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# Parties, Models, Mobsters

## A New Implementation of Model-Based Recursive Partitioning in R

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# Overview

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# Motivation: Trees

Breiman (2001, *Statistical Science*) distinguishes two cultures of statistical modeling.

- **Data models:** Stochastic models, typically parametric.
  $\rightarrow$ Classical strategy in statistics. Regression models are still the workhorse for many empirical analyses.
- **Algorithmic models:** Flexible models, data-generating process unknown. $\rightarrow$ Still few applications in many fields, e.g., social sciences or economics.

**Classical example**: Trees, i.e., modeling of dependent variable $y$ by "learning" a recursive partition w.r.t explanatory variables $z_1, \ldots, z_l$.

# Motivation: Leaves

**Key features**:

1. Predictive power in nonlinear regression relationships.
2. Interpretability (enhanced by visualization), i.e., no “black box” methods.

**Typically:** Simple models for univeriate $y$, e.g., mean.

**Idea:** More complex models for more complex $y$, e.g., regression models, multivariate normal model, item responses, etc.

**Here:** Synthesis of parametric data models and algorithmic tree models.

**Goal:** Fitting local models by partitioning of the sample space.

# Recursive partitioning

**Model-based (MOB) algorithm**:

1. Fit the parametric model in the current subsample.
2. Assess the stability of the parameters across each partitioning variable $z_j$.
3. Split sample along the $z_{j*}$ with strongest instability: Choose breakpoint with highest improvement of the model fit.
4. Repeat steps 1–3 recursively in the subsamples until some stopping criterion is met.

# Recursive partitioning

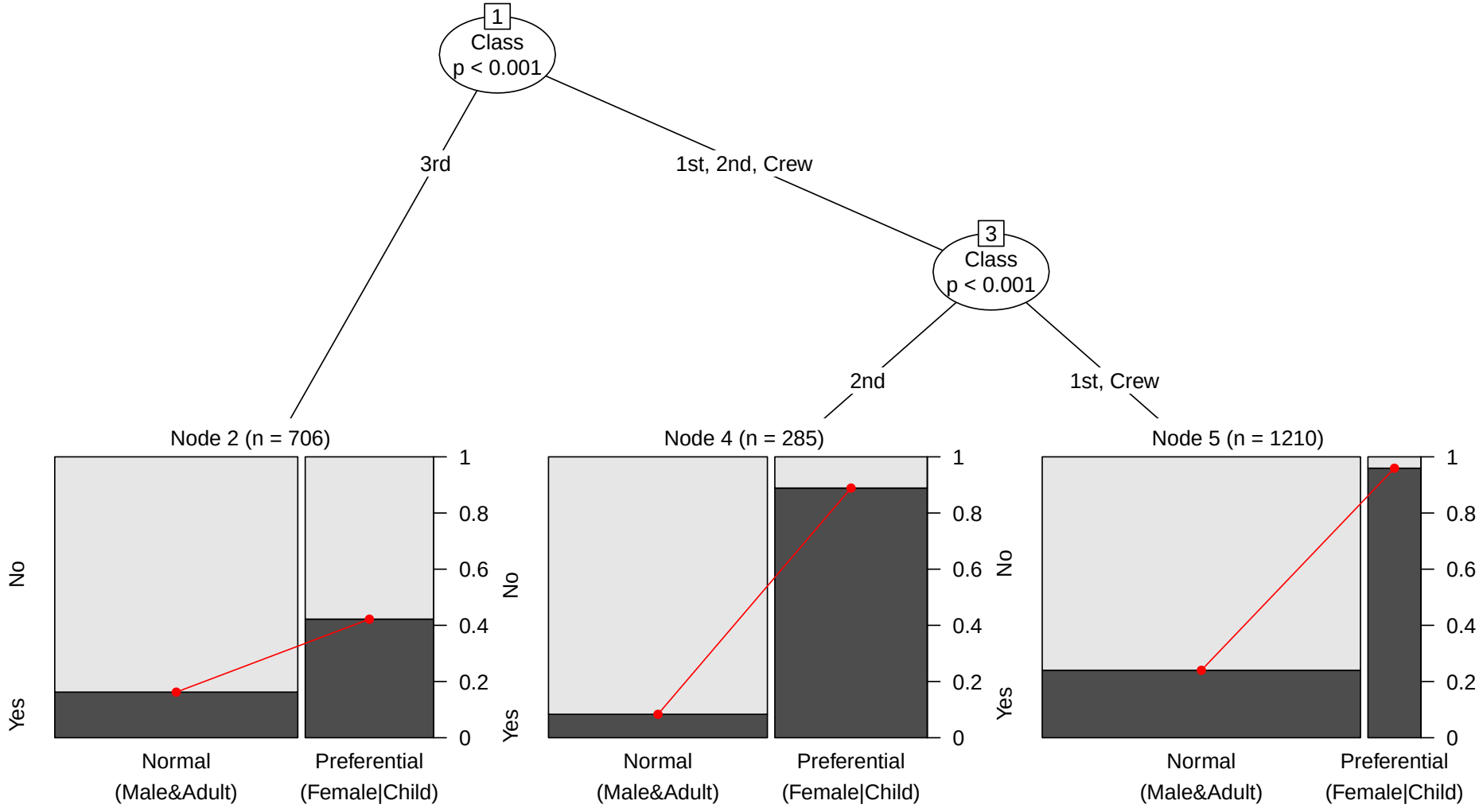
**Example:** Logistic regression, assessing differences in the effect of "preferential treatment" ("women and children first"?) in the Titanic survival data.

**In R:** Generalized linear model tree with binomial family (and default logit link).

```
R> mb <- glmtree(Survived ~ Treatment | Age + Gender + Class,
+    data = ttnc, family = binomial, alpha = 0.05, prune = "BIC")
R> plot(mb)
R> print(mb)
```

**Result:** Log-odds ratio of survival given treatment differs across classes (slope), as does the survival probability of male adults (intercept).

