Termination theorem and Ramsey's theorem

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Outline

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Program

A program is a pair (I, R) where I is a set and $R \subseteq I^2$.

- I is a set of states. Each state $a \in I$ denotes the values of variables, e.g., $a = \langle x_0 = 3, x_1 = 2, y = 100 \rangle$.

  In this talk, we fix $I = \mathbb{N}$.
- R is said to be well-founded if there is no infinite sequence $\langle a_i \in I \mid i \in \omega \rangle$ such that $a_i R a_{i+1}$.
- Usually, R is generated by a computable transition function $\delta : I \to [I]^{<\omega}$ as $aRb \Leftrightarrow b \in \delta(a)$.
  (R is deterministic if $|\delta(a)| \leq 1$ for any $a \in I$.)
- A program is said to be terminating if R is well-founded.

To consider "termination" in this abstract setting, we just study well-foundedness of binary relations.

Termination theorem

For given $R \subseteq \mathbb{N}^2$, we write $\mathrm{tcl}(R)$ for the transitive closure of R.

The following termination theorem is a basic tool of the study of program termination.

Theorem (Podelski/Rybalchenko)

For given $k \in \mathbb{N}$, we have the following.

> TT_k: *for any $R \subseteq \mathbb{N}^2$, R is well-founded if and only if there exists $T_0, \ldots, T_{k-1} \subseteq \mathbb{N}^2$ such that $\mathrm{tcl}(R) \subseteq T_0 \cup \cdots \cup T_{k-1}$ and each T_i is well-founded.*

Example

Consider the following "program".

- variables: $x_0, \ldots, x_{n-1}$ (input), y (output).
- $x_0, \ldots, x_{n-1}$ are given by input, set $y := 0$.
- calculation:
  $\langle x_0, \ldots, x_{n-1}, y \rangle$
  $\Downarrow R$
  $\langle x_0 := x_0 + y, \ldots, x_{i-1} := x_{i-1} + y, x_i := x_i - 1, x_{i+1} := x_{i+1}, \ldots, x_{n-1} := x_{n-1}, y := y + 1 \rangle$ for some $i = 0, \ldots, n-1$.
- Output y if $x_0 = \cdots = x_{n-1} = 0$.

Does this program terminate?

Example

$\Rightarrow$ Yes!

- Define T_i ($i = 0, \ldots, n-1$) as
  $\langle x_0, \ldots, x_{n-1}, y \rangle T_i \langle x'_0, \ldots, x'_{n-1}, y' \rangle$
  $\Longleftrightarrow x'_i < x_i$ and $x'_j = x_j$ for any $j > i$.
- Then, $\mathrm{tcl}(R) = T_0 \cup \cdots \cup T_{n-1}$, and each T_i is well-founded.
- Thus, by the termination theorem, R is well-founded.

Termination theorem

Recently, Berardi and Steila generalize the previous termination theorem as follows.

$R \subseteq \mathbb{N}^2$ is said to be H-well-founded if there is no infinite chain $\langle a_0, a_1, \dots \rangle$ such that $a_i R a_j$ if $i < j$.

Theorem (Berardi/Steila)

For given $k \in \mathbb{N}$, we have the following.

> HTT_k: *for any $R \subseteq \mathbb{N}^2$, R is well-founded if and only if there exists $T_0, \dots, T_{k-1} \subseteq \mathbb{N}^2$ such that $\mathrm{tcl}(R) \subseteq T_0 \cup \dots \cup T_{k-1}$ and each T_i is H-well-founded.*

Outline

Proof-theoretic analysis for termination theorem

Project by Stefano Berardi, Paulo Oliva and Silvia Steila

> "Obtain a priori-bounds for the termination of computer programs, and compare these with bounds obtained via other intuitionistic proofs of the Termination Theorem."

Roughly speaking,

- if we would know the strength of the Termination theorem in some proof-theoretic settings, e.g., constructive mathematics or reverse mathematics, one would extract some information of a bound of termination from the proof.

$\Rightarrow$ try reverse mathematics for the Termination theorem.

Termination theorem (review)

Theorem (Podelski/Rybalchenko)

For given $k \in \mathbb{N}$, we have the following.

