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Deduction of non–commutativity from commutativity

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Main theses of the present talk.

1) Emergence of a **generalized quantum mechanics** from 19–th century classical analysis:
the theory of orthogonal polynomials.

2) The **microscopic structure** of classical probability is intrinsically non commutative: **quantum decomposition of a classical random variable** with all moments.

3) Emergence of non–commutativity from commutativity.
The classical probabilistic roots of Heisenberg commutation relations.

4) Probabilistic generalization of quantum mechanics.

5) Program of a purely algebraic classification of probability measures on $\mathbb{R}^d$ with finite moments of any order.

6) Connections with the white noise programme:
– reductionism
– ultra–reductionism
– full democracy.

Orthogonal polynomials
Notations

– $d \in \mathbb{N}$,

$$D := \{1, \cdots, d\}$$

– polynomial $*$–algebra in d commuting indeteminates:

$$\mathcal{P} := \mathbb{C}[(X_j)_{j \in D}]$$

$$X_j = X_j^*$$

– generators: monomials

$$M = X_1^{n_1} \cdots X_d^{n_d} \qquad ; \qquad n_j \in \mathbb{N}$$

– degree

$$deg(M) := \sum_{j \in D} n_j = n$$

Theorem 1 For a pre–scalar product $\langle \cdot , \cdot \rangle$ on $\mathcal{P}$ the following statements are **equivalent**:

(i) The pre–scalar product $\langle \cdot , \cdot \rangle$ **is induced by a state** φ on $\mathcal{P}$, i.e. there exists a state φ on $\mathcal{P}$ such that:

$$\varphi(f^*g) = \langle f, g \rangle \qquad ; \qquad f, g \in \mathcal{P}$$

(ii) There **exists a probability measure** μ on $\mathbb{R}^d$ with finite moments of all orders such that $\forall b \in \mathcal{P}$:

$$\varphi(b) := \int_{\mathbb{R}^d} b(x_1, \cdots, x_d) d\mu(x_1, \cdots, x_d) \qquad (1)$$

(iii) The pre–scalar product $\langle \cdot , \cdot \rangle$ satisfies

$$\langle 1_{\mathcal{P}}, 1_{\mathcal{P}} \rangle_{\varphi} = 1 \qquad (2)$$

and, for each $j \in D$, multiplication by the **coordinate** X_j **is a symmetric linear operator** on $\mathcal{P}$ with respect to $\langle \cdot , \cdot \rangle$:

$$\langle X_j f, g \rangle = \langle f, X_j g \rangle \qquad (3)$$

In the following we fix a pre–scalar product $\langle \cdot , \cdot \rangle$ **satisfying (3).**
The pair

$$(\mathcal{P} , \langle \cdot , \cdot \rangle)$$

is a a pre–Hilbert space in which **the** X_j**'s are symmetric linear operators**.

The degree filtration

$\mathcal{P}_{n]} :=$ lin–span{monomials $M : deg(M) \leq n\} =$

$= $ polynomials of degree $\leq n$

$P_{n]} : \mathcal{P} \to \mathcal{P}_{n]} =$ **pre–Hilbert** space orthogonal proj.

$m \leq n \Rightarrow P_{m]}P_{n]} = P_{m]}$ (**increasing** filtration)

$$P_n := P_{n]} - P_{n-1]}$$

$$P_n : \mathcal{P} \to \mathcal{P}_{n-1]}^{\perp} \cap \mathcal{P}_{n]} = \mathcal{P}_n$$