# Recursive partitioning



# Recursive partitioning

```
Generalized linear model tree (family: binomial)

Model formula:
Survived ~ Treatment | Age + Gender + Class

Fitted party:
[1] root
|   [2] Class in 3rd: n = 706
|                 (Intercept) TreatmentPreferential
|                      -1.641                  1.327
|   [3] Class in 1st, 2nd, Crew
|   |   [4] Class in 2nd: n = 285
|   |                 (Intercept) TreatmentPreferential
|   |                      -2.398                  4.477
|   |   [5] Class in 1st, Crew: n = 1210
|   |                 (Intercept) TreatmentPreferential
|   |                      -1.152                  4.318

Number of inner nodes:    2
Number of terminal nodes: 3
Number of parameters per node: 2
Objective function (negative log-likelihood): 1061
```

# 1. Model estimation

**Models:** $\mathcal{M}(y, x, \theta)$ with (potentially multivariate) observations $y$, optionally regressors $x$, and $k$-dimensional parameter vector $\theta \in \Theta$.

**Parameter estimation:** $\widehat{\theta}$ by optimization of additive objective function $\Psi(y, x, \theta)$ for $n$ observations $y_i$ ($i = 1, \ldots, n$):

$$\widehat{\theta} \quad = \quad \underset{\theta \in \Theta}{\operatorname{argmin}} \sum_{i=1}^{n} \Psi(y_i, x_i, \theta).$$

**Special cases:** Maximum likelihood (ML), weighted and ordinary least squares (OLS and WLS), quasi-ML, and other M-estimators.

# 1. Model estimation

**Estimating function:** $\widehat{\theta}$ can also be defined in terms of

$$\sum_{i=1}^{n} \psi(y_i, x_i, \widehat{\theta}) = 0,$$

where $\psi(y, x, \theta) = \partial \Psi(y, x, \theta)/\partial\theta$.

**Central limit theorem:** If there is a true parameter $\theta_0$ and given certain weak regularity conditions:

$$\sqrt{n}(\widehat{\theta} - \theta_0) \quad \xrightarrow{\text{d}} \quad \mathcal{N}(0, V(\theta_0)),$$

where $V(\theta_0) = \{A(\theta_0)\}^{-1} B(\theta_0) \{A(\theta_0)\}^{-1}$. $A$ and $B$ are the expectation of the derivative of $\psi$ and the variance of $\psi$, respectively.

# 1. Model estimation

**Idea:** In many situations, a single global model $\mathcal{M}(y, x, \theta)$ that fits **all** $n$ observations cannot be found. But it might be possible to find a partition w.r.t. the variables $z_1, \ldots, z_l$ so that a well-fitting model can be found locally in each cell of the partition.

**Tools:**

- Assess parameter instability w.r.t to partitioning variables $z_j$ $(j = 1, \ldots, l)$.
- A general measure of deviation from the model is the estimating function $\psi(y, x, \theta)$.

# 2. Tests for parameter instability

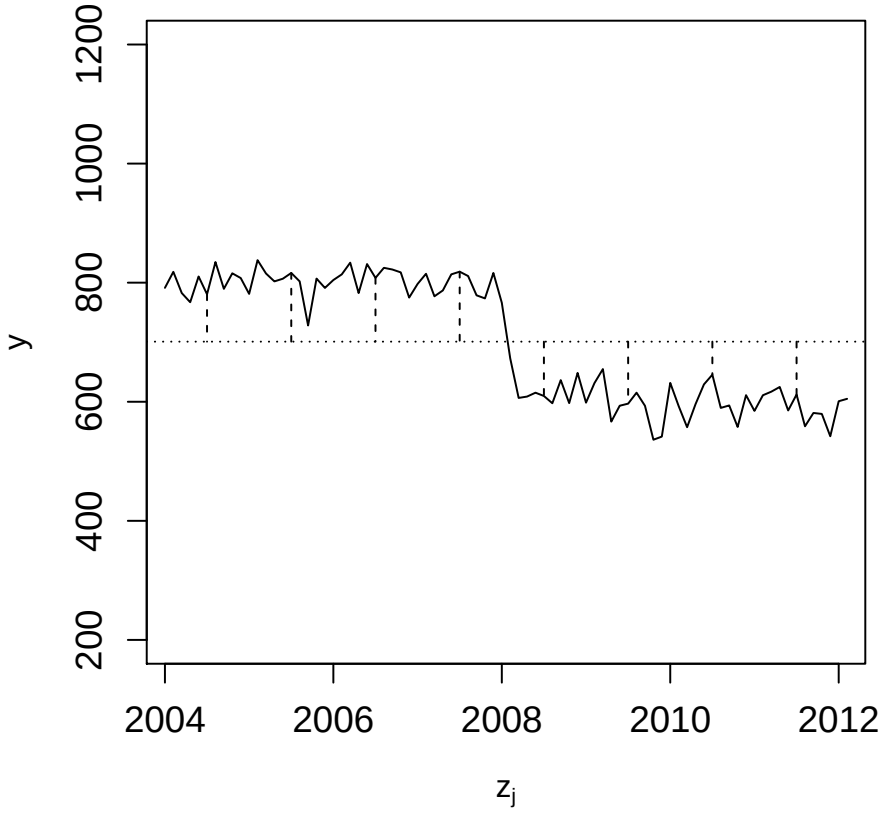
Generalized M-fluctuation tests capture instabilities in $\widehat{\theta}$ for an ordering w.r.t $z_j$.

