Some of future directions of white noise theory

Takeyuki Hida
Professor Emeritus at Nagoya University and at Meijo University

March 3,2014

AMS Subject Classification: 60H40 White Noise Theory

Abstract

Some of future directions of white noise analysis will be presented. First, we shall revisit the space of generalized white noise functionals with a view point of analogue calculus. It is reasonable, in fact, not too large, once the system of variables is taken to be white noise $\{\dot{B}(t), t \in R^1\}$. We then proceed to group theoretic observation of some notions in white noise analysis.

§1. Introduction

The plan of my talk

1. The idea.

The basic idea of white noise theory is the **Reductionism** for random complex systems. Namely, given a random complex phenomenon, we first find a system of *independent* random variables that has the same information as the given random phenomenon. Then, we form functions (usually functionals) of the independent random variables. The analysis of those functions leads us to the study of probabilistic properties of the system in question. Thus

Reduction $\rightarrow$ Functions $\rightarrow$ Analysis $\rightarrow$ Identification, Applications.

2. Our presentation is as follows:

i) Reductionism and noise.

ii) Functionals of a noise.

iii) Harmonic analysis. Some useful transformation groups for the noises.

iv) Towards non commutative calculus.

v) Concluding remarks.

§2. Reductionism and noise

This section is devoted to a brief interpretation of the background of white noise analysis.

Following the reductionism, we first discuss on how to find a system of independent random variables. In most cases we take stochastic processes as random complex systems, so that we take Lévy's infinitesimal equation for a process $X(t)$. It is expressed in the form

$$\delta X(t) = \Phi(X(s), s \le t, Y(t), t, dt), \tag{1}$$

where $Y(t)$ stands for the (possibly infinitesimal) random variable, independent of $X(s), s \le t$ and contains the information that the $X(t)$ gains in the time interval $[t, t + dt)$. It is called the innnovation.

This is, of course, a formal equation, but illustrate the idea to get the *innovation* of a process $X(t)$. The $\{Y(t)\}$ is exactly what we wish to have . See e.g. [**?**]

[Note] One may ask how about the case where t is a discrete parameter case, i.e. digital case. To get innovation we have difficulties of different kind, It is, however, rather easy to discuss independent sequence $\{X_n\}$ and functions of the X_n. On the other hand digital sequences will be used to approximate continuous parameter case, that is an analogue case.

The innovation $Y(t)$ is an idealized random variable. We shall focus our attention to favorable cases,namely, satisfying stationary in t, It is an i.i.d. system. In adition, each $Y(t)$ is atomic in the usual sense.

Such a system $\{Y(t)\}$ is called an *idealized elemental random variables*, abbr. i.e.r.v., or simply called a "noise".

It is possible to classify the system of various noises. See e.g. [**?**]. A noise is parametrized in two ways: one is "time" and the other is "space". Making a long story short, there is a table:

time parameter :	Gaussian $\dot{B}(t)$, Poisson $\dot{P}(t)$
space parameter :	$P'(\lambda)$

The $B(t)$ denotes a Brownian motion, and $P(t)$ does a Poisson process with an intensity parameter λwhich is viewed as a space parameter. The $P(\lambda)$ is obtained by fixing the time and letting the intensity be a variable, so that the derivative $P'(\lambda)$ can be defined.

To fix the idea we take a white noise $\dot{B}(t)$ as the i.e.r.v.'s, which is to be the innovation of some process $X(t)$.

§3. Space of generalized white noise functionals

1) Once a variables, that is $\dot{B}(t)$'s, are given, we come to discuss functions of them. It seems natural to begin with *polynomials* in the given variables.

Start with the simplest case, where we take linear functions of the $\dot{B}(t)$'s. Since the the noise is parametrized by t which is continuous variable, a linear function, in fact functional, is expressed in the form

$$\varphi(\dot{B}) = \int f(u)\dot{B}(u)du. \tag{2}$$

If f is an $L^2(R^1)$-function, it can be expressed in a classical manner as a Wiener integral

$$\int f(u)dB(u).$$

It is well known that such integrals form a Hilbert space $\mathcal{H}_1$ which is isomorphic to $L^2(R^1)$ through the correspondence mentioned above.

