for every $n = 1, 2, \ldots X_{n1}, X_{n2}, \ldots, \nu$ are independent.
- Let W is a Laplace distributed random variable $W \sim L(0, \sigma)$. Moreover, assume that

$$\boxed{E| X_{n1} |^k = E| W |^k, k = 1, 2, \ldots, r; r = 1, 2, \ldots.}$$

Continuous

Theorem 4

Then, for $f \in C_B^r(\mathbb{R})$

$$\boxed{\parallel A_{\sqrt{p}S_\nu} f - A_W f \parallel = o(p^{\frac{r}{2}-1}), \quad \text{as} \quad p \to 0.}$$

Main Results

Corollary 4

- Let $(X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots)$ be a row-wise triangular array of independent identically distributed random variables with
$\sqrt{p}a = E(X_{n1}); \sigma^2 = D(X_{n1}) < +\infty, n = 1, 2, \ldots.$
- Let ν be a geometric random variable with parameter $p, p \in (0, 1)$ and for every $n = 1, 2, \ldots X_{n1}, X_{n2}, \ldots, \nu$ are independent.
- Let W is a Laplace distributed random variable $W \sim L(0, \sigma)$.

Then,

$$\boxed{\sqrt{p}S_\nu \xrightarrow{d} W, \quad \text{as} \quad p \to 0.}$$

Main Results

Corollary 5

- Let $(X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots)$ be a row-wise triangular array of independent identically distributed random variables with $\sqrt{p}a = E(X_{n1}); \sigma^2 = D(X_{n1}) < +\infty, E| X_{nj} |^3 = \gamma < \infty; n = 1, 2, \ldots.$
- Let ν be a geometric random variable with parameter $p, p \in (0, 1)$ and for every $n = 1, 2, \ldots X_{n1}, X_{n2}, \ldots, \nu$ are independent.
- Let W is a Laplace distributed random variable $W \sim L(a, \sigma)$.

Continuous

Corollary 5

Then, for every $f \in C_B^2(\mathbb{R})$,

$$\boxed{\| A_{\sqrt{p}S_\nu} f - A_W f \| \leq \frac{\omega(f^{"}; \sqrt{p})}{2} \left[2\sigma^2 + \frac{3\sigma^3}{\sqrt{2}} + \gamma \right], \quad \text{as} \quad p \to 0.}$$

If $f^{"} \in Lip(\alpha, M)$ with $0 < \alpha < 1$, Then

$$\boxed{\| A_{\sqrt{p}S_\nu} f - A_W f \| \leq \frac{M p^{\frac{\alpha}{2}})}{2} \left[2\sigma^2 + \frac{3\sigma^3}{\sqrt{2}} + \gamma \right], \quad \text{as} \quad p \to 0.}$$

Negative-binomial random sum

Definition

- Let $(X_{n1}, X_{n2}, \ldots)$ be a row-wise triangular array of independent identically distributed random variables
- Let τ be a negative-binomial random variable with parameter $l, p, p \in (0,1), l = 1, 2, \ldots$ such that $P(\tau = k) = C_{k-1}^{l-1} p^l (1-p)^k - l$. Assume that for every $n = 1, 2, \ldots X_{n1}, X_{n2}, \ldots, \tau$ are independent.
- Denote by $S_\tau = X_{n1} + X_{n2} + \ldots + X_{n\tau}$ the negative-binomial sum of i.i.d. random variables

Main Results

Theorem 5

- Let $(X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots)$ be a row-wise triangular array of independent identically distributed random variables with $E(\mid X_{n1} \mid = m, n = 1, 2, \ldots, k = 1, 2, \ldots, r; r = 1, 2, \ldots.$
- Let τ be a negative-binomial random variable with parameter $l, p; l \geq 1, p \in (0, 1)$ and for every $n = 1, 2, \ldots X_{n1}, X_{n2}, \ldots, \tau$ are independent.
- Let G is a Gamma distributed random variable $G \sim \Gamma(l, \frac{l}{m})$.

Continuous

Theorem 5

Then, for every $f \in C_B^r(\mathbb{R})$

$$\boxed{\| A_{\frac{p}{l}} f - A_G f \| = o\left(\left[\frac{p}{l}\right]^{r-1} \right), \quad \text{as} \quad p \to 0.}$$

Main Results

Corollary 6 (Generalized Renyi Theorem)

- Let $(X_{nj}, j = 1, 2, \ldots, n; n = 1, 2, \ldots)$ be a row-wise triangular array of independent identically distributed random variables with
  $E(\mid X_{n1} \mid = m, n = 1, 2, \ldots, k = 1, 2, \ldots, r; r = 1, 2, \ldots.$
- Let τ be a negative-binomial random variable with parameter $l, p; l \geq 1, p \in (0, 1)$ and for every
  $n = 1, 2, \ldots X_{n1}, X_{n2}, \ldots, \tau$ are independent.
- Let G is a Gamma distributed random variable $G \sim \Gamma(l, \frac{l}{m})$.

Then,

$$\boxed{\frac{p}{l} S_\tau \xrightarrow{d} G, \quad \text{as} \quad p \to 0^+.}$$

Conclusions

- The Trotter method is elementary and elegant (apply to multi-dimensional spaces).
- This method can be applied to a wide class of random variables (not only for continuous class)
- The rates of convergence in limit theorems for geometric sums should be estimated using a probability distance based on Trotter operator

$$\boxed{d_A(pS_\nu, Z; f) := \sup_{y \in \mathbb{R}} \mid Ef(pS_\nu + y) - Ef(Z + y) \mid}$$

- Consider the case for geometrical sums of independent non-identically distributed random variables (Toda's Theorem, 2012)

Thanks

Thanks for your attention!