TT_k: *for any $R \subseteq \mathbb{N}^2$, R is well-founded if and only if there exists $T_0, \ldots, T_{k-1} \subseteq \mathbb{N}^2$ such that $\mathrm{tcl}(R) \subseteq T_0 \cup \cdots \cup T_{k-1}$ and each T_i is well-founded.*

Theorem (Berardi/Steila)

For given $k \in \mathbb{N}$, we have the following.

HTT_k: *for any $R \subseteq \mathbb{N}^2$, R is well-founded if and only if there exists $T_0, \ldots, T_{k-1} \subseteq \mathbb{N}^2$ such that $\mathrm{tcl}(R) \subseteq T_0 \cup \cdots \cup T_{k-1}$ and each T_i is H-well-founded.*

Termination theorem

Note:

- The original proof of the Podelski/Rybalchenko theorem is due to Ramsey's theorem for pairs.
- On the other hand, it is often understood by an easy consequence of Dickson's lemma by the following easy fact.
  - $(*)$ $R \subseteq \mathbb{N}^2$ is well-founded iff it is embedded into a well-ordering with the reverse order.
- D. Figueira/S. Figueira/Schmitz/Schnoebelen gave a deeper analysis by using Dickson's lemma.

However, from the view point of reverse mathematics, we have

Fact

$(*)$ *is equivalent to* ACA_0 *over* RCA_0.

So, we should compare this with Ramsey's theorem.

Ramsey's theorem (for pairs)

Definition (Ramsey's theorem)

- Ramsey's theorem (RT^2_k): for any $P : [\mathbb{N}]^2 \to k$, there exists an infinite set $H \subseteq \mathbb{N}$ such that $|P([H]^n)| = 1$.
- Weak Ramsey's theorem (WRT^2_k): for any coloring $P : [\mathbb{N}]^2 \to k$, there exists $H = \{h_0 < h_1 < \dots\}$ such that for any $i, j \in \mathbb{N}$, $P(h_i, h_{i+1}) = P(h_j, h_{j+1})$.

Note that we have the following:

$$\mathrm{ADS} \le \mathrm{WRT}^2_2 \le \mathrm{WRT}^2_3 \le \cdots \le \mathrm{WRT}^2_k \le \mathrm{CAC} < \mathrm{RT}^2_2 = \cdots = \mathrm{RT}^2_k.$$

Reverse Mathematical result

Theorem

The following are equivalent over RCA_0.

1. WRT^2_k.
2. TT_k.

Theorem

The following are equivalent over RCA_0.

1. RT^2_k.
2. HTT_k.

Outline

Termination theorem with bound

$f : \mathbb{N} \to \mathbb{N}$ is said to be a bound of R if any R-sequence starting from a is shorter than $f(a)$.
R is said to be bounded if it has a bound.

Theorem

For given $k \in \mathbb{N}$, we have the following.

$\mathrm{TT}^{bd}_{k,f}$: *for any $R \subseteq \mathbb{N}^2$, R is bounded if there exists $T_0, \dots, T_{k-1} \subseteq \mathbb{N}^2$ such that $\mathrm{tcl}(R) \subseteq T_0 \cup \cdots \cup T_{k-1}$ and each T_i is bounded by f.*

Then, can we calculate a bound for R by f?

Termination theorem with bound

$f : \mathbb{N} \to \mathbb{N}$ is said to be an *H*-bound of *R* if any homogeneous *R*-sequence starting from *a* is shorter than $f(a)$.

Theorem

For given $k \in \mathbb{N}$, we have the following.

$\mathrm{HTT}^{bd}_{k,f}$: *for any $R \subseteq \mathbb{N}^2$, R is bounded if there exists $T_0, \ldots, T_{k-1} \subseteq \mathbb{N}^2$ such that $\mathrm{tcl}(R) \subseteq T_0 \cup \cdots \cup T_{k-1}$ and each T_i is H-bounded by f.*

Then, can we calculate a bound for *R* by *f*?

We will analyze the termination bound by Ramsey-like functions.