(P_n) is a **partition of** 1, i.e.:

$$m \neq n \to P_m P_n = 0 \qquad ; \qquad \sum_{n \in \mathbb{N}} P_n = 1$$

Therefore (P_n) defines the following **orthogonal gradation** of $\mathcal{P}$:

$$\mathcal{P} = \bigoplus_{n \in \mathbb{N}} \mathcal{P}_n = \bigoplus_{n \in \mathbb{N}} P_n \mathcal{P}$$

$$\dim(\mathcal{P}_n) := \begin{pmatrix} n+d-1 \\ d-1 \end{pmatrix} \sim n^{d-1} \qquad (4)$$

$$\sum_{n\in\mathbb{N}} P_n = 1 \qquad \Rightarrow$$

$$X_j = \left(\sum_{m\in\mathbb{N}} P_m\right) X_j \left(\sum_{n\in\mathbb{N}} P_n\right) = \sum_{m,n\in\mathbb{N}} P_m X_j P_n \Rightarrow$$

$$X_j P_n = \sum_{m\in\mathbb{N}} P_m X_j P_n$$

Theorem 2 Symmetric Jacobi Relation

$$X_j P_n = P_{n+1} X_j P_n + P_n X_j P_n + P_{n-1} X_j P_n$$

The CAP operators of a state on $\mathcal{P}$

CAP $\equiv$ Creation–Annihilation–Preservation

Start from the symmetric Jacobi Relation

$$X_j P_n = P_{n+1} X_j P_n + P_n X_j P_n + P_{n-1} X_j P_n$$

and define, for each n, the linear operators

$$a^{+}_{j;n} := P_{n+1} X_j P_n \; : \; \mathcal{P}_n \to \mathcal{P}_{n+1}$$

$$a^{0}_{j;n} := P_n X_j P_n \; : \; \mathcal{P}_n \to \mathcal{P}_n$$

$$a^{-}_{j;n} := P_{n-1} X_j P_n \; : \; \mathcal{P}_n \to \mathcal{P}_{n-1}$$

$a^{\pm}_{j;n} \;\equiv\;$ rectangular matrices
$a^{0}_{j;n} \;\equiv\;$ square matrix

Theorem 3 (The quantum decomposition of the coordinates X_j $(j \in \{1, \ldots, d\})$.
Defining the linear operators

$$a_j^+ := \sum_n P_{n+1} X_j P_n = \sum_n a_{j;n}^+ \text{ (\textbf{creation})}$$

$$a_j^- := \sum_n P_{n-1} X_j P_n = \sum_n a_{j;n}^- \text{ (\textbf{annihilation})}$$

$$a_j^0 := \sum_n P_n X_j P_n = \sum_n a_{j;n}^0 \text{ (\textbf{preservation})}$$

one has

$$(a_j^+)^* = a_j^- \quad ; \quad (a_j^0)^* = a_j^0$$

$$X_j = a_j^+ + a_j^0 + a_j^- \tag{5}$$

The decomposition (5) is called
the quantum decomposition of X_j
$(j \in \{1, \ldots, d\})$.

Non–commutativity deduced from commutativity

The coordinate operators X_j **commute**:

$$X_j X_k = X_k X_j$$

This commutativity implies that the CAP operators a_j^+, a_j^0, a_j^- **do not commute**.

Much more is possible:
we can deduce the **explicit form** of
the commutation relations!

Deduction of the commutation relations among the CAP operators

(1) quantum decomposition

$$X_j = a_j^+ + a_j^0 + a_j^- \qquad ; \qquad j \in \{1, \cdots, d\}$$

(2) commutativity of the X_j's

$$0 = [X_j, X_k] \qquad ; \qquad j,\ k \in \{1, \cdots, d\}$$

imply that:

$$= [(a_j^+ + a_j^0 + a_j^-), (a_k^+ + a_k^0 + a_k^-)] =$$

$$= [a_j^+, a_k^+] + [a_j^+, a_k^0] + [a_j^+, a_k^-] +$$

$$+ [a_j^0, a_k^+] + [a_j^0, a_k^0] + [a_j^0, a_k^-] +$$

$$+ [a_j^-, a_k^+] + [a_j^-, a_k^0] + [a_j^-, a_k^-]$$

Recall that:

– a_j^+ increases the gradation degree by 1

– a_j^0 preserves the gradation

– a_j^- decreases the gradation degree by 1

Therefore:

$[a_j^+, a_k^+]$ increases the gradation degree by 2

$[a_j^0, a_k^+] + [a_j^+, a_k^0]$
increases the gradation degree by 1

$[a_j^+, a_k^-] + [a_j^0, a_k^0] + [a_j^-, a_k^+]$
preserves the gradation

$[a_j^0, a_k^-] + [a_j^-, a_k^0]$
decreases the gradation degree by 1

$[a_j^-, a_k^-]$ decreases the gradation degree by 2

Therefore, because of the mutual orthogonality of the $\mathcal{P}_k$'s, the identity

$$0 = [a_j^+, a_k^+] +$$

$$+ \; [a_j^+, a_k^0] \; + \; [a_j^0, a_k^+] +$$

$$+ \; [a_j^+, a_k^-] \; + \; [a_j^0, a_k^0] \; + \; [a_j^-, a_k^+] +$$

$$+ \; [a_j^0, a_k^-] \; + \; [a_j^-, a_k^0] +$$

$$+ \; [a_j^-, a_k^-]$$

implies that each of the rows must be zero separately:

$$[a_j^+, a_k^+] = 0 \quad \text{commutativity of the creators} \tag{6}$$

$$[a_j^-, a_k^-] = 0 \quad \text{commutativity of the annihilators} \tag{7}$$

$$[a_j^+, a_k^0] + [a_j^0, a_k^+] = 0 \tag{8}$$

$$[a_j^0, a_k^-] + [a_j^-, a_k^0] = 0 \tag{9}$$

$$[a_j^+, a_k^-] + [a_j^0, a_k^0] + [a_j^-, a_k^+] = 0 \tag{10}$$

These relations are not independent, e.g.

$$[a_j^+, a_k^+] = 0 \;\Leftrightarrow\; [a_j^-, a_k^-] = 0$$

The independent commutation relations are:

$$[a_j^+, a_k^+] = 0 \tag{11}$$

$$[a_j^+, a_k^0] + [a_j^0, a_k^+] = 0 \tag{12}$$

$$[a_k^-, a_j^+] - [a_k^-, a_j^+]^* = [a_j^0, a_k^0] =: i\hbar_{j,k} \tag{13}$$

where

$$\hbar_{j,k} = \hbar_{j,k}^*$$

We will prove that the symmetric, gradation preserving operators $\hbar_{j,k}$ are non–trivial generalizations of **Planck's constant**.

Thus we kept our promise in the title of the present talk:
we deduced non–commutativity from commutativity .
More precisely (and more relevant):
we deduced the explicit form of the commutation relations among the CAP from commutativity.

The commutation relations for symmetric states on $\mathcal{P}$

Definition

A state φ on $\mathcal{P}$ is called **symmetric** if:

$$\varphi(\text{odd monomial}) = 0$$

For symmetric states on $\mathcal{P}$ the commutation relations are simpler due to the following (well known) theorem.

Theorem

φ is symmetric if and only if

$$a_j^0 = 0 \qquad ; \qquad \forall j \in \{1, \ldots, d\}$$

The independent **commutation relations for symmetric states** on $\mathcal{P}$ are:

$$[a_j^+, a_k^+] = 0 \tag{14}$$

$$[a_k^-, a_j^+] = [a_k^-, a_j^+]^* =: \hbar_{j,k} = \hbar_{j,k}^* \tag{15}$$

Digression on non–commutative geometry

Present approach: one **postulates** that

$$X_j X_k \neq X_k X_j$$

Quantum probability approach:
the **coordinates commute** but,
given the distribution of masses,
they have an **intrinsic microscopic structure** which is non–commutative.
The explicit form of the commutation relations is **deduced from the mutual commutativity of the coordinates**.

The case $d = 1$

In the case $d = 1$ the orthogonal gradation of $\mathcal{P}$ is made of 1–dimensional spaces (because X^n is the only monomial of degree n):

$$\mathcal{P} = \bigoplus_n \mathbb{C} \cdot \Phi_n^0$$

where

$$\Phi_n^0 := P_n(X^n \cdot 1) = P_{n]}(X^n \cdot 1) - P_{n-1]}(X^n \cdot 1)$$

and the polynomials (Φ_n^0) are orthogonal but not normalized.