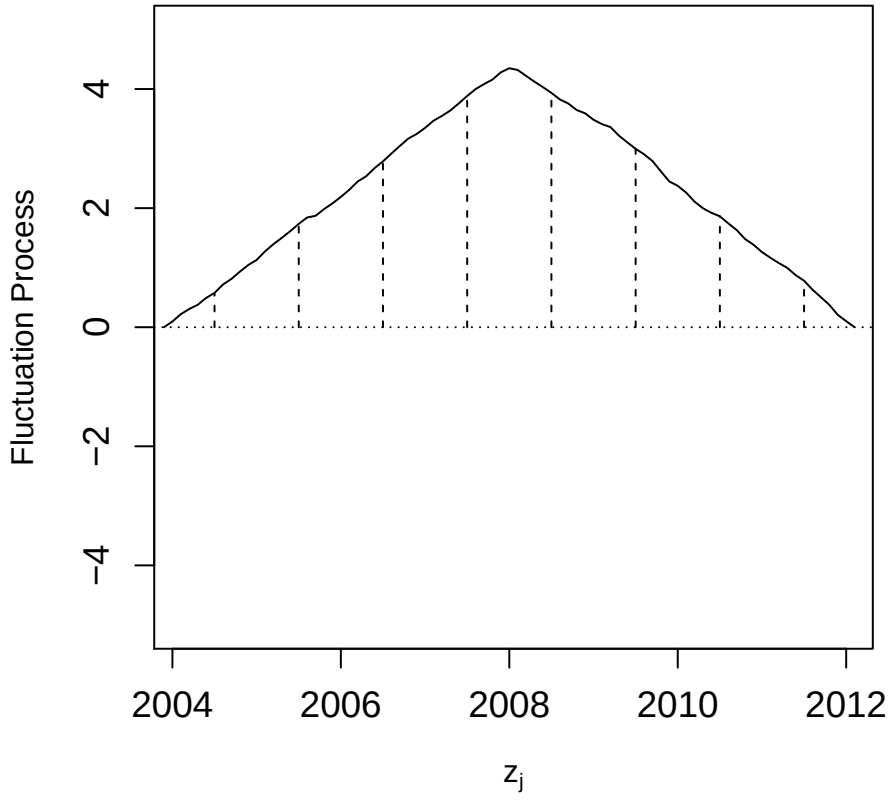
**Basis:** Empirical fluctuation process of cumulative deviations w.r.t. to an ordering $\sigma(z_{ij})$.

$$W_j(t, \widehat{\theta}) \quad = \quad \widehat{B}^{-1/2} n^{-1/2} \sum_{i=1}^{\lfloor nt \rfloor} \psi(y_{\sigma(z_{ij})}, x_{\sigma(z_{ij})}, \widehat{\theta}) \qquad (0 \leq t \leq 1)$$

**Functional central limit theorem:** Under parameter stability $W_j(\cdot) \xrightarrow{\text{d}} W^0(\cdot)$, where $W^0$ is a $k$-dimensional Brownian bridge.

# 2. Tests for parameter instability



# 2. Tests for parameter instability

**Test statistics:** Scalar functional $\lambda(W_j)$ that captures deviations from zero.

**Null distribution:** Asymptotic distribution of $\lambda(W^0)$.

**Special cases:** Class of test encompasses many well-known tests for different classes of models. Certain functionals $\lambda$ are particularly intuitive for numeric and categorical $z_j$, respectively.

**Advantage:** Model $\mathcal{M}(y, x, \widehat{\theta})$ just has to be estimated once. Empirical estimating functions $\psi(y_i, x_i, \widehat{\theta})$ just have to be re-ordered and aggregated for each $z_j$.

# 2. Tests for parameter instability

**Splitting numeric variables:** Assess instability using sup*LM* statistics.

$$\lambda_{supLM}(W_j) \quad = \quad \max_{i=\underline{i},\ldots,\bar{i}} \left(\frac{i}{n} \cdot \frac{n-i}{n}\right)^{-1} \left\| W_j\left(\frac{i}{n}\right) \right\|_2^2 .$$

**Interpretation:** Maximization of single shift *LM* statistics for all conceivable breakpoints in $[\underline{i}, \bar{i}]$.

**Limiting distribution:** Supremum of a squared, $k$-dimensional tied-down Bessel process.

**Potential alternatives:** Many other parameter instability tests from the same class of tests, e.g., a Cramér-von Mises test (or Nyblom-Hansen test), MOSUM tests, etc.

# 2. Tests for parameter instability

**Splitting categorical variables:** Assess instability using $\chi^2$ statistics.

$$\lambda_{\chi^2}(W_j) \quad = \quad \sum_{c=1}^{C} \frac{n}{|I_c|} \left|\left| \Delta_{I_c} W_j \left( \frac{i}{n} \right) \right|\right|_2^2 .$$

**Feature:** Invariant for re-ordering of the $C$ categories and the observations within each category.

**Interpretation:** Capture instability for split-up into $C$ categories.

**Limiting distribution:** $\chi^2$ with $k \cdot (C-1)$ degrees of freedom.

# 2. Tests for parameter instability

**Splitting ordinal variables:** Several strategies conceivable. Assess instability either as for categorical variables (if $C$ is low), or as for numeric variables (if $C$ is high), or via a specialized test.

$$\lambda_{maxLMo}(W_j) = \max_{i \in \{i_1, \ldots, i_{C-1}\}} \left( \frac{i}{n} \cdot \frac{n-i}{n} \right)^{-1} \left\| W_j\left(\frac{i}{n}\right) \right\|_2^2,$$

$$\lambda_{WDMo}(W_j) = \max_{i \in \{i_1, \ldots, i_{C-1}\}} \left( \frac{i}{n} \cdot \frac{n-i}{n} \right)^{-1/2} \left\| W_j\left(\frac{i}{n}\right) \right\|_\infty.$$

**Interpretation:** Assess only the possible splitpoints $i_1, \ldots, i_{C-1}$, based on $L_2$ or $L_\infty$ norm.

**Limiting distribution:** Maximum from selected points in a squared Bessel process or multivariate normal distribution, respectively.

# 3. Segmentation

**Goal:** Split model into $b = 1, \ldots, B$ segments along the partitioning variable $z_j$ associated with the highest parameter instability. Local optimization of

$$\sum_b \sum_{i \in I_b} \Psi(y_i, x_i, \theta_b).$$

$B = 2$: Exhaustive search of order $O(n)$.

$B > 2$: Exhaustive search is of order $O(n^{B-1})$, but can be replaced by dynamic programming of order $O(n^2)$. Different methods (e.g., information criteria) can choose $B$ adaptively.

