Coloring on trees and Ramsey's theorem for pairs

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Join work with C. T. Chong, National University of Singapore

Reverse Mathematics

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- Main Question: investigate *set existence axiom* required to show theorems in *ordinary mathematics*.
- Language: second order arithmetic.
- Model: $\mathcal{M} = (M, S)$, where M is the first order part and $S \subset P(M)$.
- Assumption on M: Usual axioms for Peano Arithmetic, where the induction is restricted to Σ^0_1-formulas (with set parameters).

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- $I\Sigma^0_n$: Induction principle restricted to Σ^0_n formulas;
- $B\Sigma^0_n$: Bounding principle restricted to Σ^0_n formulas;

  Bounding Principle:

$$\forall a < x\,(\exists b\,\varphi(a,b)) \to \exists u\,\forall a < x\,(\exists b < u\,\varphi(a,b))$$

Theorem (Paris and Kirby)

Over Peano Arithmetic with $I\Sigma^0_0$ and the assumption that exponential functions are total,

$$\ldots \to I\Sigma^0_{n+1} \to B\Sigma^0_{n+1} \to I\Sigma^0_n \to B\Sigma^0_n \to \ldots I\Sigma^0_1 \to B\Sigma^0_1,$$

and the arrows are not reversible.

Hierarchy of Second order principles – Big Five

Principle	Assumption on S
$\mathbf{RCA}_0$	Closed under join and Turing reduction.
$\mathbf{WKL}_0$	RCA_0+ Every infinite binary tree has an infinite path.
$\mathbf{ACA}_0$	RCA_0+ Arithmetically definable sets exist.
ATR_0	ACA_0+Transfinite induction holds.
Π^1_1-CA_0	ACA_0+ Π^1_1 definable sets exist.

Combinatorial Principles are usually between $\mathbf{RCA}_0$ and $\mathbf{ACA}_0$.

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- k-coloring: a function f from $[X]^n$ to k;
- Homogeneous set H: $H \subset X$ such that $f \upharpoonright [H]^n$ is constant;
- **Ramsey's Theorem:** For every $n, k \geq 1$, for every k-coloring on $[M]^n$, there is an infinite homogeneous set. (For a fixed pair n, k, we call this principle RT^n_k.)

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2. **From the view point of reverse mathematics, what is the logical strength of RT^n_k?**

Examples

Theorem (Specker)

There is a recursive 2-coloring of $[\mathbb{N}]^2$*, which has no recursive homogeneous set.*

Corollary

$RCA_0 \nvdash RT^2_2$.

Theorem (Jockusch)

(1) *For any recursive k-coloring of* $[\mathbb{N}]^n$, *there is a* Π^0_n *homogeneous set.*

(2) *For every* $n \geq 2$, *there is a recursive* 2*-coloring of* $[\mathbb{N}]^n$ *with no* Σ^0_n *homogeneous set.*

Corollary

Suppose $n \geq 2$, $k \geq 2$. $WKL_0 \nvdash RT^n_k$, $ACA_0 \vdash RT^n_k$.

Theorem (Simpson)

Suppose $n \geq 3$, $k \geq 2$. Over RCA_0, the following are equivalent:

(i) ACA_0;

(ii) RT^n_k;

(iii) $RT^n_{<\infty}$

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Over RCA_0, RT^2_2 does not imply WKL_0.

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Theorem (Liu)

Over RCA_0, RT^2_2 does not imply WKL_0.

Theorem (Hirst)

Over RCA_0, $RT^1_{<\infty}$ is equivalent with $B\Sigma^0_2$.

Application of Nonstandard Models

Theorem (Chong, Slaman and Yang)

Over RCA_0, SRT^2_2 (Stable version of Ramsey's theorem for pairs) does not imply RT^2_2 or $I\Sigma^0_2$.

The ideas of our results of TT^1, coloring on trees, are originated from the proof of the last theorem.

Ramsey's Theorem for trees

Definition by Chubb, Hirst and McNicholl.

