On embedding certain partial orders into the P-points under RK and Tukey reducibility

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Outline

Basic definitions

Definition

$\mathcal{U} \subseteq \mathcal{P}(\omega)$ is an ultrafilter if

1. *$\mathcal{U}$ is closed under supersets and finite intersections*
2. $\forall a \subseteq \omega$ [*either* $a \in \mathcal{U}$ *or* $(\omega \setminus a) \in \mathcal{U}$]
3. *All sets in $\mathcal{U}$ are infinite (non-principal).*

- Think of $\mathcal{U}$ as a finitely additive $\{0, 1\}$ valued measure on $\mathcal{P}(\omega)$.

- Some special classes of ultrafilters on ω have been considered:

Definition

An ultrafilter $\mathcal{U}$ on ω is called Ramsey or Selective if for every $c : [\omega]^2 \to 2$, there exists $a \in \mathcal{U}$ and $i \in 2$ such that $c''[a]^2 = \{i\}$.

Definition

An ultrafilter $\mathcal{U}$ on ω is called a P-point if for any countable collection $\{a_n : n \in \omega\} \subseteq \mathcal{U}$, there exists $a \in \mathcal{U}$ such that $\forall n \in \omega\, [a \subseteq^* a_n]$.

- It is not hard to see that Ramsey ultrafilters are P-points.

- P-points have an illustrious role in the recent history of set theory.

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- There were independently considered by several people (Daguenet, W. Rudin, Puritz, Choquet).
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- W. Rudin used them to show that $\beta\omega \setminus \omega$ need not be homogeneous.
- W. Rudin (in 1956) proved that P-points exist under CH (so $\beta\omega \setminus \omega$ fails to be homogeneous under CH).
- Shelah (in 1977)showed that it is consistent that there are no P-points.

- P-points have several other characterizations:

Fact

$\mathcal{U}$ is a P-point iff every G_δ set in $\beta\omega \setminus \omega$ containing $\mathcal{U}$ is a neighborhood of $\mathcal{U}$ in $\beta\omega \setminus \omega$.

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Fact

Let $\mathfrak{A}$ be the structure whose universe is ω and whose relations and functions are all relations and functions on ω. $\mathcal{U}$ is a P-point iff every non-standard elementary submodel of $\mathfrak{A}^\omega / \mathcal{U}$ is cofinal in $\mathfrak{A}^\omega / \mathcal{U}$.

Definition

Let $\mathcal{F}$ be a filter on a set X and $\mathcal{G}$ a filter on a set Y. We say that $\mathcal{F}$ is Rudin-Keisler(RK) reducible to $\mathcal{G}$ or Rudin-Keisler(RK) below $\mathcal{G}$, and we write $\mathcal{F} \leq_{RK} \mathcal{G}$, if there is a map $f : Y \to X$ such that for each $a \subseteq X$, $a \in \mathcal{F}$ iff $f^{-1}(a) \in \mathcal{G}$.

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- This is a quasi-order on filters and gives rise to an equivalence relation in the usual way: $\mathcal{F}$ and $\mathcal{G}$ are *RK equivalent*, written $\mathcal{F} \equiv_{RK} \mathcal{G}$, if $\mathcal{F} \leq_{RK} \mathcal{G}$ and $\mathcal{G} \leq_{RK} \mathcal{F}$.

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- If $\mathcal{F}$ and $\mathcal{G}$ are ultrafilters on ω, then $\mathcal{F} \equiv_{RK} \mathcal{G}$ if and only if there is a permutation $f : \omega \to \omega$ such that $\mathcal{F} = \{a \subseteq \omega : f^{-1}(a) \in \mathcal{G}\}$.
- For this reason, ultrafilters that are RK equivalent are sometimes said to be *(RK) isomorphic*.

- The Rudin-Keisler order on ultrafilters on ω was considered by Mary Ellen Rudin from a topological perspective.
- Analyzing this ordering and the stronger Rudin-Frolik ordering led to a proof in ZFC that $\beta\omega \setminus \omega$ is not homogeneous.

