

# Definability of the ground model and large cardinals

Toshimichi Usuba (薄葉 季路)

Kobe University

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This is partially joint work with

- Joan Bagaria, Universitat de Barcelona
- Joel David Hamkins, The City University of New York
- Konstantinos Tsaprounis, University of East Anglia

## Conventions

Throughout this talk, a **model of ZFC** means a transitive  $\in$ -model of ZFC.

For a model  $M$  of ZFC and an ordinal  $\alpha \in M$ , let  $M_\alpha = V_\alpha \cap M$ , the set of all sets in  $M$  with rank  $< \alpha$ .

Recall that for a model  $M$  of ZFC and a poset  $\mathbb{P} \in M$ , a filter  $G \subseteq \mathbb{P}$  is  **$(M, \mathbb{P})$ -generic** if  $G$  intersects with every dense subset  $D \in M$  in  $\mathbb{P}$ .

We assume all poset are **non-trivial**, that is, every  $(M, \mathbb{P})$ -generic does not belongs to  $M$ .

## Motivation and back grounds

In set-theory, the following kinds of statements are frequently appeared:

- $V$  is a forcing extension of the constructible universe  $L$ ,
- $V \neq \text{HOD}$ , the class of hereditary ordinal definable sets.
- There is a Cohen real over  $L$ .

These statements commit second order objects,  $L$ ,  $\text{HOD}$ , but it does not cause any problems: These are definable classes, and these statements are in fact expressed by first order sentences.

## Motivation and back grounds

On the other hand,

- There is a real which is generic over the ground model.
- The universe is a generic extension of some ground model.

It is not clear that these are first order statements, so we do not justify such statements in ZFC.

Of course, almost all cases, we can justify such statements.

### Question 1

Can we fully justify it?

In other words, the ground model is definable in the forcing extension?

## Definability of the ground model

### Fact 2 (Laver, Woodin)

In the forcing extension  $V[G]$  of  $V$ ,  $V$  is a **definable** class with some parameters:

There is a 1st-order formula  $\varphi(x, y)$  and  $r \in V$  such that for every  $x \in V[G]$ ,

$$x \in V \iff V[G] \models \varphi(x, r)$$

Indeed the ground model is uniformly  $\Sigma_2$ -definable:

There is a  $\Sigma_2$ -formula  $\varphi_\Sigma(x, y, z, w)$  such that:

For every model  $M$  of ZFC, poset  $\mathbb{P} \in M$ , and  $(M, \mathbb{P})$ -generic  $G$ ,

$$x \in M \iff M[G] \models \varphi_\Sigma(x, \mathbb{P}, \mathcal{P}(\mathbb{P}) \cap M, G)$$

# Definability of the ground models

## Fact 3 (Reitz)

There is a 1st-order formula  $\psi(x, y)$  such that for every transitive (possibly proper class) model  $M$  of ZFC:

1. For each set  $r \in M$ ,  $W_r = \{x : M \models \psi(x, r)\}$  is a transitive model of ZFC such that  $W_r$  is a **ground** of  $M$ , i.e., there is a poset  $\mathbb{P} \in W_r$  and a  $W_r$ -generic  $G \subseteq \mathbb{P}$  with  $M = W_r[G]$ .
2. For every model  $N \subseteq M$ , if  $N$  is a ground of  $M$ , then  $N = W_r$  for some  $r$ .

So the statement:

The universe is a forcing extension of some ground

can be expressed by a 1st-order formula  $\exists x \exists r (\neg \psi(x, r))$ .

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## Question 4

- The complexity  $\Sigma_2$  is optimal?
- What is the complexity of the statement that “the universe is a forcing extension of some ground”?