**Here:** Binary partitioning. Optionally, $B = C$ can be chosen (without search) for categorical variables.

# 4. Pruning

**Goal:** Avoid overfitting.

**Pre-pruning:**

- Internal stopping criterium.
- Stop splitting when there is no significant parameter instability.
- Based on Bonferroni-corrected $p$ values of the fluctuation tests.

**Post-pruning:**

- Grow large tree (e.g., with high significance level).
- Prune splits that do not improve the model fit based on information criteria (e.g., AIC or BIC).

**Hyperparameters:** Significance level and information criterion penalty can be chosen manually (or possibly through cross-validation etc.).

# Local models

**Goals:**

- Detection of interactions and nonlinearities in regressions.
- Add explanatory variables for models without regressors.
- Detect violations of parameter stability (measurement invariance) across several variables adaptively.

**Mobsters:**

- Linear and generalized linear model trees (Zeileis *et al.* 2008).
- Censored survival regression trees: parametric proportional hazard and accelerated failure time models (Zeileis *et al.* 2008).
- Beta regression trees (Grün *et al.* 2012).
- Bradley-Terry trees for paired comparisons (Strobl *et al.* 2011).
- Item response theory (IRT) trees: Rasch, rating scale and partial credit model (Strobl *et al.* 2014, Abou El-Komboz *et al.* 2014).

# Implementation: Building blocks

**Workhorse function:** `mob()` for

- data handling,
- calling model fitters,
- carrying out parameter instability tests and
- recursive partitioning algorithm.

**Required functionality:**

- *Parties:* Class and methods for recursive partytions.
- *Models:* Fitting functions for statistical models (optimizing suitable objective function).
- *Mobsters:* High-level interfaces (`lmtree()`, `bttree()`, `...`) that call lower-level `mob()` with suitable options and methods.

# Implementation: Old `mob()` in *party*

**Parties:** S4 class '`BinaryTree`'.

- Originally developed only for `ctree()` and somewhat "abused".
- Rather rigid and hard to extend.

**Models:** S4 '`StatModel`' objects.

- Intended to conceptualize unfitted model objects.
- Required some "glue code" to accomodate non-standard interface for data handling and model fitting.

**Mobsters:**

- `mob()` already geared towards (generalized) linear models.
- Other interfaces in *psychotree* and *betareg*.
- Hard to do fine control due to adopted S4 classes: Many unnecessary computations and copies of data.

# Implementation: New `mob()` in *partykit*

**Parties:** S3 class '`modelparty`' built on '`party`'.

- Separates data and tree structure.
- Inherits generic infrastructure for printing, predicting, plotting, ...

**Models:** Plain functions with input/output convention.

- Basic and extended interface for rapid prototyping and for speeding up computings, respectively.
- Only minimial glue code required if models are well-designed.

**Mobsters:**

- `mob()` completely agnostic regarding models employed.
- Separate interfaces `lmtree()`, `glmtree()`, ...
- New interfaces typically need to bring their model fitter and adapt the main methods `print()`, `plot()`, `predict()` etc.

# Implementation: New `mob()` in *partykit*

**New inference options:** Not used by default by optionally available.

- New parameter instability tests for ordinal partitioning variables. Alternative to unordered $\chi^2$ test but computationally intensive.
- Post-pruning based on information criteria (e.g., AIC or BIC), especially for very large datasets where traditional significance levels are not useful.
- Multiway splits for categorical partitioning variables.
- Treat weights as proportionality weights and not as case weights.

# Implementation: Models

**Input:** Basic interface.

```
fit(y, x = NULL, start = NULL, weights = NULL,
  offset = NULL, ...)
```

`y`, `x`, `weights`, `offset` are (the subset of) the preprocessed data.
Starting values and further fitting arguments are in `start` and `...`.

**Output:** Fitted model object of class with suitable methods.

- `coef()`: Estimated parameters $\hat{\theta}$.
- `logLik()`: Maximized log-likelihood function $-\sum_i \Psi(y_i, x_i, \hat{\theta})$.
- `estfun()`: Empirical estimating functions $\Psi'(y_i, x_i, \hat{\theta})$.

# Implementation: Models

**Input:** Extended interface.

```
fit(y, x = NULL, start = NULL, weights = NULL,
  offset = NULL, ..., estfun = FALSE, object = FALSE)
```

**Output:** List.

- `coefficients`: Estimated parameters $\hat{\theta}$.
- `objfun`: Minimized objective function $\sum_i \Psi(y_i, x_i, \hat{\theta})$.
- `estfun`: Empirical estimating functions $\Psi'(y_i, x_i, \hat{\theta})$. Only needed if `estfun = TRUE`, otherwise optionally `NULL`.
- `object`: A model object for which further methods could be available (e.g., `predict()`, or `fitted()`, etc.). Only needed if `object = TRUE`, otherwise optionally `NULL`.

**Internally:** Extended interface constructed from basic interface if supplied. Efficiency can be gained through extended approach.

# Implementation: Parties

**Class:** '`modelparty`' inheriting from '`party`'.

**Main addition:** Data handling for regressor and partitioning variables.

- The *Formula* package is used for two-part formulas, e.g., `y ~ x1 + x2 | z1 + z2 + z3`.
- The corresponding terms are stored for the combined model and only for the partitioning variables.

**Additional information:** In `info` slots of '`party`' and '`partynode`'.

- `call`, `formula`, `Formula`, `terms` (partitioning variables only), `fit`, `control`, `dots`, `nreg`.
- `coefficients`, `objfun`, `object`, `nobs`, `p.value`, `test`.

**Reusability:** Could in principle be used for other model trees as well (inferred by other algorithms than MOB).

# Bradley-Terry trees






**Questions:** Which of these women is more attractive? How does the answer depend on age, gender, and the familiarity with the associated TV show Germany's Next Topmodel?