Since there is only one index, we omit it from notations:

$$a_j^\varepsilon \ \to \ a^\varepsilon \qquad ; \qquad \varepsilon \in \{-1, 0, +1\}$$

In this case, since trivially

$$[a^+, a^+] = [a^0, a^0] = 0$$

the (independent) **commutation relations** are reduced to a single one:

$$[a^-, a^+] = a^- a^+ - a^+ a^- =: \hbar = \hbar^* \qquad (16)$$

We know that $\hbar$ is a symmetric, gradation preserving operator.
Since the gradation is made of 1–dimensional spaces, this implies that
the Φ_n^0 are eigenvectors of $\hbar$
Since they are an orthogonal basis there cannot be other eigenvectors.

We want to determine the corresponding eigenvalues.
To this goal we normalize this sequence:

$$\Phi_n := \begin{cases} \Phi_n^0 / ||\bar{\Phi}_n^0|| & \text{if} \quad \Phi_n^0 \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

With this notation and recalling that

$$\Phi \in \mathcal{P} \cdot 1 \ \mapsto \ \Phi^* \in (\mathcal{P} \cdot 1)^*$$

denotes the embedding of $\mathcal{P} \cdot 1$ into its dual:

$$\Phi^*\Phi := \langle \Phi, \Phi \rangle$$

one has

$$P_n = \Phi_n \Phi_n^* : Q \cdot 1 \in \mathcal{P} \cdot 1 \ \rightarrow \ \langle \Phi_n, Q \cdot 1 \rangle \Phi_n \in \mathcal{P}_n$$

and the symmetric Jacobi relation becomes:

$$X\Phi_n\Phi_n^* = P_{n+1}XP_n + P_nXP_n + P_{n-1}XP_n$$

$$= \langle \Phi_{n+1}, X\Phi_n \rangle \Phi_{n+1}\Phi_n^* +$$

$$+ \langle \Phi_n, X\Phi_n \rangle \Phi_n\Phi_n^* + \langle \Phi_{n-1}, X\Phi_n \rangle \Phi_{n-1}\Phi_n^*$$

Define

$$\langle \Phi_{n+1}, X\Phi_n \rangle =: \beta_n$$

$$\langle \Phi_n, X\Phi_n \rangle =: \alpha_n \in \mathbb{R}$$

Fact: also β_n is real

$$\beta_n = \bar{\beta}_n \in \mathbb{R}$$

With these notations the symmetric Jacobi Relation for polynomials in one variable becomes:

$$X\Phi_n = \langle \Phi_{n+1}, X\Phi_n \rangle \Phi_{n+1} +$$

$$+ \langle \Phi_n, X\Phi_n \rangle \Phi_n + \langle \Phi_{n-1}, X\Phi_n \rangle \Phi_{n-1}$$

$$= \; \beta_n \Phi_{n+1} \; + \; \alpha_n \Phi_n \; + \; \beta_{n-1} \Phi_{n-1}$$

The CAP operators in one variable

Recalling the definition of the CAP operators in one variable, one finds

$$a^{+} = \sum P_{n+1} X P_n = \sum \Phi_{n+1} \Phi_{n+1}^{*} X \Phi_n \Phi_n^{*} =$$

$$= \sum \langle \Phi_{n+1}, X\Phi_n \rangle \Phi_{n+1} \Phi_n^{*} = \sum \beta_n \Phi_{n+1} \Phi_n^{*}$$

Therefore

$$a^{-} = (a^{+})^{*} = \sum \beta_n \Phi_n \Phi_{n+1}^{*}$$

$$a^{0} = \sum \alpha_n \Phi_n \Phi_n^{*}$$

Commutation relations in symmetric form

The operators $a^{-}a^{+}$ and $a^{+}a^{-}$ preserve the orthogonal gradation:

$$a^{-}a^{+}\Phi_n = a\beta_n\Phi_{n+1}\Phi_n^*\Phi_n = \beta_n a^{-}\Phi_{n+1} = \beta_n^2\Phi_n$$

$$a^{+}a\Phi_n = \beta_{n-1}a^{+}\Phi_{n-1} = \beta_{n-1}^2\Phi_n$$

$$(a^{-}a^{+} - a^{+}a^{-})\Phi_n = [a^{-}, a^{+}]\Phi_n = (\beta_n^2 - \beta_{n-1}^2)\Phi_n$$