## $\Pi_2$ -definability of the ground model

### Theorem 5

There is a  $\Pi_2$ -formula  $\varphi_{\Pi}(x, y, z, w)$  such that for every model  $M$  of ZFC, poset  $\mathbb{P}$ ,  $(M, \mathbb{P})$ -generic  $G$ ,

$$x \in M \iff M \models \varphi_{\Pi}(x, \mathbb{P}, \mathcal{P}(\mathbb{P}) \cap M, G)$$

### Theorem 6

There is a  $\Pi_2$ -formula  $\varphi^*(y, z, w)$  such that:

1. If  $M = N[G]$  for some non-trivial  $\mathbb{P} \in N$  and  $N$ -generic  $G \subseteq \mathbb{P}$ , then  $\varphi^*(\mathbb{P}, G, r)$  holds in  $M$ , where  $r = \mathcal{P}(\mathbb{P}) \cap N$ .
2. If  $\varphi^*(y, z, w)$  holds in  $M$  for some  $y, z, w \in M$ , then there is a model  $N \subsetneq M$  such that  $N$  is definable in  $M$ ,  $y$  is a poset,  $z \subseteq y$  is  $N$ -generic,  $w = \mathcal{P}(\mathbb{P}) \cap N$ , and  $M = N[F]$ .

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Consequently, the statement that

The universe is a forcing extension of some ground

can be expressed by a  $\Sigma_3$ -formula  $\exists xyz \varphi^*(x, y, z)$ .

## Some remarks

### Remark 7

Our result does not mean that the ground model is  $\Delta_2$ -definable:  
The equivalence between  $\varphi_\Sigma$  and  $\varphi_\Pi$  is not provable from ZFC.

### Remark 8

The statement:

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# Ker properties

## Definition 9

$M, N$ : models of ZFC.

$\kappa$ : cardinal in  $N$ .

1.  $(M, N)$  has the  $\kappa$ -covering property if for every set of ordinals  $a \in N$  with  $|a|^N < \kappa$ , there is  $b \in M$  with  $a \subseteq b$  and  $|b|^M < \kappa$ .
2.  $(M, N)$  has the  $\kappa$ -approximation property if for every set of ordinals  $a \in N$ , if  $a \cap b \in M$  for every  $b \in M$  with  $|b|^M < \kappa$ , then  $a \in M$ .

## Key properties

### Fact 10 (Hamkins)

$M, N$ : models of ZFC.

$\kappa$ : cardinal.

If  $(M, V)$  and  $(N, V)$  have the  $\kappa$ -covering and the  $\kappa$ -approximation properties,  $\kappa^+ = (\kappa^+)^M = (\kappa^+)^N$ , and  $\mathcal{P}(\kappa) \cap M = \mathcal{P}(\kappa) \cap N$ , then  $M = N$ .

### Fact 11

$\kappa$ : cardinal.

$\mathbb{P}$ : poset,  $|\mathbb{P}| < \kappa$ .

$G \subseteq \mathbb{P}$ : generic.

Then  $(V, V[G])$  satisfies the  $\kappa$ -covering and the  $\kappa$ -approximation properties.



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## Definition 12

$\kappa$ : cardinal.

$\text{ZFC}_\kappa := \text{ZFC} - \text{Replacement} + \leq \kappa\text{-Replacement}$

+Every set can be coded by some set of ordinals.

If  $\theta$  is a fixed point of  $\beth$ -function and  $\text{cf}(\theta) > \kappa$ ,  $V_\theta \models \text{ZFC}_\kappa$ .

## Fact 13 (Hamkins)

$M, N, W$ : models of  $\text{ZFC}_\kappa$ ,  $M, N \subseteq W$ .

$\kappa$ : cardinal in  $W$ .

If  $(M, W)$  and  $(N, W)$  have the  $\kappa$ -covering and the  $\kappa$ -approximation properties,  $(\kappa^+)^W = (\kappa^+)^M = (\kappa^+)^N$ , and  $\mathcal{P}(\kappa) \cap M = \mathcal{P}(\kappa) \cap N$ , then  $M = N$ .