Following the reductionism, we need to have single $\dot{B}(t)$ without smearing, or need to define (2) for the case $f(u) = \delta_t(u)$. for this purpose, we restrict $\mathcal{H}_1$ to $\mathcal{H}_1^{(1)}$ involving Wiener integral of f in the Sobolev space $K^1(R^1)$ to have test function space $\mathcal{H}_1^{(1)}$. Hence, we have a Gel'fand triple

$$\mathcal{H}_1^{(1)} \subset \mathcal{H}_1 \subset \mathcal{H}_1^{(-1)}$$

Needless to say,

$$\mathcal{H}_1^{(-1)} \cong K^{(-1)}(R^1).$$

and $\dot{B}(t)$ is a member of $\mathcal{H}_1^{(-1)}$.

2) We then come to the next step.

Suppose we are given a system of polynomials. Since the number of the variables is continuously many (i.e. analogue), so that a polynomial looks like an integral as in the linear case. It is defined intuitively. The system of polynomials generate a vector space, which is denoted by $\mathbf{A}$, and it forms a ring, in fact an integral domain.

We are, however, not so much interested in the algebraic side, but we wish to proceed to the analysis. Note that there is a Gaussian measure μ behind, based on which we shall carry on the analysis. Hence, it is natural to take a system of the *Hermite polynomials* that is a complete orthonormal system in L^2-space defined by the Gaussian measure. Then, we can proceed to the analysis.

3) Once again, we note the variables are idealized random variables, so we it is necessary to give a necessary interpretation to the nonlinear functions. Even quadratic monomials, say $\dot{B}(t)^2$ we have to give some interpretation beyond $\dot{B}(t)^2 = \frac{1}{dt}$ which comes from the convenient equality $(dB(t))^2 = dt$. We should note the difference $(dB(t))^2 - dt$ is still random although it is infinitesimal, so we can not ignore the difference. We should magnify it by multiplying $\frac{1}{(dt)^2}$ to have $(\dot{B}(t))^2 - \frac{1}{dt^2}$, that is the Hermit polynomial $H_2(\dot{B}(t), 1/(dt)$ with variance parameter. Such a direction is in agreement with the method of taking the Hermite polynomials.

4) We can remind the classical case. If we take a complete orthonormal system, say $\{\xi_k\}$, then the ordinary random variables of the product of Hermite polynomials in $< \dot{B}, \xi_k >$ of degree n. Then, we have subspaces $\mathcal{H}_n$ with $n \geq 0$ to establish the Fock space:

$$(L^2) = \bigoplus \mathcal{H}_n,$$

where (L^2) is the Hilbert space of functionals of white noise with finite variance.There is an isomorphism

$$\mathcal{H}_n \cong \sqrt{n!}\hat{L}^2(R^n),$$

where $\hat{L}^2$ means the symmetric L^2. Based on this decomposition, we can define the Gel'fand triple such that

$$\mathcal{H}_n^{(n)} \subset \mathcal{H}_n \subset \mathcal{H}_n^{(-n)},$$

where

$$\mathcal{H}_n^{(n)} \ \cong \ \sqrt{n!}\hat{K}^{(n+1)/2}(R^n),$$

with $K^m(R^n)$ the Sobolev space of order m over R^n. The spaces $\mathcal{H}_n^{(-n)}$, $n \geq 0$ is spanned by the Hermite polynomials in $\dot{B}(t)$'s of degree n. The weighted sum $(L^2)^-$ of $H_n^{(-n)}$'s is the space of *generalized white noise functionals*.

The space $(L^2)^+$ of test functionals can be defined as a suitable weighted sum of $\mathcal{H}_n^{(n)}$ so as to be the dual space of $(L^2)^-$.