# Bradley-Terry trees

**Task:** Preference scaling of attractiveness.

**Data:** Paired comparisons of attractiveness.

- *Germany's Next Topmodel 2007* finalists: Barbara, Anni, Hana, Fiona, Mandy, Anja.
- Survey with 192 respondents at Universität Tübingen.
- Available covariates: Gender, age, familiarty with the TV show.
- Familiarity assessed by yes/no questions: (1) Do you recognize the women?/Do you know the show? (2) Did you watch it regularly? (3) Did you watch the final show?/Do you know who won?

# Bradley-Terry trees

**Model:** Bradley-Terry (or Bradley-Terry-Luce) model.

- Standard model for paired comparisons in social sciences.
- Parametrizes probability $\pi_{ij}$ for preferring object *i* over *j* in terms of corresponding "ability" or "worth" parameters $\theta_i$.

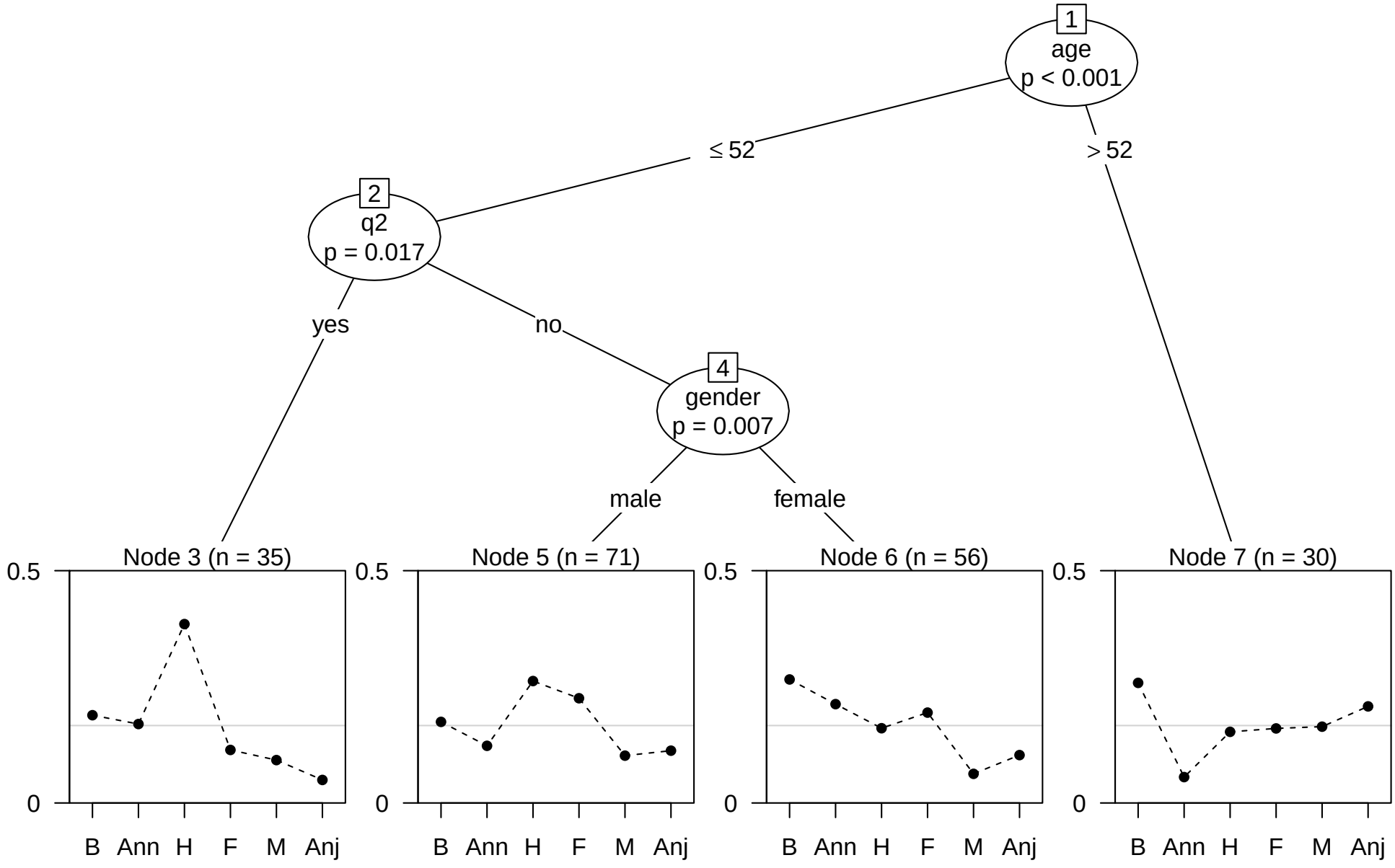
$$\begin{aligned} \pi_{ij} &= \frac{\theta_i}{\theta_i + \theta_j} \\ \text{logit}(\pi_{ij}) &= \log(\theta_i) - \log(\theta_j) \end{aligned}$$

- Maximum likelihood as a logistic or log-linear GLM.

**Mobster:** `bttree()` in *psychotree* (Strobl *et al.* 2011).

**Here:** Use `mob()` directly to build model from scratch using `btReg.fit()` from *psychotools*.

# Bradley-Terry trees



# Bradley-Terry trees

Data, packages, and `estfun()` method:

```
R> data("Topmodel2007", package = "psychotree")
R> library("partykit")
R> library("psychotools")
R> estfun.btReg <- function(x, ...) x$estfun
```

Basic model fitting function:

```
R> btfit1 <- function(y, x = NULL, start = NULL, weights = NULL,
+    offset = NULL, ...) btReg.fit(y, weights = weights, ...)
```

Fit Bradley-Terry tree:

```
R> system.time(bt1 <- mob(
+    preference ~ 1 | gender + age + q1 + q2 + q3,
+    data = Topmodel2007, fit = btfit1))

   user  system elapsed
  5.148   0.036   5.215
```

# Bradley-Terry trees

Extended model fitting function:

```
R> btfit2 <- function(y, x = NULL, start = NULL, weights = NULL,
+    offset = NULL, ..., estfun = FALSE, object = FALSE) {
+    rval <- btReg.fit(y, weights = weights, ...,
+      estfun = estfun, vcov = object)
+    list(
+      coefficients = rval$coefficients,
+      objfun = -rval$loglik,
+      estfun = if(estfun) rval$estfun else NULL,
+      object = if(object) rval else NULL
+    )
+  }
```