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## Sketch of a proof of Theorems

Let  $\varphi_{\Pi}(x, y, z, w)$  be the conjunction of following statements:

1.  $y$  is a non-trivial poset,
2.  $z \subseteq y$  is a filter on  $y$ ,
3.  $w \subseteq \mathcal{P}(y)$ ,
4. For every  $Z$ ,  $\kappa$ , and  $\gamma > \text{rank}(y) + \text{rank}(x)$ ,
  - If  $\kappa = |y|$ ,  $Z = V_{\gamma+2}$  and  $Z$  knows that  $V_{\gamma}^Z (= V_{\gamma})$  is a model of  $\text{ZFC}_{\kappa}$ ,

THEN

- There is  $M \subseteq V_{\gamma}^Z$  such that  $x \in M$ ,  $(M, V_{\gamma}^Z)$  satisfies the  $\kappa$ -covering,  $\kappa$ -approximation properties,  $\mathcal{P}(y) \cap M = z$ ,  $\kappa^+ = (\kappa^+)^M$ , and  $M[y] = V_{\gamma}^Z$ .

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# Application 1

## Definition 14

Let  $n < \omega$ . An infinite cardinal  $\kappa$  is  $\Sigma_n$ -correct if  $V_\kappa \prec_{\Sigma_n} V$ .

## Remark 15

1. If  $\kappa$  is  $\Sigma_2$ -correct, then  $\kappa$  is strong limit, a  $\kappa$ -th fixed point of  $\aleph$ -function, a  $\kappa$ -th fixed point of  $\beth$ -function, etc.
2. The class of all  $\Sigma_n$ -cardinals  $C^{(n)}$  is a closed unbounded class of ON.
3. If  $\kappa$  is supercompact, then  $\kappa$  is  $\Sigma_2$ -correct.

## Application 1

Let  $\kappa$  be an infinite cardinal.

- $\mathbb{P}$  is  $\kappa$ -closed if every descending sequence in  $\mathbb{P}$  of length  $< \kappa$  has a lower bound.
- $\mathbb{P}$  is  $\kappa$ -directed closed if for every downward directed  $D \subseteq \mathbb{P}$  with size  $< \kappa$  (i.e.,  $\forall p, q \in D \exists r \in D, (r \leq p, q)$ ),  $D$  has a lower bound.

### Fact 16 (Laver)

It is consistent that there exists a supercompact cardinal  $\kappa$  such that every  $\kappa$ -directed forcing preserves the supercompactness of  $\kappa$ . In particular,  $\Sigma_2$ -correct cardinal  $\kappa$  can be preserved by  $\kappa$ -directed closed forcings.

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# Application 1

## Question 17

Can  $\Sigma_3$ -correct cardinals be Laver indestructible?

## Theorem 18

Let  $\kappa$  be a  $\Sigma_2$ -correct cardinal. Then for every non-trivial  $\kappa$ -closed forcing  $\mathbb{P}$ ,  $\mathbb{P}$  forces that “ $\kappa$  is not  $\Sigma_3$ -correct”.

This theorem says that  $\Sigma_3$ -correct cardinals cannot be preserved by closed forcing, in fact **any** closed forcing destroys the  $\Sigma_3$ -correctness.

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# Application 1

## Corollary 19

1. Superstrong cardinals are never Laver indestructible.
2. Consequently, almost huge, huge, superhuge and rank-into-rank cardinals are never Laver indestructible.
3. Similarly, extendible cardinals, 1-extendible and even 0-extendible cardinals are never Laver indestructible.

## Very rough sketch of the theorem

$\kappa$ :  $\Sigma_2$ -correct cardinal.

$\mathbb{P}$ :  $< \kappa$ -closed poset.

$G \subseteq \mathbb{P}$ :  $(V, \mathbb{P})$ -generic.

Suppose to the contrary that  $\kappa$  is  $\Sigma_3$ -correct in  $V[G]$ .

$V[G]$  thinks that

(\*) I am a non-trivial forcing extension of some model.

(\*) is expressed by a  $\Sigma_3$ -formula  $\exists xyz \varphi^*(x, y, z)$ .

$V[G]_\kappa \prec_{\Sigma_3} V[G]$ . Hence  $V[G]_\kappa$  also thinks that

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cont.