Thus defining:

$$N\Phi_n := n\Phi_n$$

one obtains from the spectral theorem:

$$\hbar = [a^{-}, a^{+}] = (\beta_N^2 - \beta_{N-1}^2) = \sum(\beta_n^2 - \beta_{n-1}^2)\Phi_n\Phi_n^*$$

Summing up:
the eigenvalue of $\hbar$, corresponding to the eigenvector Φ_{n+1} (or Φ_{n+1}^0) is

$$\beta_n^2 - \beta_{n-1}^2$$

$$\langle\Phi_{n+1}, X\Phi_n\rangle =: \beta_n = \bar{\beta}_n$$

Monic Jacobi relations

From now on, to avoid verbal complications, we assume that

$$\Phi_n \neq 0 \qquad ; \qquad \forall n \in \mathbb{N}$$

$\Leftrightarrow$ the support of the measure(s) associated to the state φ is not reduced to a finite number of points.
In this case

$$\Phi_n := \frac{\Phi_n^0}{\|\bar{\Phi}_n^0\|} \qquad ; \qquad \forall n$$

and one can always assume that the polynomial Φ_n^0 is monic.
Dividing both sides of the identity

$$X\Phi_n = \beta_n \Phi_{n+1} + \alpha_n \Phi_n + \beta_{n-1}\Phi_{n-1}$$

by $\|\bar{\Phi}_n^0\|$ one finds:

$$X\Phi_n^0 = \beta_n \frac{||\Phi_n^0||}{||\Phi_{n+1}^0||}\Phi_{n+1}^0 + \alpha_n \Phi_n^0 + \beta_{n-1}\frac{||\Phi_n^0||}{||\Phi_{n-1}^0||}$$

The fact that the polynomial Φ_n^0 is monic implies that

$$\beta_n \frac{||\Phi_n^0||}{||\Phi_{n+1}^0||} = 1 \Leftrightarrow \beta_n = \frac{||\Phi_{n+1}^0||}{||\Phi_n^0||}$$

In particular (additional proof that the β_n are real and in fact positive)

$$\beta_n \geq 0$$

Therefore

$$\frac{||\Phi_n^0||}{||\Phi_{n-1}^0||} = \beta_{n-1}$$

and the Symmetric Jacobi relations become equivalent to

$$\Leftrightarrow X\Phi_n^0 = \beta_n \frac{||\Phi_n^0||}{||\Phi_{n+1}^0||} \Phi_{n+1}^0 + \alpha_n \Phi_n^0 + \beta_{n-1} \frac{||\Phi_n^0||}{||\Phi_{n-1}^0||}$$

$$\Leftrightarrow X\Phi_n^0 = \Phi_{n+1}^0 + \alpha_n \Phi_n^0 + \beta_{n-1}^2 \Phi_{n-1}^0$$

Commutation relations in monic form

Defining:

$$\omega_n := \beta_{n-1}^2 \geq 0 \qquad (> 0 \text{ in our case})$$

the symmetric Jacobi relations

$$X\Phi_n^0 = \Phi_{n+1}^0 + \alpha_n \Phi_n^0 + \beta_{n-1}^2 \Phi_{n-1}^0 =$$

become **the monic Jacobi relations**

$$\Phi_{n+1}^0 + \alpha_n \Phi_n^0 + \omega_n \Phi_{n-1}^0$$

This is the form of the Jacobi relations found in the books.
It must be complemented by the condition:

$$\omega_n = 0 \;\Rightarrow\; \omega_{n+k} = 0 \quad ; \quad \forall k \geq 0$$

The sequence (ω_n) is called **the principal Jacobi sequence**.