Fit Bradley-Terry tree again:

```
R> system.time(bt2 <- mob(
+    preference ~ 1 | gender + age + q1 + q2 + q3,
+    data = Topmodel2007, fit = btfit2))

   user  system elapsed
  4.132   0.000   4.142
```

# Bradley-Terry trees

```
Model-based recursive partitioning (btfit2)

Model formula:
preference ~ 1 | gender + age + q1 + q2 + q3

Fitted party:
[1] root
|   [2] age <= 52
|   |   [3] q2 in yes: n = 35
|   |       Barbara    Anni    Hana   Fiona   Mandy
|   |        1.3378  1.2318  2.0499  0.8339  0.6217
|   |   [4] q2 in no
|   |   |   [5] gender in male: n = 71
|   |   |        Barbara     Anni     Hana    Fiona    Mandy
|   |   |        0.43866  0.08877  0.84629  0.69424 -0.10003
|   |   |   [6] gender in female: n = 56
|   |   |       Barbara    Anni    Hana   Fiona   Mandy
|   |   |        0.9475  0.7246  0.4452  0.6350 -0.4965
|   [7] age > 52: n = 30
|       Barbara    Anni    Hana   Fiona   Mandy
|        0.2178 -1.3166 -0.3059 -0.2591 -0.2357
```

# Bradley-Terry trees

```
Number of inner nodes:    3
Number of terminal nodes: 4
Number of parameters per node: 5
Objective function: 1829
```

Standard methods readily available:

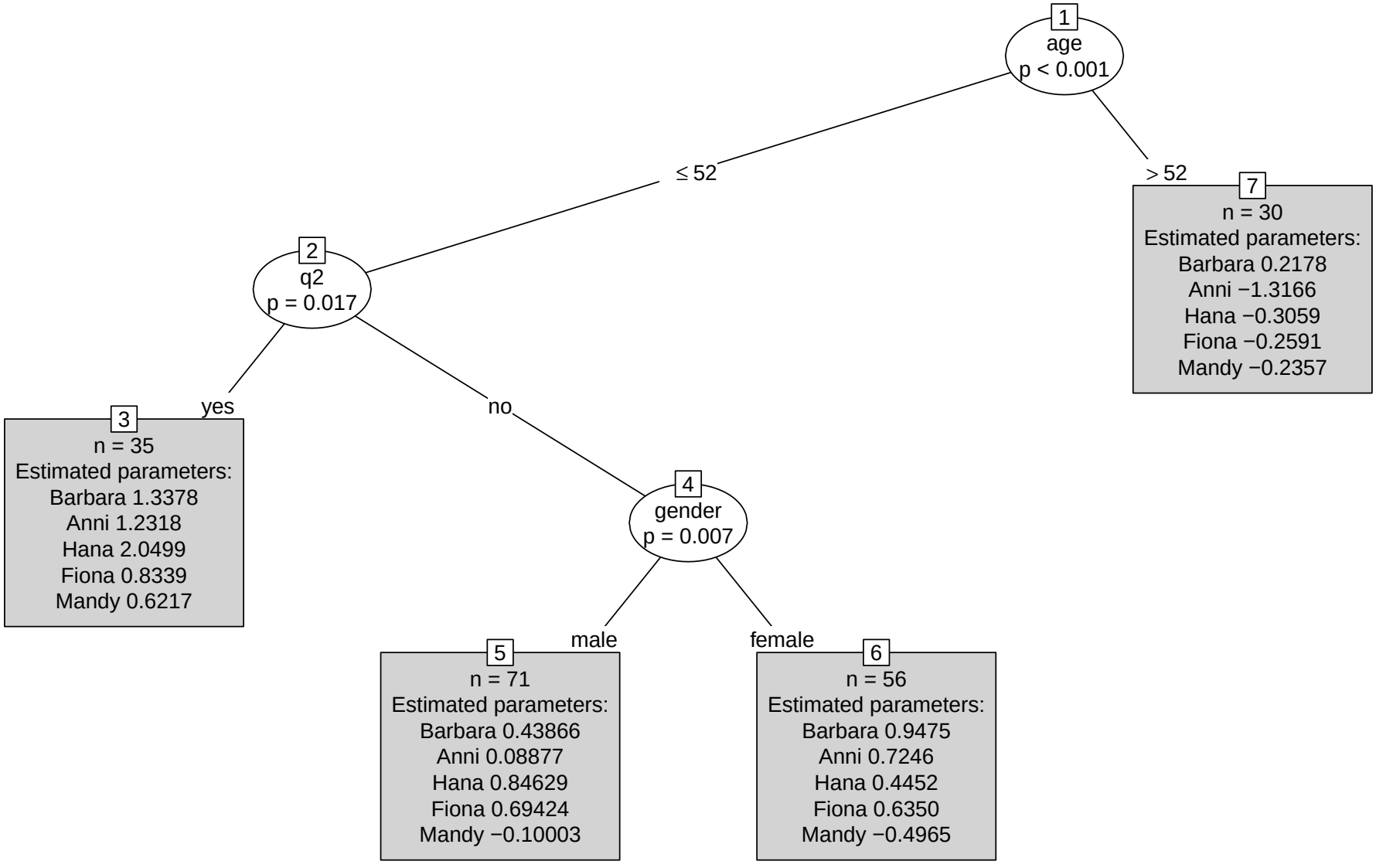
```
R> plot(bt2)
R> coef(bt2)
```

```
  Barbara     Anni    Hana   Fiona   Mandy
3  1.3378  1.23183  2.0499  0.8339  0.6217
5  0.4387  0.08877  0.8463  0.6942 -0.1000
6  0.9475  0.72459  0.4452  0.6350 -0.4965
7  0.2178 -1.31663 -0.3059 -0.2591 -0.2357
```

