The sequential price of anarchy of network congestion games

José Correa Jasper de Jong Bart de Keijzer (me) Marc Uetz

University of Chile
University of Twente
Sapienza University of Rome

Stochastic Methods in Game Theory / Workshop on Congestion Games
Singapore
17 December 2015

Topic

Sequential games
Price of Anarchy
Congestion games

Cost Minimization Games

A *cost minimization game* is a triple (n, c, Σ) where

- n is the number of players.
- $\Sigma = (\Sigma_1, \ldots, \Sigma_n)$ are the action sets of the players.
- $c = (c_1, \ldots, c_n)$ are the cost functions of the players, where $c_i : \Sigma \rightarrow \mathbb{R}$.

The Traditional Price of Anarchy

Consider a cost minimization game (n, c, Σ) and let s^* be its *social optimum*.
I.e., s^* minimizes the sum of costs $C : \sum_{i=1}^{n} c_i$.
The *price of anarchy (PoA)* quantifies the quality of the equilibria of a game by comparing the one with worst social cost to optimal social welfare. $C(s^*)$.

Definition

Let S be the set of pure equilibria of a cost minimization game. Let $s \in \arg_s \max\{C(s) : s \in S\}$. The *price of anarchy (PoA)* is

$$PoA(\Gamma) = \frac{C(s)}{C(s^*)}.$$

The Sequential Price of Anarchy (1/2)

In the *sequential version* of Γ, players instead arrive one by one, and choose their action upon arrival.

Each player i must specify an action in Σ_i *for every choice of actions of the previous players $j < i$*.
A strategy of i is a function $t_i : \times_{j<i}\Sigma_j \to \Sigma_i$.

The Sequential Price of Anarchy (2/2)

Definition

A *subgame perfect equilibrium* (SPE) of a sequential game is a strategy profile $t = (t_1, \ldots, t_n)$ such that for all i and $s_{<i} \in \Sigma_1 \times \cdots \times \Sigma_{i-1}$, players i to n play a pure equilibrium:

Strategy profile $(t_i(s_{<i}, \cdot), \ldots, t_n(s_{<i}, \cdot))$ is a pure equilibrium in the (sequential) subgame of Γ when actions of the first $i-1$ players are fixed to $s_{<i}$.

Definition

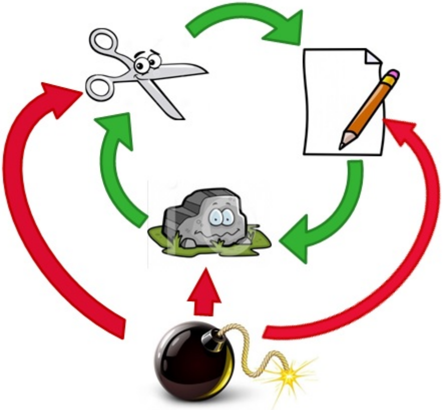
The *sequential price of anarchy (SPoA)* of Γ is

$$SPoA(\Gamma) = \frac{C(s)}{C(s^*)},$$

where s is the action profile resulting from an SPE that has max social cost.

Example (1/3)

A variation on Rock-Paper-Scissors:



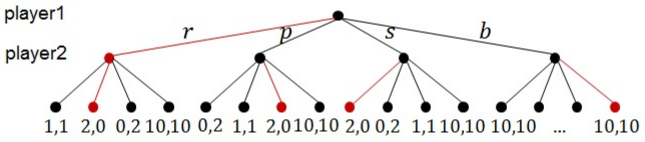
Example (2/3)

Rock-Paper-Scissors-Bomb in normal form:

	r	p	s	b
r	$1, 1$	$2, 0$	$0, 2$	$10, 10$
p	$0, 2$	$1, 1$	$2, 0$	$10, 10$
s	$2, 0$	$0, 2$	$1, 1$	$10, 10$
b	$10, 10$	$10, 10$	$10, 10$	$10, 10$

- (b, b) is the only pure equilibrium.
- PoA = 10.
- The PoA is too pessimistic here.
- What about the SPoA?