Note that  $V_\kappa = V[G]_\kappa$  since  $\mathbb{P}$  is  $< \kappa$ -closed.  
Pick  $\mathbb{Q}, F, r \in V_\kappa$  such that  $V_\kappa \models \varphi^*(\mathbb{Q}, F, r)$ .  
Again,  $V_\kappa = V[G]_\kappa \prec_{\Sigma_3} V[G]$ .  
So

$$V[G] \models \varphi^*(\mathbb{Q}, F, r).$$

By by theorem, there is  $N \subsetneq V[G]$  such that

$$r = \mathcal{P}(\mathbb{Q}) \cap N \text{ and } V[G] = N[F].$$

In particular,  $V[G]$  is a **small-forcing** extension of  $N$ .

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cont.

Now  $V_\kappa \prec_{\Sigma_2} V$  and  $V_\kappa \models \varphi^*(\mathbb{Q}, F, r)$ .

So  $V \models \varphi^*(\mathbb{Q}, F, r)$ , and there is  $M \subsetneq V$  such that

$$r = \mathcal{P}(\mathbb{Q}) \cap M \text{ and } V = M[F].$$

Hence  $V[G] = M[F][G]$ .

Consequently, we have

$$M[F][G] = V[G] = N[F].$$

## Fact 20 (Hamkins)

$W_0, W_1$ : models of ZFC,  $\delta \in W_0 \cap W_1$ .

Suppose  $W_0[G_0][H_0] = W_1[G_1][H_1]$ , where

1.  $G_0 \subseteq \mathbb{P}_0 \in W_0$  is non-trivial poset,  $|\mathbb{P}_0|^{W_0} < \delta$ .
2.  $G_1 \subseteq \mathbb{P}_1 \in W_1$  is a non-trivial poset,  $|\mathbb{P}_1|^{W_1} < \delta$ .
3.  $H_0 \subseteq \mathbb{Q}_0 \in W_0[G_0]$  is  $< \delta$ -closed.
4.  $H_1 \subseteq \mathbb{Q}_1 \in W_1[G_1]$  is  $< \delta$ -closed.

If  $\mathcal{P}(\delta) \cap W_0 = \mathcal{P}(\delta) \cap W_1$ , then  $W_0 = W_1$ .

1.  $M[F][G]$  is a small-forcing  $* < \kappa$ -closed forcing extension,
2.  $N[F]$  is a small-forcing extension, and
3.  $M \cap \mathcal{P}(\mathbb{Q}) = r = N \cap \mathcal{P}(\mathbb{Q}) = N \cap \mathcal{P}(\mathbb{Q})$ .

By Hamkins' theorem, we have  $M = N$ . So

$$N[F] = M[F] \subsetneq M[F][G] = N[F].$$

This is a contradiction. □

## Application 2

We have known that the statement “the universe is a forcing extension of some ground” is a  $\Sigma_3$ -statement.

### Theorem 21

There is no  $\Pi_3$ -sentence which always represents “the universe is a forcing extension of some ground.”

## Sketch of a proof

Suppose there is some  $\Pi_3$ -formula  $\psi$  which always represents “the universe is a forcing extension of some ground.”

Suppose  $\kappa$  is regular  $\Sigma_2$ -correct, and every (in fact some)  $\kappa$ -directed closed forcing preserves the  $\Sigma_2$ -correctness.

Now  $V[G] \models \psi$ .

Since  $\psi$  is  $\Pi_3$  and  $V[G]_{\kappa} \prec_{\Sigma_2} V[G]$ ,  $V[G]_{\kappa}$  also satisfies  $\psi$ . So there are  $\mathbb{Q}, F, r \in V[G]_{\kappa}$  such that  $\varphi^*(\mathbb{Q}, F, r)$  holds in  $V[G]_{\kappa}$ . Then we can derive a contradiction as before.

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## Indestructibility of $\Sigma_3$ -correctness

We have known that  $\Sigma_3$ -correct cardinals must be destroyed by closed forcing. However the following is unknown:

### Question 22

Can the  $\Sigma_3$ -correctness be preserved by **some** non-small forcing? that is, is it possible that  $\kappa$  is  $\Sigma_3$ -correct,  $\mathbb{P}$  is a poset which is not forcing equivalent to any forcing of size  $< \kappa$ , and  $\mathbb{P}$  forces that “ $\kappa$  is  $\Sigma_3$ -correct ”?