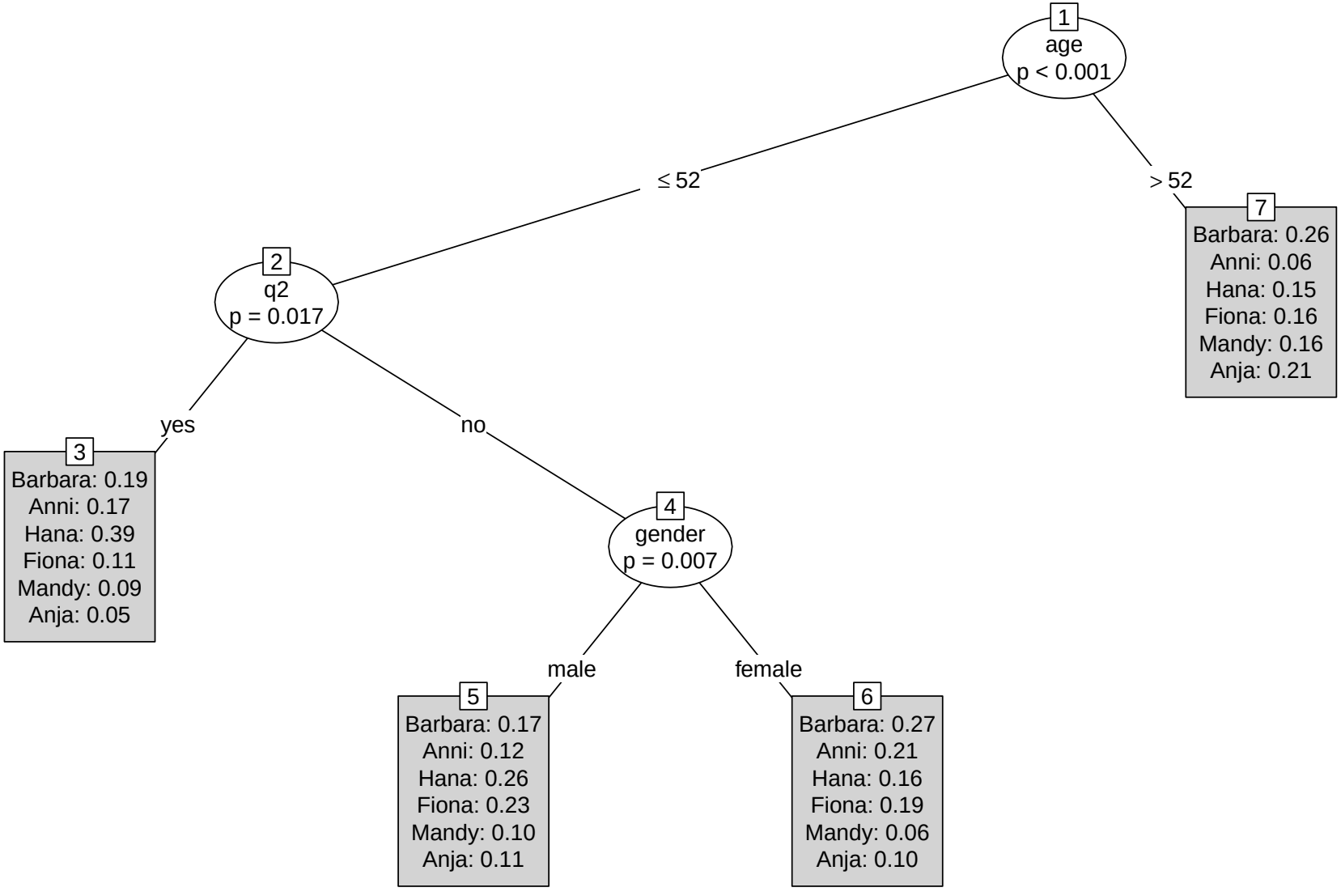
Customization:

```
R> worthf <- function(info) paste(info$object$labels,
+    format(round(worth(info$object), digits = 2)), sep = ": ")
R> plot(bt2, FUN = worthf)
```