Example (3/3)



SPoA = 1

Central Question

- Is the *SPoA* always better than the *PoA*?
- Answer is *no*.
- But: $SPoA \leq PoA$ for some important classes of games.
- *Our question: What about congestion games?*

Some Literature

- Paes Leme, Syrgkanis, and Tardos (2012) (Machine cost sharing, Unrelated machine scheduling, Consensus, and Cut games. Complexity.)
- Angelucci, Bilò, Flammini, and Moscardelli (2013) (Isolation games)
- De Jong and Uetz (2014) (Congestion games with small numbers of players)

Paes Leme, Syrgkanis, and Tardos (2012)

- For machine cost sharing games: $SPoA \in O(\log n)$ (compared to $PoA \in \Theta(n)$).
- Unrelated machine scheduling games: $SPoA \in O(m2^n)$ (compared to PoA unbounded even for two players and machines).
- For Consensus games: $SPoA = 1$. For Cut games: $SPoA = 4$.
- *Believe: The merits of sequential equilibria carry over to classes of games that are natural.*
- Computational:
  - SPE computable in polynomial time for machine cost sharing games.
  - Computing an SPE is PSPACE-hard for unrelated machine scheduling games.

De Jong and Uetz (2014)

For affine congestion games:

- *SPoA* $= 1.5$ for two players.
- *SPoA* $= 2 + 63/488 \approx 2.13$ for three players.
- *SPoA* ≥ 2.46 for 4 or more players.
- Various results for singleton and symmetric singleton special cases.
- *Conjecture: In congestion games, the SPoA is at most the PoA.*

Our Results

We study the SPoA of affine (network) congestion games further. In particular *symmetric* ones.

- Main result: The SPoA is unbounded.
- The PoA is $5/2$
- Computing a two player SPE is NP-hard.
- For two players the SPoA is $7/5$.

Symmetric Network Congestion Games

Our class of congestion games is as follows.

- There is a directed network $G = (V, E)$ with two special nodes s, t.
- The arcs are the facilities/resources.
- The latency function on each arc $e \in E$ is *affine*, i.e., $\ell_e(x) = a_e x + b_e$ where $a_e, b_e \in \mathbb{R}_{\geq 0}$.
- Players choose an (s, t)-path.

The Two-Player Case (1/2)

I have drawn an example on the whiteboard.

The Two-Player Case (1/2)

I have drawn an example on the whiteboard.
It has a SPoA of $7/5$. The SPE is:

- Player 1 chooses path (s, a, b, c, t).
- Player 2 chooses:
  - (s, t) if player 1 chooses (s, a, b, c, t),
  - (s, a, c, t) if player 1 chooses (s, a, b, t),
  - (s, a, b, t) if player 1 chooses (s, a, c, t),
  - Any (best response) path for all remaining choices of player 1.

The Two-Player Case (2/2)

So we conclude:

Corollary

The SPoA of two player symmetric affine network congestion games is at least $7/5$.

We can prove a matching upper bound:

Lemma

The SPoA of two player symmetric affine network congestion games is at most $7/5$.

Upper Bound Proof Sketch (1/2)

- Assume w.l.o.g. that all latency functions are of the form $x \mapsto x$.
- Derive various constraints that must hold in an SPE, in terms of:
  - Minimum cardiniality of a strategy.
  - Relative sizes of the strategies and the intersections of the strategies under both the optimum and the SPE.
  - Relative costs of the strategies in the optimum and the SPE.

Upper Bound Proof Sketch (2/2)

Formulate these constraints as a mathematical program. Relax them into a linear program and solve.

$$\max \quad \left\{ z - \frac{7}{5}(2+c) \right. \\ : z \le 3 + c, z \le 2 + b + d + c + a, z \le 3 + 3c - b - d - a, \\ \left. 0 \le c \le \frac{6}{7}, 0 \le d \le 1, 0 \le b \le \frac{c}{2}, 0 \le a \le 1 \right\}.$$

Computational Hardness

Theorem

Computing an action profile resulting from a subgame perfect equilibrium of symmetric linear network congestion games is (strongly) NP*-hard for two players.*

Proof: by a reduction from Hamiltonian path.

NP-Hardness Construction

Given a graph $G = (V, E)$. Make a new graph G''.

- For each $v \in V$ introduce two nodes v' and v'' arc (v', v'') with latency $1 \cdot x$.
- For each edge $(u, v) \in E$, introduce arc (u'', v') with latency $\epsilon \cdot x$.
- Add three nodes s, s', t.
- Add arc (s, s') with latency $(M + \epsilon) \cdot x$.
- For each $v \in V$, add arcs (s', v') and (v'', t) with latency 0.
- Add arc (s, t) with latency $2M + 1$.

General Lower Bound (1/8)

Theorem

The sequential price of anarchy of symmetric linear network congestion games is unbounded.

Main proof ideas:

- Define a network.
- Create a *master plan* that the players should play.
- Define appropriate *punishing* action.
- Player applies punishing action when preceding player disrespects master plan.

General Lower Bound (2/8)

First the network. I have drawn its construction on the whiteboard.