- The Rudin-Keisler order on ultrafilters on ω was considered by Mary Ellen Rudin from a topological perspective.
- Analyzing this ordering and the stronger Rudin-Frolik ordering led to a proof in ZFC that $\beta\omega \setminus \omega$ is not homogeneous.
- Around the same time Keisler independently considered the same order from a model-theoretic point of view.

Fact

Let $\mathfrak{A}$ be the structure whose universe is ω and whose relations and functions are all relations and functions on ω. If $\mathcal{U}$ and $\mathcal{V}$ are ultrafilters on ω, then $\mathcal{U} \leq_{RK} \mathcal{V}$ iff $\mathfrak{A}^{\omega}/\mathcal{U}$ elementarily embeds into $\mathfrak{A}^{\omega}/\mathcal{V}$.

- The Rudin-Keisler order can also be used to characterize some special ultrafilters.
- For example, the Ramsey ultrafilters are precisely the RK-minimal ones: $\mathcal{U}$ is Ramsey iff for every $\mathcal{V}$, $\mathcal{V} \leq_{RK} \mathcal{U}$ implies $\mathcal{V} \equiv_{RK} \mathcal{U}$.

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- For example, the Ramsey ultrafilters are precisely the RK-minimal ones: $\mathcal{U}$ is Ramsey iff for every $\mathcal{V}$, $\mathcal{V} \leq_{RK} \mathcal{U}$ implies $\mathcal{V} \equiv_{RK} \mathcal{U}$.
- Blass studied the global structure of the P-points under the Rudin-Keisler ordering in 1973 (see [1]).
- Such a study makes sense under some assumption that guarantees lots of P-points.
- The most natural such assumption is $\mathrm{MA}(\sigma - \mathrm{centered})$ (used by Blass in [1]).

- Recall that a subset X of a poset $\mathbb{P}$ is *centered* if any finitely many elements of X have a lower bound in $\mathbb{P}$.
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- A subset D of a poset $\mathbb{P}$ is called *dense* if $\forall p \in \mathbb{P} \exists q \in D \, [q \leq p]$.
- A subset G of a poset $\mathbb{P}$ is called a *filter* if
  1. $\forall q \in G \forall p \in \mathbb{P} \, [q \leq p \implies p \in G]$
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Definition

$\mathrm{MA}(\sigma - \mathit{centered})$ *is the following statement: for every* σ*-centered poset* $\mathbb{P}$ *and every collection* $\mathcal{X}$ *of fewer than* $\mathfrak{c}$ *many dense subsets of* $\mathbb{P}$*, there is a* $G \subseteq \mathbb{P}$ *which is a filter such that* $\forall D \in \mathcal{X}\, [G \cap D \neq 0]$.

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- So do standard forcing axioms like PFA.
- $\mathrm{MA}(\sigma - \mathit{centered})$ makes it easy to build P-points: guarantees existence of $2^{\mathfrak{c}}$ of them.

Definition

A family $F \subseteq [\omega]^\omega$ *is said to have the finite intersection property (FIP) if for any* $a_0, \ldots, a_k \in F$, $a_0 \cap \cdots \cap a_k$ *is infinite.*

Definition

$\mathfrak{p} = \min\{|F| : F \subseteq [\omega]^\omega \wedge F$ *has the FIP* $\wedge \neg \exists b \in [\omega]^\omega \, \forall a \in F \, [b \subseteq^* a]\}$.

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Fact

$\mathrm{MA}(\sigma - \mathit{centered})$ *holds iff* $\mathfrak{p} = \mathfrak{c}$.

Question (Blass, 1973)

Assuming $\mathrm{MA}(\sigma - centered)$*, what partial orders can be embedded into the P-points under the Rudin-Keisler ordering?*

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Theorem (Blass, 1973)

Assuming $\mathrm{MA}(\sigma - centered)$*, there are* $2^{\mathfrak{c}}$ *pairwise non-isomorphic Ramsey ultrafilters.*

- So the above gives “large antichains”.
- He also showed there are “large chains”.