The sequence (α_n) is called **the secondary Jacobi sequence**.

The monic commutation relations

The monic form of the commutation relations is then:

$$[a^-, a^+] = (\beta_N^2 - \beta_{N-1}^2) = (\omega_N - \omega_{N-1}) =$$

$$= \sum (\omega_n - \omega_{n-1}) \Phi_n \Phi_n^*$$

The commutation relation

$$[a^-, a^+] \ = \ (\omega_N - \omega_{N-1})$$

are called **the probabilistic commutation relations in monic form**.

The Gaussian case

It is known that the principal Jacobi sequence for the standard Gaussian measure on $\mathbb{R}$ are given by

$$\omega_n = n \qquad ; \qquad \alpha_n = 0$$

With this choice the commutation relations become

$$[a^-, a^+] = (\omega_N - \omega_{N-1}) = (N - (N-1)) =$$

$$= id_{\mathcal{P}} =: 1$$

These are the
Heisenberg commutation relations

$$[a^-, a^+] = 1$$

If, instead of the Gaussian, one chooses other measures one obtains different generalizations of the Heisenberg commutation relations.

The following table provides table provides some examples.

Measure	Polynomials	Jacobi parameters
Gaussian $N(0,\sigma^2)$	Hermite $H_n(x;\sigma^2)$ $= (-\sigma^2)^n e^{x^2/2\sigma^2} \partial_x^n e^{-x^2/2\sigma^2}$	$\alpha_n = 0$ $\omega_n = \sigma^2 n$ $(\lambda_n = \sigma^{2n} n!)$
Poisson Poi (a)	Charlier $C_n(x;a) =$ $(-1)^n a^{-x} \Gamma(x+1) \Delta^n \left[\frac{a^x}{\Gamma(x-n+1)}\right]$	$\alpha_n = n + a$ $\omega_n = an$ $(\lambda_n = a^n n!)$
Gamma $\Gamma(\alpha)$, $(\alpha > -1)$ $\frac{1}{\Gamma(\alpha+1)} x^\alpha e^{-x}$, $x > 0$	Laguerre $\mathcal{L}_n^{(\alpha)}(x)$ $= (-1)^n x^{-\alpha} e^x \partial_x^n [x^{n+\alpha} e^{-x}]$	$\alpha_n = 2n + 1 + \alpha$ $\omega_n = n(n+\alpha)$ $(\lambda_n =$ $n!(n+\alpha)\cdots(1+\alpha))$
Uniform on $[-1,1]$	Legendre $\tilde{L}_n(x) = \frac{1}{2^n (2n-1)!!} \partial_x^n [(x^2-1)^n]$	$\alpha_n = 0$ $\omega_n = \frac{n^2}{(2n+1)(2n-1)}$ $\left(\lambda_n = \frac{(n!)^2}{[(2n-1)!!]^2 (2n+1)}\right)$
Arcsine $\frac{1}{\pi\sqrt{1-x^2}}$, $\|x\| < 1$ $n \geq 1$	Chebyshev (1st kind) $\tilde{T}_0(x) = 1$ $\tilde{T}_n(x) = \frac{1}{2^{n-1}} \cos(n \cos^{-1} x)$, $\left(\lambda_n = \frac{1}{2^{2n-1}}\right)$	$\alpha_n = 0$ $\omega_n = \begin{cases} \frac{1}{2}, & n = 1 \\ \frac{1}{4}, & n \geq 2 \end{cases}$
Semicircle $\frac{2}{\pi}\sqrt{1-x^2}$, $\|x\| < 1$	Chehyshev (2nd kind) $\tilde{U}_n(x) = \frac{1}{2^n} \frac{\sin[(n+1)\cos^{-1} x]}{\sin[\cos^{-1} x]}$	$\alpha_n = 0$ $\omega_n = \frac{1}{4}$ $\left(\lambda_n = \frac{1}{4^n}\right)$
$\frac{1}{\sqrt{\pi}} \frac{\Gamma(\beta+1)}{\Gamma\left(\beta+\frac{1}{2}\right)} (1-x^2)^{\beta-\frac{1}{2}}$ $\|x\| < 1$, $\beta > -\frac{1}{2}$ $= C_n^{(\beta)} (1-x^2)^{\frac{1}{2}-\beta}$ $\partial_x^n [(1-x^2)^{n+\beta-\frac{1}{2}}]$	Gegenbauer $\tilde{G}_n^{(\beta)}(x)$ $\omega_n = \frac{n(n+2\beta-1)}{4(n+\beta)(n+\beta-1)}$	$\alpha_n = 0$ $C_n^{(\beta)} = \frac{(-1)^n 2^n \Gamma(2\beta+n)}{\Gamma(2\beta+2n)}$