What we have discussed can give many significant motivations to problems from related fields. In particular,this is related to the subjects in the next section.

It is noted that we can see useful connections towards the recent works by L. Accardi (the Lecture at Nagoya white noise seminar, January 2014).

§4. Renormalization

1) We have understood that the $\dot{B}(t)$ is not an ordinary random variable, but an idealized variable, and we have given it a place to live, i.e. $\mathcal{H}_1^{(-1)}$. A monomial in $\dot{B}(t)$ of higher degree is made to be the Hermite polynomial of the same degree, because of the Gaussian measure. This modification is often understood to be a trick to subtract off awkward variables which are infinity. This has, of course, only a formal meaning. Even it says in the case where only polynomials are concerned. We wish to find a general policy to do that.

Making a long story short, we take a reproducing kernel Hilbert space $\mathbf{F} = \mathbf{F}(C)$ with the kernel $C(\xi - \eta)$, where $C(\xi$ is the characteristic functional of the Gaussian measure μ, that is $C(\xi) = \exp[-\frac{1}{2}\|\xi\|^2]$.

Allow me to speak of a formal statement; "$\dot{B}(t)$is a stochastic square root of the δ-function". Indeed, $E(\dot{B}(t)\dot{B}(s)) = \delta(t-s)$. In addition $C(\cdot - \xi)$ plays the role of the δ-function in $\mathbf{F}(C)$. More explicitly: for $f \in \mathbf{F}$

$$(f(\cdot), C(\cdot - \xi)) = f(\xi).$$

With this structure, we shall be able to give a reasonable interpretation on taking Hermite polynomials instead of just monomials, or on the modification that we did. Basic method that we use can be simplified in such a way that is illustrated below.

The reproducing kernel $C(\xi - \eta)$ is the inner product of two exponential functionals $e^{i<x,\xi>}$ and $e^{i<x,\xi>}$. Fir notational convenience, we use a letter x to be a sample function of $\dot{B}(t)$.

Remind that $e^{i<x,\xi>}$ can play the role of a test functional, while $e^{i<x,\eta>}$, letting η run through a wider function space, generates general white noise functionals. We are, therefore, suggested to consider the so-called T-transform

$$\varphi(x) \ \to \ (T\varphi)(\xi) = \int e^{i<x,\xi>}\varphi(x)d\mu.$$

We prefer to discuss within the real world not complex valued, so that we switch from T-transform to S- transform

$$\varphi(x) \ \to \ (S\varphi)(\xi) = C(\xi)\int e^{<x,\xi>}\varphi(x)d\mu.$$

The functional $C(\xi)$ is put in front of the above integral for normalization: that is a common functional, with which a constant function is transformed to itself.

Now we can carry on the analysis exactly in manner of what we have expected.

Since the exponential function $e^{<x,\xi>}$ is taken to be a test functional, so that the inner product with a generalized functional can be well defined. Take the simplest functional $\dot{B}(t)$. Then, we have

$$(S\dot{B}(t)(\xi) = \xi(t).$$

rigorously. For $\dot{B}(t)^2$, we have only formal answer. i.e.

$$(S\dot{B}(t)^2)(\xi) = \xi(t)^2 + \frac{1}{dt}.$$

This formula tells us that $\dot{B}(t)^2$ can not be a generalized functional, but it is so after the *renormalization*, i.e. subtraction from monomial. Indeed, this is the additive renormalization, and it says almost the same trick as before, so far as polynomials are concerned. The additive renormalization is linear on the ring $\mathcal{A}$ and multiplicative for a monomial if it is a product of monomials. Thus, we have seen a consistent methods of renormalization for the ring $\mathcal{A}$.

There is a short remark. There exists a class of white noise functionals outside of $\mathbf{A}$. They can be renormalized in some way to be generalized functionals.

2) Exponentials of linear functionals.