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Theorem (Blass, 1973)

Assume $\mathrm{MA}(\sigma - centered)$ both ω_1 and $\mathbb{R}$ can be embedded into the P-points under Rudin-Keisler ordering.

- Much later similar results were proved for the notion of Tukey reducibility.

Definition

Given (upward) directed posets D and E, a map $g : E \to D$ is called a convergent map if the image of every (upward) cofinal subset of E is cofinal in D. D is said to be Tukey reducible to or Tukey below E if there is a convergent map $g : E \to D$. We write this as $D \leq_T E$.

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Given (upward) directed posets D and E, a map $g : E \to D$ is called a convergent map if the image of every (upward) cofinal subset of E is cofinal in D. D is said to be Tukey reducible to or Tukey below E if there is a convergent map $g : E \to D$. We write this as $D \leq_T E$.

- Again we get an equivalence relation in the usual way: D is *Tukey equivalent* to E, written $D \equiv_T E$ if $D \leq_T E$ and $E \leq_T D$.
- This notion arose in the Moore-Smith theory of convergence.
- If $D \leq_T E$, then any D-net on a topological space contains an E-subnet.

- If $\mathcal{U}$ is any ultrafilter, then $\langle \mathcal{U}, \supseteq \rangle$ is a directed set.
- If $\mathcal{U}$ and $\mathcal{V}$ are ultrafilters, then $\mathcal{U} \leq_{RK} \mathcal{V}$ implies $\mathcal{U} \leq_{T} \mathcal{V}$.

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- If $\mathcal{U}$ and $\mathcal{V}$ are ultrafilters, then $\mathcal{U} \leq_{RK} \mathcal{V}$ implies $\mathcal{U} \leq_{T} \mathcal{V}$.

Fact (R. and Todorcevic[4])

Ramsey ultrafilters are also Tukey minimal.

Theorem (Dobrinen and Todorcevic[2])

Assuming $\mathrm{MA}(\sigma - centered)$, ω_1 can be embedded into the P-points under Tukey reducibility.

Theorem (R.)

Assuming $\mathrm{MA}(\sigma - centered)$, $\mathbb{R}$ can be embedded into the P-points under Tukey reducibility.

Theorem (Dobrinen and Todorcevic[2])

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Theorem (R.)

Assuming $\mathrm{MA}(\sigma - centered)$, $\mathbb{R}$ *can be embedded into the P-points under Tukey reducibility.*

- Recall that in general a convergent $g : \mathcal{V} \to \mathcal{U}$ need not be definable.

Theorem (Dobrinen and Todorcevic[2])

If $\mathcal{U}$ *and* $\mathcal{V}$ *are P-points and* $\mathcal{U} \leq_T \mathcal{V}$, *then there exists a continuous monotone map* $\varphi : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ *such that* $\varphi \restriction \mathcal{V} : \mathcal{V} \to \mathcal{U}$ *is convergent (monotone means* $\forall a, b \in \mathcal{P}(\omega) \, [a \subseteq b \implies \varphi(a) \subseteq \varphi(b)]$*).*

Question

What partial orders can be embedded into the P-points under Tukey reducibility under $\mathrm{MA}(\sigma - (centered))$*?*

Main Result

Definition

$\mathcal{P}(\omega)/\mathrm{FIN}$ is the Boolean algebra gotten by quotienting $\mathcal{P}(\omega)$ by the ideal of finite sets.

Theorem (R. and Shelah [3])

Assume $\mathrm{MA}(\sigma - centered)$. Then there is a sequence of P-points $\langle \mathcal{U}_{[a]} : [a] \in \mathcal{P}(\omega)/\mathrm{FIN} \rangle$ such that

1. *if $a \subseteq^* b$, then $\mathcal{U}_{[a]} \leq_{RK} \mathcal{U}_{[b]}$;*
2. *if $b \not\subseteq^* a$, then $\mathcal{U}_{[b]} \not\leq_{T} \mathcal{U}_{[a]}$.*

Thus $\mathcal{P}(\omega)/\mathrm{FIN}$ is embeddable into P-points both under RK and Tukey reducibility.