$[a^-,a^+]e_n=\hbar e_n$	Coherent vector	Generating function
$\sigma^2 I$	$e^{\frac{zx}{\sigma^2}-\frac{z^2}{2\sigma^2}}$	$e^{tx-\frac{1}{2}\sigma^2t^2}=\sum_{n=0}^{\infty}\frac{H_n(x;\sigma^2)}{n!}t^n$
aI	$e^{-z}\left(1+\frac{z}{a}\right)^x$	$e^{-at}(1+t)^x=\sum_{n=0}^{\infty}\frac{C_n(x;a)}{n!}t^n$
$(2n+\alpha+1)e_n$	$\sum_{n=0}^{\infty}\frac{\mathcal{L}_n^{(\alpha)}(x)}{n!(n+\alpha)\cdots(1+\alpha)}z^n$	$(1+t)^{-\alpha-1}e^{\frac{tx}{1+t}}$ $=\sum_{n=0}^{\infty}\frac{t^n}{n!}\mathcal{L}_n^{(\alpha)}(x)$
$-\frac{1}{(2n+3)(2n+1)(2n-1)}e_n$	$\sum_{n=0}^{\infty}\frac{((2n-1)!!)^2(2n+1)}{(n!)^2}$ $\tilde{L}_n(x)z^n$	$\frac{1}{\sqrt{1-2tx+t^2}}$ $=\sum_{n=0}^{\infty}\frac{(2n-1)!!}{n!}\tilde{L}_n(x)t^n$
$\begin{cases}\frac{1}{2}e_0, & n=0\\ -\frac{1}{4}e_1, & n=1\\ 0, & n\geq 2\end{cases}$	$\frac{1-2xz}{1-4xz+4z^2}$	$\frac{4-t^2}{4-4tx+t^2}$ $=\sum_{n=0}^{\infty}\tilde{T}_n(x)t^n$
$\begin{cases}\frac{1}{4}e_0, & n=0\\ 0, & n\geq 1\end{cases}$	$\frac{1}{1-4xz+4z^2}$	$\frac{4}{4-4tx+t^2}=\sum_{n=0}^{\infty}\tilde{U}_n(x)t^n$
$\frac{\beta^2-\beta}{2(n+1+\beta)(n+\beta)(n-1+\beta)}$	not in closed form	$\frac{1}{(1-2tx+t^2)^{\beta}}$ $=\sum_{n=0}^{\infty}\frac{2^n\Gamma(\beta+n)}{\Gamma(\beta)n!}\tilde{G}_n^{(\beta)}(x)t^n$

Probabilistic extensions of quantum mechanics: the free evolution

Start from the probabilistic commutation relations in monic form:

$$[a^-, a^+] \ = \ (\omega_N - \omega_{N-1})$$

$$[a^+, a^+] = [a^-, a^-] = 0$$

From

$$[a^-, a^+a^-] = [a^-, a^+]a^- = (\omega_N - \omega_{N-1})a^-$$

one deduces that

$$\frac{d}{dt}e^{ita^+a^-}a^-e^{-ita^+a^-} = ite^{ita^+a^-}[a^+a^-, a^-]e^{-ita^+a^-}$$