As before we use the notation x, a member of E^*, the sample function of $\dot{B}$. Let

$$\varphi(x) = \text{exp}[< x, \xi >].$$

Modify by multiplying the constant to have

$$f(x) = \exp[< x, \xi > -\frac{1}{2}\|\xi\|^2 t^2],$$

which is the generating function of the Hermite polynomial with parameter $\|\xi\|^2$. Indeed,

$$f(x) = \sum_n t^n H_n(x, \|\xi\|^2),$$

each term of which is the renormalized variable of $< x, \xi >^n$. We can therefore understand that $f(x)$ is the renormalized variable with multiplication by $\exp[-\frac{1}{2}\|\xi\|^2 t^2]$. By the power series expansion, we understand that it is the result of term by term additive renomalization. We call this effect the multiplicative renormalization.

3) Exponentials of quadratic functionals.

In this case the use of the S-transform is essential. Both idea and computations depend on the literature [**?**], in particular §.4.6.

Let $\varphi(x)$ be a real-valued $\mathcal{H}_2$ functional,i.e.a quadratic polynomial in $\dot{B}(t)$'s. The $\varphi(x)$ can be expressed in the form

$$\varphi(x) = \int\int F(u,v)x(u)X(v)dudv$$

with a symmetric $L^2(R^2)$-function. Define $g(x)$ by

$$g(x) = \exp[\varphi(x)].$$

We assume that the eigenvalues of the kernel (the integral operator) are all outside of the interval $(0,4]$, so that $g(x)$ is in the space (L^2).

The S-transform of $g(x)$ is expressed in the form

$$(Sg)(\xi) = \delta(2; F)^{-1/2} \exp[\int\int \cap G(u,v)\xi(u)\xi(v)dudv],$$

where

$$\cap G(u,v) = \sum_n (-\lambda_n + 2)^{-1} \eta_n(u)\eta_n(v),$$

with the eigen-system of F: (λ_n, η_n) and where $\delta(2, F)$ is the modified Fredholm determinant.

It is important to note that we can shift this form to the case where φ is a generalized functional of white noise. Since ξ is in a nuclear space, exponential part of the Sg has no problem, but the modified Fredholm determinant will change. Removing such awkward factor is to be the *multiplicative* renormalization.

It is therefore a good problem to give some plausible interpretation to the modified Fredholm determinant.

It is noted that there is an important contribution in this direction made by M. Grothaus and L. Streit,[**?**]. There we can see significance of this approach.

Although some problems are left, let us now sum up what we have discussed so far.

Theorem The renormalization is algebraically idempotent and it is defined so as to be a surjective onto the space $(L^2)^-$ of generalized white noise functionals.

§5. Some useful transformation groups for the noises

Having been motivated by the established theory of transformation groups, we meet, very often, the significant roles played by the groups in white noise theory. We shall choose three topics.

Before we come to the main topics, we need to make some general remark. A noise that we are concerned with has a parameter,either the whole line R^1 or the positive half space. In any case, we are led to take the Affine group involving the shift and the dilation.

A noise has the probability distribution, so that the group of the measure-preserving transformations acting on the measure space.

Thus, we can think of two ways that lead us to the, as it were, harmonic analysis in two ways.

1) Gaussian case.

If we make one point compactification so that $R^1 \cong S^1$, then we are given projective transformations, or conformal group. We may proceed some one-parameter groups coming from the automorphisms of the parameter space.

Infinite dimensional rotation group $O(E)$

After H. Yoshizawa, we define the rotation g of a nuclear space E.

Definition A continuous linear transformation g acting on E is called a *rotation* of E, if it is an orthogonal transformation:

$$\|g\xi\| = \|\xi\|, \ \xi \in E,$$

where $\| \cdot \|$ is the $L^2(R^1)$-norm.

Naturally we are given the adjoint group $O^*(E^*)$, which can characterize the white noise measure μ, which is the probability distribution of the $\{\dot{B}(t)\}$ that is introduced on E^*. This is the reason why the group $O(E)$ is important in the study of white noise analysis.