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Ramsey's Theorem for trees

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- $[2^{<M}]^n =$ the collection of all size n linearly ordered subsets of $2^{<M}$;
- k-coloring: a function f from $[2^{<M}]^n$ to k;
- Monochromatic tree H: $H \subset 2^{<M}$ such that (1) $H \cong 2^{<M}$ and (2) $f \upharpoonright [H]^n$ is constant; [A monochromatic tree may not be a real "tree".]

- **Ramsey's Theorem for trees:** For every $n, k \geq 1$, for every k-coloring on $[2^{<M}]^n$, there is a monochromatic tree. (For a fixed pair n, k, we call this principle TT^n_k. We also write TT^n for $\mathrm{TT}^n_{<\infty}$.)

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Theorem (Jockusch)

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For any recursive k-coloring of $[2^{<\mathbb{N}}]^n$, there is a Π^0_n monochromatic tree.

Corollary

Suppose $n, k \geq 2$. $WKL_0 \nvdash TT^n_k$, $ACA_0 \vdash TT^n_k$.

Theorem (Simpson)

Suppose $n \geq 3$, $k \geq 2$. Over RCA_0, the following are equivalent:

(i) ACA_0;

(ii) RT^n_k;

(iii) TT^n_k;

(iv) $RT^n_{<\infty}$;

(v) $TT^n_{<\infty}$.

Corollaries for $n = 1, 2$

Corollary

- $WKL_0 \nvdash TT^2_2$;
- $RCA_0 + TT^1 \vdash B\Sigma^0_2$.

Questions

(1) $TT^1 \vdash I\Sigma^0_2$?

(2) $\mathsf{RCA}_0 + TT^2 \vdash \mathsf{ACA}_0$?

(3) $\mathsf{RCA}_0 + TT^2_2 \vdash TT^2$?

What happens on tree colorings?

Lemma

$RCA_0 + I\Sigma^0_2 \vdash TT^1$.

Corollary

$RCA_0 + TT^1 \nvdash RT^2_2$.

We already know that TT^1 implies $B\Sigma^0_2$. So we only consider models of $\mathrm{RCA}_0 + B\Sigma^0_2 + \neg I\Sigma^0_2$.

Lemma

Given b. There is a recursive coloring of $2^{<M}$ such that for every $X \subset [0, b]$, there is no $\emptyset' \oplus X$-recursive monochromatic tree.

Corollary

$RCA_0 + SRT^2_2 \not\vdash TT^1$.

Main Theorem

Theorem (Chong and Li)

$RCA_0 + RT^2_2 \nvdash TT^1$.

Sketch of the proof.

- We prove this theorem in a $B\Sigma^0_2$ reflection model.

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- We prove this theorem in a $B\Sigma^0_2$ reflection model.
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- It is well known that $\mathrm{RT}^2_2 \leftrightarrow \mathrm{SRT}^2_2 + \mathrm{COH}$.

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$RCA_0 + RT^2_2 \nvdash TT^1$.

Sketch of the proof.

- We prove this theorem in a $B\Sigma^0_2$ reflection model.
- We fix a recursive coloring with no recursive monochromatic tree.
- It is well known that $\mathrm{RT}^2_2 \leftrightarrow \mathrm{SRT}^2_2 + \mathrm{COH}$.
- In general, we split the construction into two sorts of stages:

  Sort I: Solve one SRT^2_2 problem;

  Sort II: Solve one COH problem.

At each stage, we make sure that

- $B\Sigma^0_2$ is preserved, and
- no monochromatic tree is added to the second order part.

A Lemma to avoid monochromatic trees

Lemma

Suppose there is no Y-recursive monochromatic tree and $M \models B\Sigma_2^0[Y]$ and $T_2 \leq_T Y$. Then there is a string $\sigma \in T_2$ such that for every $e \in B$ either (1) $\Phi_{e,|\sigma|}^{\sigma \oplus Y}$ is not a finite monochromatic tree, or (2) there is an $n \in M$ such that for all $\tau \supset \sigma$ in T_2, either $\Phi_{e,|\tau|}^{\tau \oplus Y} \upharpoonright n \downarrow$ is not a finite monochromatic tree or $\Phi_{e,|\tau|}^{\tau \oplus Y} \upharpoonright n \uparrow$.

Corollary

Over RCA_0, TT^1 and RT^2_2 are independent.

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Thank you!