New noise depending on the space parameter and the concept of multiplicity

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Abstract

Our work is in line with the reductionism applied to the study of random complex system, which may be expressed as functionals of a noise obtained by reducing the given phenomena. A noise, we understand a system of idealized elemental random variables formed by independent identically distributed random variables.

We are particularly interested in the noise which is depending on a continuous space parameter, which is an or-

dered set. We can define a system $E(\lambda), \lambda \in (0, \infty)$ of projections and therefore appeal to the Stone-Hellinger-Hahn type Theorem, where the notion of the multiplicity arises as a characteristic of the noise in question.

1. Introduction

A noise, we understand it is a system of idealized elemental random variables depending on a continuous parameter. Standard noises are classified according to the probability distribution and type. We know that there are classes

i) Gaussian depending on time,

ii) Poisson type depending on time,

iii) Poisson type depending on space.

We recognize these noises by approximation, because a system of continuously many independent identically distributed random variables is not easy to be dealt with. We may understand by approximating such a system by a sequence of countably many independent random variables, that is by digital systems. The limit with reasonable assumptions directs us to the three cases mentioned above. For details, we refer to the paper [1].

The noises i) and ii) are well known, however the case iii) is not so popular. We shall particularly focus our attention to the class iii).

2. The noise of the type iii)

A Poisson distribution, as is well-known, may arise in the study of the law of small probability, where we recognize that there is a freedom to choose the intensity, denoted by λ, of the limiting Poisson distribution. It is a parameter different from the time t. An interesting interplay between t and λ may be found in a Poisson process, but this is not a topic to be discussed in this report.

We are interested in the characterization of probability distributions of various noises. For this purpose, transformation of the probability measure space plays an important role. Some general observation will be given in the next section.

Then, we come to the main topic; namely investigations of the space noise. There, to fix the idea, we forget the time t and will discuss the probabilistic roles of λ, viewed as a *space parameter*. Finally, we shall discuss some connections with decomposition of a Lévy process with a special emphasis on classification of components due to the type of the probability distribution.

3. Invariance of probability distributions of noises.

Probability distributions of noises, which are systems of idealized random variables, are introduced on the space of generalized functions. We are, therefore, use characteristic functionals. Since the noise may be considered to be additive, in a sense, we prefer to take the so-called ψ-functional, which is the logarithm of the characteristic functional. In fact, the ψ-functional is additive for sum of independent random variables, regardless they are ordinary or idealized.

We now remind

i) Gaussian case, that is $\dot{B}(t)$.

The ψ functional is $-\frac{1}{2}\|\xi\|^2$. It involves the L^2-norm, which immediately implies the invariance of Euclidean distance, that is rotations or orthogonal transformations acting on the space of generalized functions on which probability distribution of $\dot{B}(t), t \in R^1$ is introduced.

We know that the infinite dimensional rotation group plays extremely important roles in the white noise analysis.

A trivial note is that we may introduce scale, that is to have $\psi(\xi) = -\frac{\sigma^2}{2}\|\xi\|^2$. We then have a distribution of the same type.

ii) Poisson type depending on time,

We now come to Poisson noises. Most elemental distribution is a single Poisson distribution with intensity λ. Single

Poisson noise is not so much interested at present.

iii) Poisson type depending on space parameter.

We wish to introduce the space parameter to a collection of Poisson type distributions which is a family of probability distributions and two of every pair are not of the same type. We can therefore expect to find an important technique fitting for the discussion on the *invariance* depending on the parameter. Actually, the parameter to be introduce is the *intensity* of Poisson distribution. The intensity can be viewed as a space variable, this can be illustrated by the construction of noise of Poisson type, cf Lévy [1] and our recent report [3].

Details will be discussed in the next section.

4. Space noise

The basic idea of this section is to find how to combine suitably parameterized Poisson distributions (each component is atomic in type) so that the compound distribution satisfies invariance under a certain group acting on the probability measure space.

An atomic distribution with space parameter is a Poisson distribution with intensity λ. We now introduce a slack variable u which denotes the scale (in reality, u does not play essential role from the viewpoint of the classification according to the type) and plays the role of a "label", as it were, of the intensity. Thus, we have a characteristic function $\varphi(z)$ of the distribution in question expressed in the form:

$$\varphi(z) = \exp[\lambda(e^{izu} - 1)],$$

where $z \in R^1$ and $\lambda, u > 0$.