Most of significant members of the group $O(E)$ are divided into two classes. One is the class I, which is determined as follows. Take a complete orthonormal system $\{\xi_n\}$ each member of which is in E. If

$g \in O(E)$ is defined by a linear isometric (that is orthogonal) transformation, then g is in class I. A member of class I is, in fact, *digital*. While, a member g of class II is a transformation that comes from the change of the parameter t. More explicitly

$$(g\xi)(t) = \xi(f(t))\sqrt{|f'(t)|},$$

where f is a smooth function that defines a surjection of R^1. A one-parameter subgroup of class II members is called a "whisker". Whiskers, often a group of them can serve essential roles in white noise analysis. Examples of such groups are those isomorphic to $PGL(2)$, $SO(d.1)$, $Aff(R^1)$ and so on. Still, we in search of new subgroups of class II members.

2) For a Poisson noise.

Beside the time shift, we can find a simple group that describes the duality between the time t and the intensity λ.

3) The Affine group $Aff(R^1)$ for a new noise.

Consider a noise depending on the space parameter, say λ for Poisson type noise.

Observing the action of the group Since the distribution is viewed as a convergent sequence of positive numbers, we shall take the generating function. Up to constant factor, it plays the key role of the representation of the Affine group. We are thus given an interesting problem to clarify the analytic role of the generating function in connection with the affine group.

The result is due to [17].

Compound noise. A space noise has rather simple invariant properties under transformation groups, however, compound (space) noise enjoys interesting characters. For instance, if the intensity measure parametrized by scale parameter is formed so as to satisfy the dilation invariant, then stable distribution arises. Such a formulation suggest us to find the so-called underlying process, when a long tail distribution is observed as a statistics.

§6. Towards non-commutative white noise analysis.

There are many directions that lead us to non- commutative white noise analysis. To find a road to a systematic approach is a very important to propose future problems, since we already have several examples. We shall show some of them.

1) As for non-commutative analysis which is useful for us, it is natural to be back to the famous literature [2].

2) Hamiltonian path integrals.

Here is a short note on the Hamiltonian path integrals. We also we state briefly the Lagrangian path integral that we have established before. It is known that the relations between Lagrangian and Hamiltonian from mechanics. We wish to know any good probabilistic relations between the two kinds of path integrals.

In the Hamiltonian dynamics, different from Lagrangian dynamics, the variables position (configuration) x and p (momentum) are independent variables, so that we do not understand as $p = m\frac{dq}{dt}$.

The relationship between x and q are connected by $dx \wedge dp$, where non-commutativity appears.

Before we come to the Hamiltonian path integral, we shall have a quick review of earlier approach to the path integral using Lagrangian. See [18].

We set

$$S(t_0, t_1) = \int_{t_0}^{t_1} L(t)dt. \tag{3}$$

and set

$$\exp\left[\frac{i}{\hbar}\int_{t_0}^{t_1} L(t)dt\right] = \exp\left[\frac{i}{\hbar}S(t_0, t_1)\right] = B(t_0, t_1).$$

Then, we have (see Dirac [2]), for $0 < t_1 < t_2 < \cdots < t_n < t$,

$$B(0, t) = B(0, t_1) \cdot B(t_1, t_2) \cdots B(t_n, t).$$

Theorem The quantum mechanical propagator $G(0, t; y_1, y_2)$ is given by the following average

$$G(0, t; y_1, y_2) = \left\langle N e^{\frac{i}{\hbar}\int_0^t L(y,\dot{y})ds + \frac{1}{2}\int_0^t \dot{B}(s)^2 ds} \delta_o(y(t) - y_2) \right\rangle, \qquad (4)$$

where N is the amount of multiplicative renormalization. The average $\langle\ \rangle$ is understood to be the integral with respect to the white noise measure μ.

The basic idea of using white noise analysis seems to be the same as we have reviewed above. Important part is to see how we come to non-commutative calculus.