## Theorem 23

It is consistent that  $\kappa$  is  $\Sigma_3$ -correct (in fact it can be a large large cardinal), and every non-small forcing forces that “ $\kappa$  is not  $\Sigma_3$ -correct”.



## Definition 24 (Reitz)

**Continuum Coding Axiom (CCA)** is the assertion that for every set  $x$  of ordinals, there is an ordinal  $\alpha$  such that for every  $\xi < \sup(x)$ ,

$$\xi \in x \iff 2^{\aleph_{\alpha+\xi+1}} = \aleph_{\alpha+\xi+2}.$$

## Fact 25

1. There is a class forcing  $\mathbb{P}$  which forces CCA and preserves almost all large cardinals.
2. CCA implies  $V = \text{HOD}$  and the negation of GCH.
3. CCA implies “the universe does not have a proper ground”.

## Lemma 26

$M_0, M_1$ : models of ZFC.

If  $M_0$  and  $M_1$  satisfy CCA, and there is a common forcing extension of  $M_0$  and  $M_1$ , then  $M_0 = M_1$ .

Why;

Suppose  $N = M_0[G] = M_1[F]$  for some  $G \subseteq \mathbb{P} \in M_0$  and  $F \subseteq \mathbb{Q} \in M_1$ .

If  $x \in M_0$ , there is a large  $\alpha > |\mathbb{P}| + |\mathbb{Q}|$  such that

$$\xi \in x \iff (2^{\aleph_{\alpha+\xi+1}})^{M_0} = \aleph_{\alpha+\xi+2}^{M_0}.$$

But  $\aleph_{\alpha+\xi}^{M_0} = \aleph_{\alpha+\xi}^N = \aleph_{\alpha+\xi}^{M_1[F]}$ , hence  $x$  can be computed in  $N_1$ , and  $x \in N_1$ .

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If  $x \in M_0$ , there is a large  $\alpha > |\mathbb{P}| + |\mathbb{Q}|$  such that

$$\xi \in x \iff (2^{\aleph_{\alpha+\xi+1}})^{M_0} = \aleph_{\alpha+\xi+2}^{M_0}.$$

But  $\aleph_{\alpha+\xi}^{M_0} = \aleph_{\alpha+\xi}^N = \aleph_{\alpha+\xi}^{M_1[F]}$ , hence  $x$  can be computed in  $N_1$ , and  $x \in N_1$ .

## Sketch of a proof

Suppose CCA. Suppose  $\kappa$  is  $\Sigma_3$ -correct. Let  $\mathbb{P}$  be a non-small poset,  $G$  be  $(V, \mathbb{Q})$ -generic, and suppose  $V[G]_\kappa \prec_{\Sigma_3} V[G]$ .  
 $V[G]$  thinks:

I am a forcing extension of some ground, and the element of the ground are coded by the continuum function. (so the ground satisfy the CCA).

This is a  $\Sigma_3$ -statement, so  $V[G]_\kappa$  also satisfies this statement.

Then we can find  $\mathbb{Q}, F, r \in V[G]_{\kappa}$  and  $N \subseteq V[G]_{\kappa}$  such that  $N$  satisfies CCA and with  $N[F] = V[G]_{\kappa}$ . Again,  $N[F] = V[G]_{\kappa} \prec_{\Sigma_2} V[G]$ , so  $V[G] = M[F]$  for some  $M \subseteq V[G]$  with  $M \models \text{CCA}$ .

Now  $V$  and  $M$  satisfy CCA, then we have  $V = M$ . Hence  $V[G] = V[F]$ , and  $V[G]$  can be obtained as a small forcing. This is a contradiction.

## Question 27

1. Is the ground model always uniformly  $\Delta_2$ -definable?
2. Can  $\Sigma_3$ -correct cardinal (or extendible cardinal) be preserved by non-small forcing?

Thank you for your attention!