Non-principal ultrafilters, program extraction and higher order reverse mathematics

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Outline

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Higher order arithmetic

Definition (RCA_0^ω, Recursive comprehension, Kohlenbach '05)

RCA_0^ω is the finite type extension of RCA_0:

- Sorted into type 0 for $\mathbb{N}$, type 1 for $\mathbb{N}^\mathbb{N}$, type 2 for $\mathbb{N}^{\mathbb{N}^\mathbb{N}}$, $\dots$,
- contains basic arithmetic: 0, successor, $+$, $\cdot$, λ-abstraction,
- quantifier-free axiom of choice restricted to choice of numbers over functions ($\mathsf{QF\text{-}AC}^{1,0}$), i.e.,

$$\forall f^1 \, \exists y^0 \, \mathsf{A}_{\mathit{qf}}(f, y) \rightarrow \exists G^2 \, \forall f^1 \, \mathsf{A}_{\mathit{qf}}(f, G(f))$$

- and a recursor R_0, which provides primitive recursion (for numbers),

$$R_0(0, y^0, f) = y, \qquad R_0(x+1, y, f) = f(R_0(x, y, f), x),$$

- Σ_1^0-induction.

The closed terms of RCA_0^ω will be denoted by T_0.
In Kohlenbach's books this system is denoted by $\widehat{\mathsf{E\text{-}PA}}^\omega\!\upharpoonright + \mathsf{QF\text{-}AC}^{1,0}$.

Functional interpretation

Theorem (Functional interpretation)

If

$$\mathsf{RCA}_0^\omega \vdash \forall x\, \exists y\, \mathsf{A}_{\mathit{qf}}(x,y)$$

the one can extract a term $t \in T_0$, *such that*

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Sketch of proof.

Apply the following proof translations:

- Elimination of extensionality,
- a negative translation,
- Gödel's Dialectica translation. □

See Kohlenbach: Applied Proof Theory.

The intuition behind the functional interpretation

Each formula can be assigned an equivalent $\forall\exists$-formula.
E.g.

$$A :\equiv \forall x\, \exists y\, \forall z\, A_{qf}(x, y, z)$$

will be assigned

$$A^{ND} \equiv \forall x\, \forall f_z\, \exists y\, A_{qf}(x, y, f_z(y)).$$

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- This assignment preserves logical rules, like

$$\frac{A \qquad A \to B}{B},$$

  and exhibits programs.
- Thus, to prove the program extraction theorem we only have to provide programs for the axioms.

Arithmetical comprehension

Let Π^0_1-CA be the schema

$$\forall f\,\exists g\,\forall n\ (g(n)=0 \leftrightarrow \forall x\,f(n,x)=0)\,.$$

Define ACA^ω_0 to be $\mathsf{RCA}^\omega_0 + \Pi^0_1$-CA.

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Let Feferman's μ be

$$\mu(f) := \begin{cases} \min\{x \mid f(x)=0\} & \text{if } \exists x\, f(x)=0, \\ 0 & \text{otherwise.} \end{cases}$$

Denote by (μ) be the statement that μ exists.

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Denote by (μ) be the statement that μ exists.

Theorem

- $\mathsf{RCA}^\omega_0 + (\mu) \vdash \Pi^0_1$-CA
- $\mathsf{RCA}^\omega_0 + (\mu)$ *is* Π^1_2*-conservative over* ACA^ω_0

Theorem (Functional interpretation relative to μ)

If

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the one can extract a term $t \in T_0[\mu]$, *such that*

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We interpreted ACA_0^ω non-constructively using μ.
One can also interpret ACA_0^ω directly using bar recursion.
See Avigad, Feferman in Handbook of Proof Theory

Outline

Filter

Filter

A set $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ is a *filter over* $\mathbb{N}$ if

- $\forall X, Y \ (X \in \mathcal{F} \wedge X \subseteq Y \to Y \in \mathcal{F})$,
- $\forall X, Y \ (X, Y \in \mathcal{F} \to X \cap Y \in \mathcal{F})$,
- $\emptyset \notin \mathcal{F}$

Ultrafilter

A filter $\mathcal{F}$ is an *ultrafilter* if it is maximal, i.e., $\forall X \left(X \in \mathcal{F} \vee \overline{X} \in \mathcal{F}\right)$

$\mathcal{P}_n := \{X \subseteq \mathbb{N} \mid n \in X\}$ is an ultrafilter. These filters are called *principal*.