We now modify the characteristic function. One thing, change imaginary variable iz to real t since we stick to real. As has been mentioned before, we take the φ-function. Moreover, we modify it a little to have

$$\varphi(t) = \lambda e^{tu}. \tag{1}$$

This is acceptable, since ψ-functions of two positive sequences $\{a_n\}$ and $\{ca_n\}, c > 0$, are the same up to constant $\log c$. We are interested only in analytic properties. Namely, we take the moment generating function of the sequence $\frac{\lambda^n}{n!}, n = 0, 1, 2, \cdots$.

Our discussion now starts with the key function (1). Since the variable t runs through R^1, which is one-dimensional Euclidean space, so that we can immediately think of the Affine group $Aff(R^1)$, which is denoted simply by A.

Now define the operators $g(a,b)$, parametrized by (a,b) acting on t-space R^1 such that for $g \in A$

$$g = \{g(a,b) : a, b \in R^1\}.$$

determined by

$$g(a,b)t = at + b.$$

In terms of the matrix form, we may write in the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

$$\psi(g(a,b)t) = \lambda e^{bu} e^{atu} \tag{2}$$

We can see that, by the action $g(a,b)$, a Poisson distribution or its generating function changes in such a way that the intensity λ changes to λe^{bu} and the scale u goes to au.

Thus we have a new function of **different type**, so that we have to have a sum of them. This is the reason why we have to introduce “multiplicity”.

The generating function tells us

Proposition

By the action of the dilation a the intensity does not change.

Here is an important remark. The above proposition means that the scale parameter u cannot control the intensity. This fact gives us an important suggestion on the decomposition of a Lévy process. Namely, if the intensity does not change, then the type of the distribution remains within the *same type*. So, changing u does not contribute to the decomposition.

Now consider a representation of the group A on the convex hull $G = \{\psi(g(\cdot, \cdot)t), g \in A\}$ spanned by the generating functions applying the action of the Affine group.

First, fix the parameter λ of the generating function. Take finitely many dilations, say a_j's. Then, we may form

$$\sum_j \psi(g(a_j, 0)t)$$

which corresponds to a sum of *independent* Poisson type random variables. Obviously the sum can not be a linear combination, since we do not touch the intensity. Also the sum should be only finite sum because of convergence.

Definition

Let λ (discrete spectrum) be fixed. The number of the independent variables is called the **multiplicity** of the representation.

In fact, the multiplicity may be called discrete multiplicity corresponding to the point spectrum of the intensity.

It is easy to establish a relationship between the generating functions just obtained and Lévy process with the intensity being fixed but with different jumps as many as the number of the multiplicity.

5. The intensity measure

Coming back to the affine group, we now restrict our attention to the shift, i.e. the action of b. Then, we have

$$\psi(g(1,b)t) = \lambda e^{bu} e^{tu}$$

We may repeat such operations as many times as we wish by changing the amount of the shift and by choosing different u's, we can conclude

Theorem

The mapping $\psi(g(1,b)t)$ coming from the shift of t generates a measure, denote it by $dn(\lambda)$ sitting in front of e^{tu} of the ψ function.

Outline of the proof

1) member $g(1,b)$ in the Affine group A acts in such a way that

$$\psi(g(1,b)t) = \exp(\lambda e^{bu} e^{tu})$$

Taking various b in R^1 and apply $g(1,b)$ repeatedly. Then form a *convex hull*

$$K = \{\lambda(\alpha e^{bu} + \beta e^{b'u'}); \alpha, \beta > 0, \alpha + \beta = 1\}$$

.

Let $K^1(R^1)$ be the Sobolev space of order 1 over R^1. A continuous positive generalize function defined on $K^1(R^1)$ is a measure.

Take $f(u)$ in the convex hull K and ξ in $K^1(R^1)$, respectively. Form

$$\mathcal{F} = \{f; < f, \xi >= \int f(u)\xi(u)du, \geq 0, \int \frac{u^2}{1+u^2} f(u)du < \infty.\}.$$

where f is continuous.

Existence of such f's is shown by examples. .