Now we are in a position to discuss the Hamiltonian path integral. We follow the Klauder-Grothaus-Bock line. Hamiltonian $H(x,p,t)$ is given by

$$H(x,p,t) = \frac{1}{2m}p^2 + V(x,p,t).$$

Hamiltonian action $S(x,p,t)$ is given by

$$S(x,p,t) = \int_0^t p(\tau)\dot{x}(\tau) - H(x(\tau),p(\tau),\tau)d\tau$$

First we give the configuration (coordinate space) path integral, then come to that on the momentum space, using white noise analysis in both cases.

1. Configuration space.

A possible trajectory starting from x_0 is denoted by

$$x(\tau) = x_0 + \sqrt{\hbar/m}B(\tau), 0 \le \tau \le t \tag{5}$$

Momentum p is taken to be

$$p(\tau) = \sqrt{\hbar m}\omega(\tau), 0 \le \tau \le t,$$

where ω is another white noise.

Then, the Feynman integrand I_c is given by :

$$\begin{aligned} I_c &= N\exp[\frac{i}{\hbar}\int_0^t p(\tau)\dot{x}(\tau)-\frac{p(\tau)^2}{2m}d\tau+\frac{1}{2}\int_0^t \dot{x}(\tau)^2+p(\tau)^2 d\tau) \\ &\cdot \exp[-\frac{i}{\hbar}\exp[-\frac{i}{\hbar}\int_0^t V(x(\tau),p(\tau),\tau)d\tau]\delta(x(t)-y), \end{aligned}$$

where N is the (multiplicative) renormalizing constant, the idea of which is as was discussed before. Note that quadratic functionals are there in the exponential. The delta function is put for the pinning effect instead of taking a Brownian bridge.

2. Hamiltonian path integral in momentum space .

The variable $p(\tau)$ has a fluctuation simply as a Brownian motion :

$$p(\tau)=p_0+\frac{\sqrt{\hbar m}}{t}B(\tau), 0\leq\tau\leq t.$$

The same for the space variable $x(\tau)$:

$$x(\tau) = \sqrt{\hbar/m} t\omega(\tau), 0 \leq \tau \leq t.$$

The Feynman integrand I_m is given by

$$\begin{aligned} I_m &= N \exp[\frac{i}{\hbar}\int_0^t (-x(\tau)\dot{p}(\tau) - \frac{p(\tau)^2}{2m}) d\tau + \frac{1}{2}\int_0^t (\omega(\tau)^2 + B(\tau)^2) d\tau] \\ &\quad \cdot \exp[-\frac{i}{\hbar}\int_0^t V(x(\tau), p(\tau), \tau) d\tau] \delta(p(t) - p') \end{aligned}$$

Taking the expectation that is the integral with respect to the white noise measure, we obtain the quantum mechanical propagator.

§7. Concluding remarks

I. Decomposition of Lévy processes; revisited.

In connection with a representation of Affine group.

A noise with the space parameter. Give it a reasonable position to the noise dependeng on the space variable, too. cf. the H. Weyl's literature "Raum, Zeit, Materie", I would say an analogy to this : "Raum, Zeit, Rauschen", at the suggestion of L. Streit.

II. Passage from digital to analogue, and Infinite.

The transition will produce profound, sometimes implicit figure, so that we need deep considerations.

Polynomials in continuously many variables, similarly differential operators.

One more matter to be noted is the number of the variables is continuum, so that we must be careful on the sums (actually continuous sums) of functions and operators acting on them.

[Note] A quotation from Hermann Weyl [20].

”Mathematics is the science of the infinite.”

We may emphasize the significance of continuously infinite (not countable).

We have a system of continuously many variables, that is white noise $\{\dot{B}(t), t \in R^1\}$. Basic functionals are polynomials in the $\dot{B}(t)$'s. The space $H_n^{(-n)}$ of generalized white noise functionals of degree n is spanned by homogeneous polynomials (actually Hermite polynomials) of degree n. In particular, a monomial is expressed in the form

$$\varphi(\dot{B}) = \prod_{\sum p_j = n} : \dot{B}(t_j)^{p_j} :,$$

where t_j's are different.