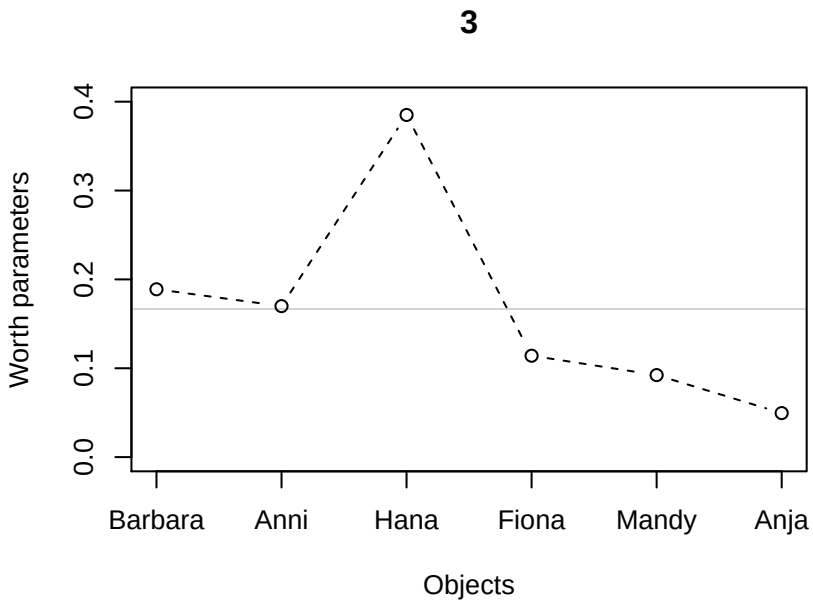
# Bradley-Terry trees

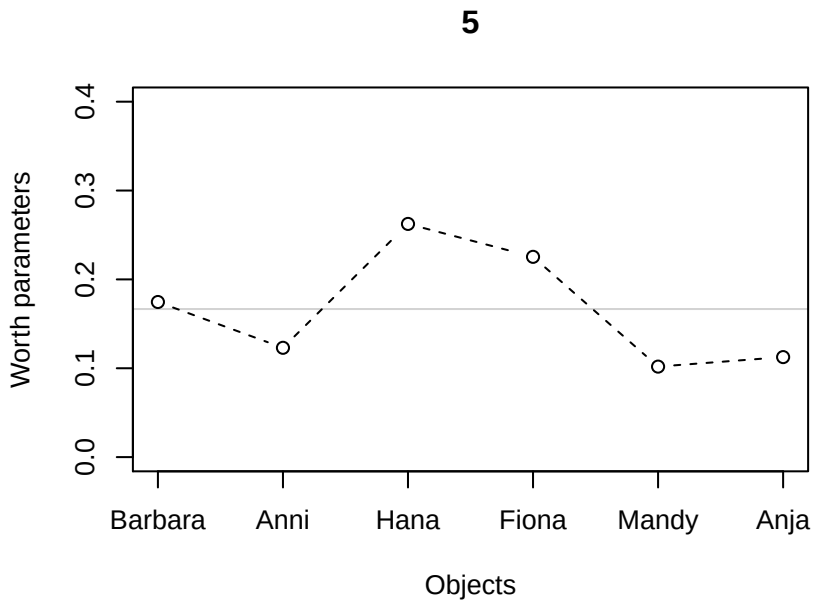


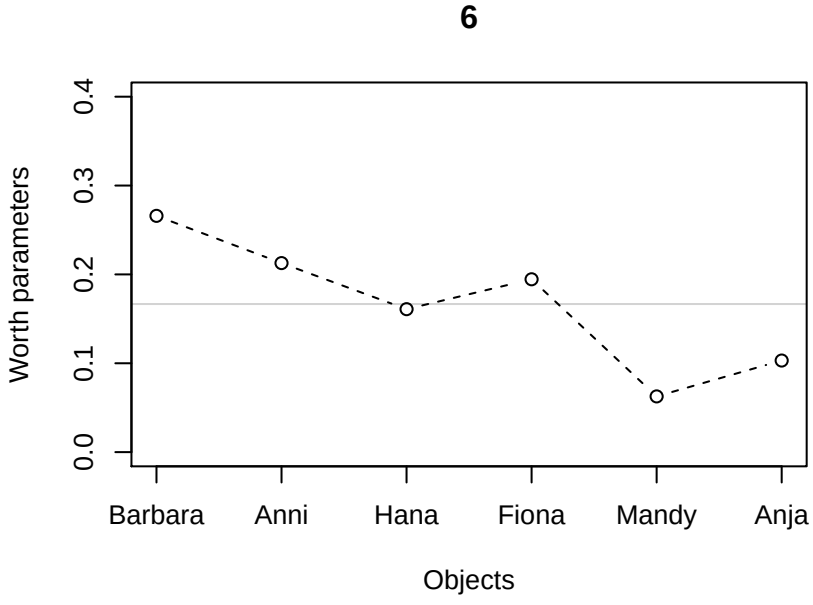
# Bradley-Terry trees

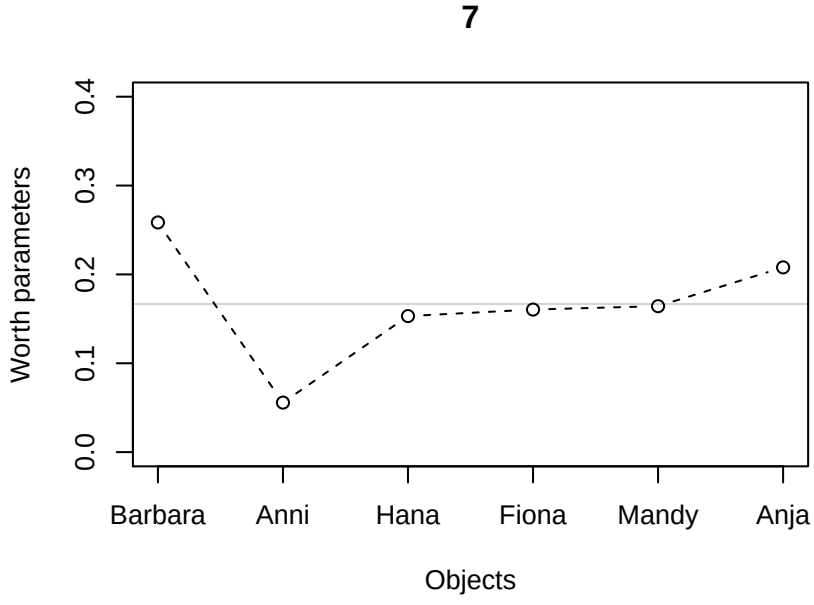


# Bradley-Terry trees



# Bradley-Terry trees

Apply plotting in all terminal nodes:

```
R> par(mfrow = c(2, 2))
R> nodeapply(bt2, ids = c(3, 5, 6, 7), FUN = function(n)
+    plot(n$info$object, main = n$id, ylim = c(0, 0.4)))
```

Predicted nodes and ranking:

```
R> tm
```

```
  age gender q1  q2 q3
1  60   male no  no no
2  25 female no  no no
3  35 female no yes no
```

```
R> predict(bt2, tm, type = "node")
```

```
1 2 3
7 3 5
```

```
R> predict(bt2, tm, type = function(object) t(rank(-worth(object))))
```

```
  Barbara Anni Hana Fiona Mandy Anja
1       1    6    5     4     3    2
2       2    3    1     4     5    6
3       3    4    1     2     6    5
```

# Summary

- Synthesis of parametric data models and algorithmic tree models.
- Based on modern class of parameter instability tests.
- Aims to minimize clearly defined objective function by greedy forward search.
- Can be applied general class of parametric models.
- Alternative to traditional means of model specification, especially for variables with unknown association.
- All new implementation in *partykit*.
- Enables more efficient computations, rapid prototyping, flexible customization.

# References


**Software:**

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Zeileis A, Hothorn T (2014). *Parties, Models, Mobsters: A New Implementation of Model-Based Recursive Partitioning in R.* `vignette("mob", package = "partykit")`

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