Weighted Fourier algebras of non-compact Lie groups and its spectrum

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Weighted convolution algebras

- G: locally compact group $\Rightarrow (L^1(G), *)$ is a Banach algebra that can distinguish G.
- A measurable function $\omega : G \to (0, \infty)$ is called a **weight** if it is sub-multiplicative i.e.

$$\omega(st) \leq \omega(s)\omega(t), \quad s, t \in G.$$

- For a weight ω the weighted space $L^1(G;\omega)$ equipped with the norm $\|f\|_{L^1(G;\omega)} = \int_G \omega(x)\,|f(x)|\,dx$ is still **a Banach algebra w.r.t. the convolution**. $L^1(G;\omega)$ is called a **Beurling algebra on** G.
- (**Examples**) $G = \mathbb{R}$ or $\mathbb{Z}$, $\alpha \geq 0$, $\beta \geq 1$.
  $\omega_\alpha(x) = (1 + |x|)^\alpha$ (Polynomial type weights)
  $\omega_\beta(x) = \beta^{|x|}$ (Exponential type weights).

Reformulation using co-multiplication

- We begin with the co-multiplication (the adjoint of the convolution map)

$$\Gamma : L^{\infty}(G) \to L^{\infty}(G \times G)$$

given by $\Gamma(f)(s,t) = f(st)$.

- $(L^1(G;\omega))^* = L^{\infty}(G;\omega^{-1})$ with the norm

$$\|f\|_{L^{\infty}(G;\omega^{-1})} := \left\|\frac{f}{\omega}\right\|_{\infty},$$

so that $\Phi : L^{\infty}(G) \to L^{\infty}(G;\omega^{-1}),\ \ f \mapsto f\omega$ is an isometry.

Reformulation using co-multiplication: continued

- Using the convolution again on $L^1(G;\omega)$ means we will use the same Γ on $L^\infty(G;\omega^{-1})$. Then, the isometry Φ gives us the modified co-multiplication

$$\widetilde{\Gamma} : L^\infty(G) \to L^\infty(G \times G), \;\; f \mapsto \Gamma(f)\Gamma(\omega)(\omega^{-1} \otimes \omega^{-1}).$$

- Note that $\Gamma(\omega)(\omega^{-1} \otimes \omega^{-1}) \leq 1$ iff ω is a weight.
- We would like to do the same procedure in the dual (i.e. Fourier alebra) setting.

The Fourier algebra $A(G)$

- G: locally compact group.
- The group von Neumann algebra $VN(G)$ is given by

  $$\{\lambda(x) : x \in G\}'' \subseteq B(L^2(G)),$$

  where $\lambda(x)$ is the left translation (i.e. $\lambda(x)f(y) = f(x^{-1}y)$).
- $\lambda : G \to B(L^2(G))$ is called the **left regular representation**.
- **(Eymard, '64)**
  $A(G) := VN(G)_* = \{f * \check{g} : f, g \in L^2(G)\} \subseteq C_0(G)$, where $\check{g}(x) = g(x^{-1})$.
- $(A(G), \cdot)$ is known to be a commutative Banach algebra distinguishing G, which we call the **Fourier algebra** on G.
- (Example) $G = \mathbb{R}$
  $(A(\mathbb{R}), \cdot) \cong (L^1(\widehat{\mathbb{R}}), *)$

The Heisenberg group

- $H_1 = \left\{ (x, y, z) = \begin{bmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$ be the Heisenberg group with the Haar measure = the Lebesgue measure on $\mathbb{R}^3$.
- $VN(H_1)$ and $A(H_1)$ can be described concretely using representation theory of H_1.
- For any $r \in \mathbb{R}\backslash\{0\}$ we have the Schrödinger representation
$$\pi^r(x, y, z)\xi(w) = e^{2\pi i r(-wy+z)}\xi(-x+w),\ \xi \in L^2(\mathbb{R}).$$
- We have
$$\lambda \cong \int_{\mathbb{R}\backslash\{0\}}^{\oplus} \pi^r |r| dr,$$
$$VN(H_1) \cong L^\infty(\mathbb{R}\backslash\{0\}, |r|dr; B(L^2(\mathbb{R}))),$$
$$A(H_1) \cong L^1(\mathbb{R}\backslash\{0\}, |r|dr; S^1(L^2(\mathbb{R}))),$$
where $S^1(\mathcal{H})$ is the trace class on $\mathcal{H}$.