The Fréchet filter $\{X \subseteq \mathbb{N} \mid X \text{ cofinite}\}$ is a filter but not an ultrafilter.

Non-principal ultrafilters

A set $\mathcal{U} \subseteq \mathcal{P}(\mathbb{N})$ is a *non-principal ultrafilter over* $\mathbb{N}$ if

- $\forall X \left(X \in \mathcal{U} \vee \overline{X} \in \mathcal{U}\right)$,
- $\forall X, Y \ (X \in \mathcal{U} \wedge X \subseteq Y \to Y \in \mathcal{U})$,
- $\forall X, Y \ (X, Y \in \mathcal{U} \to X \cap Y \in \mathcal{U})$,
- $\forall X \ (X \in \mathcal{U} \to X$ is infinite$)$.

The existence of a non-principal ultrafilter is not provable in ZF.

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Coding sets as characteristic function, i.e, $n \in X :\equiv [X(n) = 0]$, this can be formulated in RCA_0^ω:

$$(\mathcal{U}): \begin{cases} \exists \mathcal{U}^2 \left(\forall X^1 \left(X \in \mathcal{U} \vee \overline{X} \in \mathcal{U}\right)\right. \\ \qquad \wedge \forall X^1, Y^1 \ (X \cap Y \in \mathcal{U} \to Y \in \mathcal{U}) \\ \qquad \wedge \forall X^1, Y^1 \ (X, Y \in \mathcal{U} \to (X \cap Y) \in \mathcal{U}) \\ \qquad \wedge \forall X^1 \ (X \in \mathcal{U} \to \forall n \, \exists k > n \, (k \in X)) \\ \qquad \left. \wedge \forall X^1 \ (\mathcal{U}(X) =_0 \mathrm{sg}(\mathcal{U}(X)) =_0 \mathcal{U}(\lambda n. \mathrm{sg}(X(n)))) \right) \end{cases}$$

Lower bound on the strength of $\mathsf{RCA}_0^\omega + (\mathcal{U})$

Theorem (K.)

$$\mathsf{RCA}_0^\omega + (\mathcal{U}) \vdash (\mu)$$

In particular, $\mathsf{RCA}_0^\omega + (\mathcal{U})$ *proves arithmetical comprehension.*

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Proof.

Let $f\colon \mathbb{N} \to \mathbb{N}$ and set $X_f := \{n \mid \exists m \leq n \; f(m) = 0\}$.
Then

$$\begin{aligned} \exists n\,(f(n) = 0) &\iff X_f \text{ is cofinite} \\ &\iff X_f \in \mathcal{U} \end{aligned}$$

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Thus

$$\forall f \; (X_f \in \mathcal{U} \to \exists n \;\; f(n) = 0 \qquad\qquad\qquad\qquad)$$

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Thus

$$\forall f\,(X_f \in \mathcal{U} \to \exists n\,(f(n) = 0 \wedge \forall n' < n\, f(n') \neq 0))$$

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$\mathsf{QF\text{-}AC}^{1,0}$ yields a functional satisfying (μ). $\square$

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$\mathsf{RCA}_0^\omega + (\mathcal{U})$ *is* Π^1_2*-conservative over* $\mathsf{RCA}_0^\omega + (\mu)$ *and thus also over* ACA_0^ω*.*

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Suppose $\mathsf{RCA}_0^\omega + (\mathcal{U}) \vdash \forall f\, \exists g\, \mathsf{A}(f, g)$ and A does not contain $\mathcal{U}$.

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1. The functional interpretation yields a term $t \in T_0[\mu]$, such that

$$\forall f\, \mathsf{A}(f, t(\mathcal{U}, f)).$$

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2. Normalizing t, such that each occurrence of $\mathcal{U}$ in t is of the form

$$\mathcal{U}(t'(n^0)) \qquad \text{for a term } t'(n^0) \in T_0[\mathcal{U}, \mu, f].$$

In particular, $\mathcal{U}$ is only used on countably many sets (for each fixed f).