Paris-Harrington theorem (for pairs)

Definition (Paris-Harrington theorem)

- Paris-Harrington theorem (PH^2_k): for any $a \in \mathbb{N}$, there exists $b \in \mathbb{N}$ such that for any $P : [[a,b]]^2 \to k$, there exists a set $H \subseteq [a,b]$ such that $|P([H]^n)| = 1$ and $|H| > \min H$.
- Weak Paris-Harrington theorem (WPH^2_k): for any $a \in \mathbb{N}$, there exists $b \in \mathbb{N}$ such that for any $P : [[a,b]]^2 \to k$, there exists $H = \{h_0 < h_1 < \cdots < h_m\} \subseteq [a,b]$ such that for any $i,j < m$, $P(h_i, h_{i+1}) = P(h_j, h_{j+1})$ and $|H| > \min H$.

Define

- $H_k(a) = \min\{b \mid [a,b] \text{ enjoys the condition for } \mathrm{PH}^2_k\}$.
- $W_k(a) = \min\{b \mid [a,b] \text{ enjoys the condition for } \mathrm{WPH}^2_k\}$.

Paris-Harrington theorem (for pairs)

Definition (Paris-Harrington theorem)

- $\mathrm{PH}^2_{k,f}$: for any $a \in \mathbb{N}$, there exists $b \in \mathbb{N}$ such that for any $P : [[a,b]]^2 \to k$, there exists a set $H \subseteq [a,b]$ such that $|P([H]^n)| = 1$ and $|H| > f(\min H)$.
- $\mathrm{WPH}^2_{k,f}$: for any $a \in \mathbb{N}$, there exists $b \in \mathbb{N}$ such that for any $P : [[a,b]]^2 \to k$, there exists $H = \{h_0 < h_1 < \cdots < h_m\} \subseteq [a,b]$ such that for any $i,j < m$, $P(h_i,h_{i+1}) = P(h_j,h_{j+1})$ and $|H| > f(\min H)$.

Define

- $H^f_k(a) = \min\{b \mid [a,b]$ enjoys the condition for $\mathrm{PH}^{2,f}_k\}$.
- $W^f_k(a) = \min\{b \mid [a,b]$ enjoys the condition for $\mathrm{WPH}^{2,f}_k\}$.

bdd-Termination vs PH

Theorem (WKL_0)

For any $k \in \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{N}$ the following are equivalent.

1. $\mathrm{WPH}^2_{k,f}$.
2. $\mathrm{TT}^{bd}_{k,f}$.

More precisely, for 1 → 2, if $\mathrm{tcl}(R)$ is k-disjunctively bounded by f, then R is bounded by W^f_k.

Note that if f is primitive recursive and k is standard, then $\mathrm{WPH}^2_{k,f}$ is provable within RCA_0. Thus, W^f_k is bounded by a primitive recursive function (by Person's theorem).

Corollary

R has a primitive recursive bound if and only if R is k-disjunctive primitive recursively bounded.

bdd-Termination vs PH

Theorem (WKL_0)

For any $k \in \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{N}$ the following are equivalent.

1. $\mathrm{PH}^2_{k,f}$.
2. $\mathrm{HTT}^{bd}_{k,f}$.

More precisely, for 1 → 2, if $\mathrm{tcl}(R)$ is k-disjunctively H-bounded by f, then R is bounded by H^f_k.

Note that if f is primitive recursive and k is standard, then $\mathrm{PH}^2_{k,f}$ is provable within RCA_0. Thus, H^f_k is bounded by a primitive recursive function (by Person's theorem).

Corollary

R has a primitive recursive bound if and only if R is k-disjunctive primitive recursively H-bounded.

Fast growing functions

Let F_k be the usual k-th fast growing function defined as

$$\begin{cases} F_0(x) = x + 1, \\ F_{n+1}(x) = F_n{}^{(x+1)}(x). \end{cases}$$

Theorem (Ketonen/Solovay)

- $W_k \leq H_k \leq F_{k+4}$.

  More generally, $W_k^{F_n} \leq H_k^{F_n} \leq F_{k+n+4}$.

Theorem (from recent termination analysis)

- $W_k \leq F_{k+1}$.

  More generally, $W_k^{F_n} \leq F_{k+n+1}$.

Termination theorem with bound

Theorem

For given $k \in \mathbb{N}$, we have the following.