Fact

Under $\mathrm{MA}(\sigma - centered)$, $\mathcal{P}(\omega)/\mathrm{FIN}$ *is universal for (strict) partial orders of size at most* $\mathfrak{c}$.

Corollary

Under $\mathrm{MA}(\sigma - centered)$ *any (strict) partial order of size at most* $\mathfrak{c}$ *embeds into the P-points both under RK and Tukey reducibility (in fact, the strictness requirement can be removed).*

- The proof uses creature forcing.
- In a creature forcing a condition consists of a sequence of finite sets with some measure-like structure (called a norm).
- Key point: prove that there are set with arbitrarily large norm.

Definition

Let A be a non-empty finite set. Say that u is a creature acting on A if u is a pair of sequences $\langle\langle u_a : a \subseteq A\rangle, \langle\pi_{u,b,a} : a \subseteq b \subseteq A\rangle\rangle$ such that the following things hold:

1. *each u_a is a non-empty finite set;*
2. *$\pi_{u,b,a} : u_b \to u_a$ is an onto function;*
3. *if $a \subseteq b \subseteq c$, then $\pi_{u,c,a} = \pi_{u,b,a} \circ \pi_{u,c,b}$.*

The set of all creatures acting on A is denoted $\mathcal{CR}(A)$.

Definition

For a non-empty finite set A *and* $u \in \mathcal{CR}(A)$, $\Sigma(u)$ *denotes the collection of all* $v \in \mathcal{CR}(A)$ *such that:*

1. *for each* $a \subseteq A$, $v_a \subseteq u_a$*;*
2. *for each* $a \subseteq b \subseteq A$, $\pi_{v,b,a} = \pi_{u,b,a} \restriction v_b$.

Definition

For a non-empty finite set A, define the norm of $u \in \mathcal{CR}(A)$, denoted $\mathrm{nor}(u)$, as follows. We first define by induction on $n \in \omega$, the relation $\mathrm{nor}(u) \geq n$ by the following clauses:

1. $\mathrm{nor}(u) \geq 0$ *always holds;*
2. $\mathrm{nor}(u) \geq n + 1$ *iff*
   - (a) *for each $a \subseteq A$, if $u_a = u^0 \cup u^1$, then there exist $v \in \Sigma(u)$ and $i \in 2$ such that $\mathrm{nor}(v) \geq n$ and $v_a \subseteq u^i$;*
   - (b) *for any $a, b \subseteq A$, if $b \nsubseteq a$, then there exist $v, w \in \Sigma(u)$ such that $\mathrm{nor}(v) \geq n$, $\mathrm{nor}(w) \geq n$, $v_a = w_a$, and $v_b \cap w_b = 0$.*

Define $\mathrm{nor}(u) = \max\{n \in \omega : \mathrm{nor}(u) \geq n\}$.

Lemma

For finite non-empty set A and for every $n \in \omega$ there is $u \in \mathcal{CR}(A)$ with $\mathrm{nor}(u) \geq n$.

Open Questions

- Under $\mathrm{MA}(\sigma - \mathrm{centered})$ there are $2^{\mathfrak{c}}$ P-points.
- But there are only $\mathfrak{c}$ many functions $f : \omega \to \omega$ and only $\mathfrak{c}$ many continuous monotone maps $\varphi : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$.
- So no P-point $\mathcal{U}$ can have more then $\mathfrak{c}$ predecessors (either under $\leq_{RK}$ or $\leq_T$).

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- So no P-point $\mathcal{U}$ can have more then $\mathfrak{c}$ predecessors (either under $\leq_{RK}$ or $\leq_T$).

Question

Suppose $\mathrm{MA}(\sigma - (centered))$ and $2^{\mathfrak{c}} = \mathfrak{c}^+$ holds. Suppose $\mathbb{P}$ is a (strict) partial order of size at most $\mathfrak{c}^+$ such that for each $p \in \mathbb{P}$, $\{q \in \mathbb{P} : q \leq p\}$ has size at most $\mathfrak{c}$. Does $\mathbb{P}$ embed into the P-points?

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