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3. Build in $\mathsf{RCA}_0^\omega + (\mu)$ a filter which acts on these sets as ultrafilter.

Step 1: Functional interpretation

Suppose $\mathsf{RCA}_0^\omega + (\mathcal{U}) \vdash \forall f^1 \, \exists g^1 \, \mathsf{A}(f, g)$
where A is arithmetical and does not contain $\mathcal{U}$.

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Recall $(\mathcal{U})$:

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(\mathcal{U}) : \begin{cases} \exists \mathcal{U}^2 \, (\, \forall X^1 \, \left(X \in \mathcal{U} \vee \overline{X} \in \mathcal{U} \right) \\ \qquad \wedge \forall X^1, Y^1 \; (X \cap Y \in \mathcal{U} \to Y \in \mathcal{U}) \\ \qquad \wedge \forall X^1, Y^1 \; (X, Y \in \mathcal{U} \to (X \cap Y) \in \mathcal{U}) \\ \qquad \wedge \forall X^1 \; (X \in \mathcal{U} \to \forall n \, \exists k > n \, (k \in X)) \\ \qquad \wedge \forall X^1 \; (\mathcal{U}(X) =_0 \mathrm{sg}(\mathcal{U}(X)) =_0 \mathcal{U}(\lambda n.\, \mathrm{sg}(X(n))))\,) \end{cases}
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Modulo $\mathsf{RCA}_0^\omega + (\mu)$ this is of the form $\exists \mathcal{U}^2 \, \forall Z^1 \, (\mathcal{U})_{\mathit{qf}}(\mathcal{U}, Z)$.

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$$\mathsf{RCA}_0^\omega + (\mu) \vdash \forall \mathcal{U}^2 \, \forall f^1 \, \exists Z^1 \, \exists g^1 \, \Big((\mathcal{U})_{\mathit{qf}}(\mathcal{U}, Z) \to \mathsf{A}_{\mathit{qf}}(f, g) \Big).$$

Step 1: Functional interpretation (cont.)

$$\mathsf{RCA}_0^\omega + (\mu) \vdash \forall \mathcal{U}^2 \, \forall f^1 \, \exists Z^1 \, \exists g^1 \left((\mathcal{U})_{\mathsf{qf}}(\mathcal{U}, Z) \to \mathsf{A}_{\mathsf{qf}}(f, g) \right).$$

The functional interpretation extracts terms $t_Z, t_g \in T_0[\mu]$, such that

$$\mathsf{RCA}_0^\omega + (\mu) \vdash \forall \mathcal{U}^2 \, \forall f^1 \left((\mathcal{U})_{\mathsf{qf}}(\mathcal{U}, t_Z(\mathcal{U}, f)) \to \mathsf{A}_{\mathsf{qf}}(f, t_g(\mathcal{U}, f)) \right).$$

Step 2: Term normalization

The terms t_Z, t_g are made of

- 0, successor, $+$, $\cdot$, λ-abstraction
- the primitive recursor R_0, i.e.

$$R_0(0, y, f) = y, \qquad R_0(x+1, y, f) = f(R_0(x, y, f), x),$$

- μ^2 and
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Lemma

The terms t_Z, t_g can be normalized, such that each occurrence of $\mathcal{U}$ is of the form

$$\mathcal{U}(t'(n^0)) \qquad \text{for a term } t' \text{ possible containing } \mathcal{U}, f.$$

Step 2: Term normalization (cont.)

Proof.

Consider $t[\mathcal{U}, f, n^0]$, where $\mathcal{U}, f, n^0$ are variables.
Assume that all possible λ-reductions haven been carried out. Then one of the following holds:

1. $t = 0$,
2. $t = S(t'_1)$, $t = f(t'_1)$, $t = t'_1 + t'_2$, $t(n) = t'_1 \cdot t'_2$,
3. $t = \mu(t'_g)$, $t = \mathcal{U}(t'_g)$, $t = R_0(t'_1, t'_2, t'_g)$.

Restart the procedure with t'_1, t'_2 and $t'_g m^0$. □

Step 3: Construction of (a substitute for) $\mathcal{U}$

We fix an f and construct a filter $\mathcal{F}$, such that

$$\mathsf{RCA}_0^\omega + (\mu) \vdash (\mathcal{U})_{\mathsf{qf}}(\mathcal{F}, t_Z(\mathcal{F}, f)). \tag{$*$}$$

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This yields then

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and thus the theorem.