- Parametrized by $k \in \mathbb{N}_{>0}$.
- There are k segments.
- A segment is a collection of $n - \epsilon$ parallel disjoint paths, with arcs interconnecting the paths internally.
- Arcs have cost function 0 or x.
- In a segment, any set of internal arcs can be chosen by a player.

General Lower Bound (3/8)

Analysis of the cost of the optimum.

- Every player should take 1 non-dummy arc in every segment.
- Put each player of the first $n - \epsilon$ players on disjoint path.
- This leaves ϵ players who have to share an arc with one of the $n - \epsilon$ other players.
- $n - 2\epsilon$ players have cost k.
- Other 2ϵ players have cost $2k$.
- Optimal social cost: $k((n - 2\epsilon) + 2 \cdot 2\epsilon)$.

General Lower Bound (4/8)

The master plan.

- If there are $\geq 2\epsilon$ successors, then:
  - If all predecessors have played according to this plan, play *fill*.
  - If *exactly one* predecessor did not play according to this plan, play *punish*.
  - If more than one predecessor does not play according to this plan, then play *greedy*.
- Else, play *greedy*.

Fill, *punish*, and *greedy* are called *action types*.

General Lower Bound (5/8)

Description of the action types:

- *Greedy*: In each segment, choose single resource with fewest number of players. When tied, overlap with the last player disrespecting the plan.
- *Punish*: Let j be the unique player disrespecting the plan. If there exists a non-dummy arc that j chose and is occupied by $< k$ players: Choose r and choose one free resource in each other segment. Otherwise play *greedy*.
- *Fill*: First filler chooses $\sqrt{k}$ free resources per segment, next $\sqrt{k} - 1$ fillers choose same resources as predecessor. Then process restarts.

General Lower Bound (6/8)

The master plan.

- If there are $\geq 2\epsilon$ successors, then:
  - If all predecessors have played according to this plan, play *fill*.
  - If *exactly one* predecessor did not play according to this plan, play *punish*.
  - If more than one predecessor does not play according to this plan, then play *greedy*.
- Else, play *greedy*.

Fill, *punish*, and *greedy* are called *action types*.

General Lower Bound (7/8)

Lemma

For the right choice of ϵ, the master plan is an SPE.

Proof sketch:

- Prove that for every player, for every choice of actions of previous players, following the plan is best, *conditioned on* subsequent players following the plan.
- Divide the proof for this into three parts: one for each action type.
- Subgame perfection follows from backward induction.

General Lower Bound (8/8)

Analysis of the cost of the master plan.

- First $n - 2\epsilon$ players play *fill*. Resulting in cost of $k \cdot \sqrt{k} \cdot \sqrt{k} = k^2$ per player.
- Last 2ϵ players play *greedy*. Resulting in cost of $2k$ per player.
- Total: $(n - 2\epsilon)k^2 + 2\epsilon 2k$

$$SPoA = \frac{k^2(n - 2\epsilon) + 2k2\epsilon}{k((n - 2\epsilon) + 2 \cdot 2\epsilon)} \in \Omega(k)$$

The Price of Anarchy (1/4)

What is the price of anarchy of symmetric affine network congestion games?

It turns out that this problem is open!

It is known that the PoA of symmetric affine (non-network) congestion games is $5/2$.

The Price of Anarchy (2/4)

We provide a lower bound of $5/2$ for symmetric affine network congestion games.

Theorem

The PoA of symmetric affine network congestion games is $5/2$

Proof works by constructing a sequence of examples whose PoA converges to $5/2$.

The Price of Anarchy (3/4)

- Sequence of examples is parametrized by the number of players.
- I have drawn the example for three players on the whiteboard.
- Again: disjoint *principal* (s, t)-paths with *interconnecting* dummy arcs.
- In the optimum everyone takes a disjoint path.
- In the equilibrium everyone takes part of each principal path:
  - Take a small part of a principal path,
  - and continue to the next principal path by taking an interconnecting arc,
  - wrapping around when last path is reached.

The Price of Anarchy (4/4)

- In equilibrium, players get in the way of each other.
- In equilibrium, social cost is $5n^2 - 2n$.
- Under the optimum, social cost is $2n^2 + n$.

Thus, we can make the PoA as bad as

$$\lim_{n \to \infty} \frac{5n - 2}{2n + 1} = \frac{5}{2}.$$

Our Main Open Problem

A subgame perfect equilibrium of a game is “almost always” unique.

But not in our main lower bound result. In fact: the *sequential price of stability* is 1 there.

When a SPE is unique, is the SPoA constant? What is the sequential price of stability?