The Heisenberg group: continued

- **(Fourier transform on H_1)**
  We define
  $$\mathcal{F}^{H_1} : L^1(H_1) \to VN(H_1)$$
  given by
  $$\mathcal{F}^{H_1}(f)(r) = \int_{H_1} f(x, y, z)\pi^r(x, y, z)dxdydz.$$
- **(Fourier inversion on H_1)**
  We define
  $$(\mathcal{F}^{H_1})^{-1} : A(H_1) \to L^\infty(H_1)$$
  given by for $A = (A(r))_r \in A(H_1)$
  $$(\mathcal{F}^{H_1})^{-1}(A)(x, y, z) = \int_{\mathbb{R}\setminus\{0\}} \mathrm{Tr}(A(r)\pi^r(x, y, z))|r|dr.$$

Weighted Fourier algebra

- Recall that ω on G gives us M_ω a (unbdd) closed, densely defined, positive, invertible operator affilliated to $L^\infty(G)$ acting on $L^2(G)$.
- For $VN(G) \subseteq B(\mathcal{H})$ we will consider W, a (unbdd) closed, densely defined, positive, invertible operator affilliated to $VN(G)$ acting on $\mathcal{H}$.
- We consider the weighted spaces $VN(G; W^{-1}) := \{AW : A \in VN(G)\}$, $\|AW\|_{VN(G;W^{-1})} = \|A\|_{VN(G)}$ and $A(G; W) := \{W^{-1}\phi : \phi \in A(G)\}$, $\left\|W^{-1}\phi\right\|_{A(G;W)} = \|\phi\|_{A(G)}$.
- $\Phi : VN(G) \to VN(G; W^{-1}),\ \ A \mapsto AW$ is an (complete) isometry.

Weighted Fourier algebra: continued

- The co-multiplication this time is given by

$$\Gamma : VN(G) \to VN(G \times G), \;\; \lambda(x) \mapsto \lambda(x) \otimes \lambda(x).$$

- If we use *"the same"* Γ on $VN(G; W^{-1})$, then by applying Φ we get a modified co-multiplication

$$\widetilde{\Gamma} : VN(G) \to VN(G \times G), \;\; A \mapsto \Gamma(A)\Gamma(W)(W^{-1} \otimes W^{-1}).$$

- We say W is a **weight** on the dual of G if W^{-1} is bounded and (loosely speaking)

$$\left\| \Gamma(W)(W^{-1} \otimes W^{-1}) \right\| \leq 1.$$

- Then $A(G; W)$ is a commutative Banach algebra under the pointwise multiplication.
- (**Definition**, Ludwig/Spronk/Turowska '12, L/Samei '12) We call $A(G; W)$ a **Beurling-Fourier algebra on** G.

Extension of weights

- One serious problem of $A(G;W)$ is to find a nontrivial weight W.
- **(Extension procedure)**
  $H < G$ an abelian subgroup and $\phi : \widehat{H} \to (0,\infty)$ a weight. Then the operator $W = i(M_\phi)$ is a weight on the dual of G, where i is the embedding

$$i : L^\infty(\widehat{H}) \cong VN(H) \hookrightarrow VN(G),\ \lambda_H(x) \mapsto \lambda_G(w).$$

- **(Example)** Let X be the subgroups $X = \{(x,0,0) : x \in \mathbb{R}\}$ of H_1. By applying the extension procedure we get the weight W^β_X using the weight function $\phi(t) = \beta^{|t|}$, $\beta \geq 1$ on $\mathbb{R}$. We can easily check the following.