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Let $t_1, \ldots, t_k$ be the list term with $\mathcal{U}(t_j(n))$ in t_Z, t_g.

- Assume that $t_1, \ldots$ is ordered according to the subterm ordering.
- We start with the trivial filter $\mathcal{F}_0 = \{\mathbb{N}\}$.
- For each t_i we build a refined $\mathcal{F}_i \supseteq \mathcal{F}_{i-1}$ such that $(\mathcal{U})_{\mathit{qf}}$ relativized the sets coded by $t_1, \ldots, t_i$ holds.
- $\mathcal{F} := \mathcal{F}_k$ solves then $(*)$.

Step 3: Sketch of the construction of $\mathcal{F}_1$

Let $\mathcal{A} := \{A_1, A_2, \dots\}$ be the set of subsets of $\mathbb{N}$ coded by t_1.
We assume that $\mathcal{A}$ is closed under union, intersection and inverse.

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We want a filter $\mathcal{F}_1$, such that

- $\forall X \in \mathcal{A} \left(X \in \mathcal{F}_1 \vee \overline{X} \in \mathcal{F}_1\right)$,
- $\forall X, Y \in \mathcal{A}\ (X \in \mathcal{F}_1 \wedge X \subseteq Y \to Y \in \mathcal{F}_1)$,
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Construction:

- We decide for each $i = 1, 2, \dots$ whether we put A_i or $\overline{A_i}$ into $\mathcal{F}_1$.
- We put A_i into $\mathcal{F}_1$ if the *intersection of A_i with the previously chosen sets* is infinite. Otherwise we put $\overline{A_i}$ into $\mathcal{F}_1$.

Program extraction

Corollary (to the proof)

If $\mathsf{RCA}_0^\omega + (\mathcal{U}) \vdash \forall f\, \exists g\, \mathsf{A}_{qf}(f,g)$ *and* A_{qf} *does not contain* $\mathcal{U}$ *then one can extract a term* $t \in T_0[\mu]$, *such that*

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Corollary

If $\mathsf{RCA}_0^\omega + (\mathcal{U}) \vdash \forall f\, \exists g\, \mathsf{A}_{qf}(f,g)$ *and* A_{qf} *does not contain* $\mathcal{U}$ *then one can extract a term* t *in Gödel's System* T, *such that*

$$\mathsf{A}_{qf}(f, t(f))$$

Proof.

- The previous corollary yields a term primitive recursive in μ.
- Interpreting the term using the bar recursor $B_{0,1}$ and then using Howard's ordinal analysis gives a term $t \in T$. □

Outline

The general concept

The proof theory

- Functional interpretation (Step 1)
- Term normalization (Step 2)

The combinatorics

Construction of the partial ultrafilter on the countable algebra. (Step 3)

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Extension to

- abstract types (Günzel, ongoing work),
- type 3 operators, e.g. Lebesgue measure defined on all subsets of unit interval. (K. '13)

The general concept

The proof theory

- Functional interpretation (Step 1)
- Term normalization (Step 2)

Extension to

- abstract types (Günzel, ongoing work),
- type 3 operators, e.g. Lebesgue measure defined on all subsets of unit interval. (K. '13)

The combinatorics

Construction of the partial ultrafilter on the countable algebra. (Step 3)

Extension to

- idempotent ultrafilters by using *iterated Hindman's theorem* (K. '12),
- possibly other type 2 operators.

Possible Applications

Possible Applications:

- Program extraction for ultralimit arguments e.g.,
  - from fixed point theory,
  - Ergodic theory.
- Program extraction for non-standard arguments.

Outline

Summary

- Program extraction and conservativity for non-principal ultrafilters.
- The Π^1_2-consequences of $\mathsf{RCA}^\omega_0 + (\mathcal{U})$ and the Π^1_2-consequences of ACA^ω_0 are the same.
- Combination of functional interpretation and program normalization applicable to other principles:
    - idempotent ultrafilters
    - Lebesgue measure

Summary

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Thank you for your attention!

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