Obviously $\mathcal{F}$ is a subset of $K^{(-1)}(R^1)$, so that the closure of $\mathcal{F}$ can be defined. Let it be denoted by $\mathcal{F}'$.

Each member of $\mathcal{F}'$ is a "measure" denoted by $dn(u)$.

We have a freedom to choose any measure dn to have a function $\int e^{tu}dn(u)$ which can be a generating function, since the sum (integral) preserves the property to be a ψ function because a sum of ψ-function corresponds to a sum of independent random variables.

It is easy to be back to a generating functions of a probability distribution by the normalization of the sequence to be a probability distribution.

5) Take any measure dn, By the general theory of measures we have a decomposition

$$dn = dn_c + dn_d,$$

where dn_c is the continuous par and dn_d is the discrete part.

From the discrete part we can choose atoms. For each atom u_d we can find the multiplicity. ore precisely

$$\psi(t) = \sum_k n(u_d) e^{tu_d},$$

which is a finite sum, because of the integrability.

The factors e^{tu} with λ generate a measure which may be denoted by $dn(\lambda)$.

The measure is decomposed into two parts: continuous part dn_0 and discrete parts $n_d(\lambda_k)$, which is countable.

It is easily seen that every discrete point λ_k admits the multiplicity (> 1), produced by dilations. While each λ in the continuous part produces infinitesimal random variables of Poisson type, so that further consideration will be done in this direction.

[Note 1] We have to pause for a while to remind the note made when we met a difficulty to manage continuously many independent ordinary random variables. Here has happened a similar situation. In addition, we have to consider the multiplicity by using the projections $E(\lambda)$. This fact will be discussed in a separate paper.

[Note 2] It seems to be better to express the measure dn_0 as $dn_0(\lambda(u))$ or simply by $dn_0(u)$ because the translation of t by b always in the form bu which gives the factor e^{bu}. Such a notation helps us to understand the decomposition of a Lévy process. Namely, we finally have

Theorem

The general form of the ψ-function that admits invariance under the Affine transform is expressed in the form

$$\psi(t) = \int dn_0(u)(e^{tu} - 1) + \sum_k \lambda_k (\sum_j e^{tu_{j,k}}).$$

Here is reminded once again that $dn(u)$ involves a discrete part that produces the multiplicity.

6. Concluding Remarks

1) Important Fact

So far we have discussed the characterization of the sequence of positive numbers

$$\{\frac{\lambda^k}{k!}, k = 0, 1, 2, \cdots\}.$$

by using the action of the affine group $Aff(R^1)$.

Now,let it be back to **probability distribution**. Namely, we put $e^{-\lambda}$ to have

$$\left\{\frac{\lambda^k}{k!}e^{-\lambda}, k = 0, 1, 2, \cdots\right\}$$

Having returned to the Poisson distribution, we may revisit a decomposition of a Lévy process from "our" viewpoint (by using Group action).

Since the variable t runs through R^1, it is quite natural to consider the action by $Aff(R^1)$.

Note : If we come to the n-dimensional case, i.e. the case R^n, then we use the group $Aff(R^n)$ which is the skew-product $GL(n, R)$ and R^n:

$$at + b \ \rightarrow \ A\mathbf{t} + \mathbf{h},$$

where A is $n \times n$ non-singular matrix and $\mathbf{t}$ is n-dimensional vector. Then, the meaning of the group action gets more clear.

2) Lévy process

Change the variable t of the generating function to iz to have the characteristic function. Then, we understand it correspond to a decomposition of a Lévy process.

The discrete part n_d of the intensity measure would lead us to a (new!) general decomposition of a Lévy process with the **multiplicity**.

Significant remark is the “Decomposition” means that different components should be of different type.

cf. Decomposition of a natural number n to be a product of prime number. This is the meaning of decomposition.

3) Hellinger-Hahn theory

We do not use directly the Stone-Hellinger-Hahn theorem concerning unitary group, but we have used the idea to use the resolution of the identity and the roles of the projection $E(\lambda)$.

4) We have not mentioned on the multiplicity for the case of continuous spectrum $dn(u)$.

If $dn(u)$ satisfies dilation invariant property, then $dn_c(u) = \frac{c}{u^{1+\alpha}}, 0 < \alpha < 2$. Then,certainly the multiplicity is one.

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