$$W^\beta_X(r)\xi = \widehat{\phi} * \xi = \widehat{(\phi \cdot \check{\xi})}.$$

Spectrum of $A(G)$ and $A(G;\omega)$

- Recall that $\mathrm{Spec}A(G) \cong G$, where $\mathrm{Spec}A(G)$ is the space of non-zero mutiplicative functionals on $A(G)$.
- We believe that $\mathrm{Spec}A(G;\omega)$ is actually coming from the points of the **complexification** $G_{\mathbb{C}}$ of G.
- For a (real) Lie group G we can associate its (real) Lie algebra $\mathfrak{g}$. Then the complexified Lie agebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} + i\mathfrak{g}$ might have its associated Lie group $G_{\mathbb{C}}$.
- We call $G_{\mathbb{C}}$ the **complexification** of G.
- $\mathbb{R}_{\mathbb{C}} = \mathbb{C}, \quad (H_1)_{\mathbb{C}} = \left\{ \begin{bmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}.$
- (**Ludwig/Spronk/Turowska, '12**)
  Our belief is true for compact groups!

The case of $\mathbb{R}$

- Let $\varphi : A(\mathbb{R};\omega_\beta) \cong L^1(\widehat{\mathbb{R}};\omega_\beta) \to \mathbb{C}$ is multiplicative (w.r.t. ptwise multiplication).
- Let $\tilde{\varphi} = \varphi|_{\mathcal{A}}$, where $\mathcal{A} = C_c^\infty(\widehat{\mathbb{R}})$. Then $\tilde{\varphi}$ is a distribution which is multiplicative w.r.t. convolution.
- In other words, $\tilde{\varphi}$ is a solution to the (distributional) Cauchy functional equation

  $$f(x+y) = f(x)f(y),\ x, y \in \mathbb{R},$$

  which we know that the solution must be of the form e^{icx} for some $c \in \mathbb{C}$.
- (1) The density of $\mathcal{A}$ in $A(\mathbb{R};\omega_\beta)$ (2) the norm condition for φ and complex Fourier inversion gives us the condition

  $$|Imc| \leq \log\beta.$$

- Note that $\beta = 1$ recovers Spec$A(\mathbb{R}) \cong \mathbb{R}$.

The case H_1: the real challenge

- We will take the same approach as the case of $\mathbb{R}$ by using the Euclidean structure behind H_1. First we consider a subalgebra $\mathcal{A} = \mathcal{F}^{\mathbb{R}^3}(C_c^\infty(\mathbb{R}^3))$ of $A(H_1; W_X^\beta)$.
- If $(*)$ $\mathcal{A}$ is dense in $A(H_1; W_X^\beta)$, then we can conclude that $u \in \mathrm{Spec}A(H_1; W_X^\beta)$ is uniquely determined by a point $(\tilde{x}, \tilde{y}, \tilde{z}) \in \mathbb{C}^3 \cong (H_1)_\mathbb{C}$ using distributional Cauchy functional equation on $\mathbb{R}^3$.
- If $(**)$ $\mathcal{A}$ has enough many elements that allows complex Fourier inversion of $\mathcal{F}^{H_1}$, then we can conclude as follows.
- **(Ghandehari/L./Samei/Spronk, preprint)** Let $u = u_{(\tilde{x}, \tilde{y}, \tilde{z})}$ is the character on $\mathcal{A}$ coming from $(\tilde{x}, \tilde{y}, \tilde{z}) \in (H_1)_\mathbb{C} \cong \mathbb{C}^3$. Then u is bounded on $A(H_1; W_X^\beta)$ iff (1) $|\mathrm{Im}\tilde{x}| \leq \frac{1}{2\pi} \log \beta$ and (2) $\mathrm{Im}\tilde{y} = \mathrm{Im}\tilde{z} = 0$.
- We could not check the conditions $(*)$ and $(**)$, but we found an intermediate space that fills the gap!!

Other non-compact Lie groups

- The case of the Euclidean motion group $E(2)$ can be done similarly, but easier!
- The case of $ax + b$ group is still open due to the absence of enough elements allowing complex Fourier inversions.