$$= -ite^{ita^+a^-}(\omega_N - \omega_{N-1})a^-e^{-ita^+a^-}$$

$$= -it(\omega_N - \omega_{N-1})e^{ita^+a^-}a^-e^{-ita^+a^-}$$

Because a^+a^- leaves the orthogonal gradation invariant, in particular

$$[a^+a^-, N] = 0$$

Therefore, denoting

$$a^-(t) := e^{ita^+a^-}a^-e^{-ita^+a^-}$$

since $a^-(0) = a^-$ one obtains

$$\frac{d}{dt}a^-(t) = -it(\omega_N - \omega_{N-1})a^-(t) \qquad ; \quad a^-(0) = a^-$$

whose unique solution is

$$a^-(t) = e^{ita^+a^-}a^-e^{-ita^+a^-} = e^{-it(\omega_N - \omega_{N-1})}a^-$$

Therefore

$$a^+(t) = a^+e^{it(\omega_N - \omega_{N-1})}$$

This implies, using $[a^+a^-, N] = [a^-a^+, N] = 0$, that

$$[a(t), a^+(t)] = e^{-it(\omega_N - \omega_{N-1})}a^-a^+e^{it(\omega_N - \omega_{N-1})} -$$

$$-a^+e^{it(\omega_N - \omega_{N-1})}e^{-it(\omega_N - \omega_{N-1})}a^-$$

$$= a^-a^+ - a^+a^- = [a, a^+]$$

Thus the map

$$t \in R \ \mapsto \ a^{\pm}(t) = \begin{cases} a^{+}e^{\pm it(\omega_N - \omega_{N-1})} \\ e^{-it(\omega_N - \omega_{N-1})}a^{-} \end{cases}$$

is a $*$–Lie–algebra isomorphism, hence it extends to an (associative) $*$–algebra isomorphism of the universal enveloping algebra of $(a^{+}, a^{-}, 1)$.

This means that the $*$–algebra

$$\text{Pol}(a^{\pm}) := \text{algebraic span of}\{a^{\pm}\}$$

is left invariant by its unique $*$–automorphism extending the generalized free evolution.

This map is called
the generalized free evolution.

Problem

What is the multi–dimensional analogue of the Jacobi sequences of a given probability measure μ on $\mathbb{R}$?

$$\{(\omega_n)_{n\in\mathbb{N}} \; ; \; (\alpha_n)_{n\in\mathbb{N}}\} \; , \; \omega_n \in \mathbb{R}_+ \; ; \; \alpha_n \in \mathbb{R} \; , \; n \in \mathbb{N} \tag{17}$$

subjected to the only constraint that, for any n, $k \in \mathbb{N}$,

$$\omega_n = 0 \Longrightarrow \omega_{n+k} = 0 \tag{18}$$

Which is the multi–dimensional analogue of the relation

$$\omega_n = 0 \implies \omega_{n+k} = 0$$

The answer to this question came after more than 15 years of successive approximations.
see the paper:

Luigi Accardi, Abdessatar Barhoumi,
Ameur Dhahri:
Identification of the theory of multi–dimensional orthogonal polynomials with the theory of symmetric interacting Fock spaces with finite dimensional 1–particle space,
Preprint 2013

Abstract

The talk explains how a simple and non trivial generalization of quantum mechanics emerges of from a well established topic of 19–th century classical analysis: the theory of orthogonal polynomials.

In fact every classical random variable with all moments has a unique, intrinsic, **quantum decomposition** in terms of generalized creation, annihilation and preservation (CAP) operators.

The commutation relations among the CAP operators are uniquely determined by the principal Jacobi sequence of the probability distribution of the classical random variable.

This shows that classical probability possesses a **microscopic structure** which is intrinsically non commutative. In this sense one can speak of:
emergence of non–commutativity from commutativity.

Standard quantum mechanics corresponds to the equivalence class (for the relation of having the same principal Jacobi sequence) of the standard Gaussian or Poisson measure on $\mathbb{R}$.

The recently obtained multi–dimensional generalization of these results suggests a new approach to non–commutative geometry.