- $\mathrm{TT}_k^{F_n}$: *for any $R \subseteq \mathbb{N}^2$, R is bounded by F_{k+n+1} if there exists $T_0, \ldots, T_{k-1} \subseteq \mathbb{N}^2$ such that $\mathrm{tcl}(R) \subseteq T_0 \cup \cdots \cup T_{k-1}$ and each T_i is bounded by F_n.*
- $\mathrm{HTT}_k^{F_n}$: *for any $R \subseteq \mathbb{N}^2$, R is bounded by F_{k+n+4} if there exists $T_0, \ldots, T_{k-1} \subseteq \mathbb{N}^2$ such that $\mathrm{tcl}(R) \subseteq T_0 \cup \cdots \cup T_{k-1}$ and each T_i is bounded by F_n.*

This upper bound is sharp.

Fact

The program for F_k is k-disjunctive linearly (F_0-) bounded.

Termination theorem with bound (sharper version)

If R is a relation for a deterministic program, we have the converse.

Theorem

For given $k \in \mathbb{N}$, we have the following.

> *for any deterministic program $R \subseteq \mathbb{N}^2$, R is bounded by F_k only if there exists $T_0, \ldots, T_{k+1} \subseteq \mathbb{N}^2$ such that $\mathrm{tcl}(R) \subseteq T_0 \cup \cdots \cup T_{k+1}$ and each T_i is bounded by F_0.*

Corollary

R has a primitive recursive bound if and only if R has k-disjunctive linearly bounded for some k if and only if R has k-disjunctive linearly H-bounded for some k.

Multiple application

One can apply the termination theorem many times.

Theorem

$\mathrm{TT}^{bd}_{2,l}$: for any $R \subseteq \mathbb{N}^2$, R is bounded if there exists $T_0, T_1 \subseteq \mathbb{N}^2$ such that $\mathrm{tcl}(R) \subseteq T_0 \cup T_1$ and each T_i is bounded by F_0 (linearly bounded).

Consider the case $R = T_0 \cup T_1$ but T_1 is not bounded by F_0, that is, there exists $I \subseteq \mathbb{N}$ and $a \in I$ such that there is a T_1-increasing sequence of length $\geq F_0(a)$ in I. In this situation, one can continue checking the termination of R by applying $\mathrm{TT}^{bd}_{2,l}$ again for (I, T_1).

Can we calculate the bound for R again?

Iterated Paris-Harrington

If R is shown to be well-founded by m-times application of $\mathrm{TT}^{bd}_{2,l}$, then can we calculate the bound for R?
$\Rightarrow$ Yes, by m-th iteration of WPH^2_2.

On the other hand

- m-th iteration of WPH^2_2 is provable from $\mathrm{WKL}_0 + \mathrm{CAC}$.
- $\mathrm{WKL}_0 + \mathrm{CAC}$ is Π^0_3-conservative over $\mathrm{I}\Sigma^0_1$ (Chong/Slaman/Yang).

Thus, we have the following.

Corollary

R has a primitive recursive bound if and only if R is shown to be well-founded by m-times application of $\mathrm{TT}^{bd}_{2,l}$.

Iterated Paris-Harrington

If R is shown to be well-founded by m-times application of $\mathrm{HTT}^{bd}_{2,l}$, then can we calculate the bound for R?

$\Rightarrow$ Yes? by m-th iteration of PH^2_2.

On the other hand

- m-th iteration of PH^2_2 is provable from $\mathsf{WKL}_0 + \mathrm{RT}^2_2$.
- Actually, it forms Π^0_2-part of $\mathsf{WKL}_0 + \mathrm{RT}^2_2$.

Thus, knowing which R is shown to be well-founded by m-times application of $\mathrm{HTT}^{bd}_{2,l}$ is the same as knowing Π^0_2-part of $\mathsf{WKL}_0 + \mathrm{RT}^2_2$ e.g.,

Corollary

The program for Ackerman function is shown to be well-founded by m-times application of $\mathrm{HTT}^{bd}_{2,l}$ if and only if $\mathsf{WKL}_0 + \mathrm{RT}^2_2$ proves the totality of Ackerman function.

Conclusion

- Termination theorem for programs is equivalent to Ramsey's theorem for pairs.
- A bound for a terminating program can be given by the Paris/Harrington number, and it is almost optimal.
- In general, proof-theoretic method would be applied to the study of termination.

Thank you!

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