The number operator $\Delta_\infty = \int \partial_t^* \partial_t dt$ characterizes the subspace $H_n^{(-n)}$ involving homogeneous polynomials of degree n.

A characterization of $H_n^{(-n)}$-functionals can be given by the number operator $\mathcal{N}$ (or the Laplace-Beltrami operator Δ_∞ by putting $-$ in front, i.e. $\Delta_\infty = -\mathcal{N}$.

Proposition On the space $(L^2)^-$ the pair $\{\mathcal{N}, n\}$ forms an eigensystem of the number operator: for $\varphi(\dot{B})$ in $H_n^{(-n)}$ we have

$$\mathcal{N}\varphi = n\varphi.$$

III. Space of the generalized white noise functionals. In this note we have introduce the space $(L^2)^-$, which is fitting to explain other topics in this report. There is, however, anothedr beautiful method to introduce the space of generalized functionals. That is

$$(\mathcal{S}) \subset (L^2) \subset (\mathcal{S})^*.$$

This method has another advantages to discuss the white noise analysis. We shall discuss in other opportunity.

References

[1] S.A. Albeverio, R.J. Hoegh-Krohn, and S. Mazzucchi, Mathematical theory of Feynman path integrals. An introduction.2nd ed. LNM 523, Springer, 2008.

[2] V. Arnold, Mathematical methods of classical mechanics. Springer, 1989.

[3] W. Bock, Hamiltonian path integrals in white noise analysis. Dissertation, Univ. Kaiserslautern. 2013.

[4] A.D. Barbour and Louis H.Y. Chen, Stein's method and applications. World Sci. Pub. Co. 2005.

[5] T. Hida, Analysis of Brownian functionals. Carleton Univ. Math. Notes 13, 1975,

[6] T. Hida, Brownian motion. Springer, 1980.

[7] T. Hida, H.-H. Kuo, J. Potthoff and L. Streit, White noise. An Infinite-dimensional calculua. Kluwer academic Pub. 1993.

[8] T. Hida and Si Si, Lectures on white noise functionals. World Sci. Pub. Co. 2008. [9] T.Hida, White noise (in Japanese). Maruzen Pub. Co.,2014. to appear.

[10] R. Léandre and H. Ouerdiane, Connes-Hida calculus and Bismut-Quillen superconnections.Inst. Élie Cartan.prepub.2003.

[11] P.Lévy, Sur les intégrales dont les éleménts sont des variables aléatoires indépendants. Ann. della R. Scoula Normale Superior di Pisa. Ser. II, III (1934) 337 - 366. Also three C.R. papers (1934).

[12] P. Lévy, Random functions: General theory with with special reference to Laplacian random functions. Univ. of California Pub. in Statistics 1 (1953) 331-388.

[13] D. Mumford, Dawnings of probability theory

[14] J. Potthoff and L. Streit, A characterization of Hida distribution. J. Functional Analisis. 101. (1991), 212-229.

[15] Si Si, Introduction to Hida distributions. World Scientific Pub. Co.2012.

[16] Si Si, Graded rings of homogeneous chaos generated by polynomials in noises depending on time and space parameters, respectively. Moscow Conf. (2013).

[17] Si Si, IDAQP, Singapore Workshop paper (2014)

[18] L. Streit and T. Hida, Generalized Brownian functionals and Feynman integral. Stochastic Processes and their Applications. 16 (1983), 55-69.

[19] M. Grothaus and L. Streit, Quadratic actions, semi-classical approximation, and delta sequences in Gaussian analysis. Report on Math. Phys.44 (1999), 381-405.

[20] H. Weyl, Levels of infinity. Selected writings of infinity (1930). ed. P. Pesic. Dover Pub. 2012.