Continuous Wardrop equilibria and Mean Field Games, with congestion costs or capacity constraints

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Outline

1. MFG: a coupled system of PDEs
2. Different variational problems
3. Coming back to an equilibrium
4. The analogy with continuous Wardrop equilibria
5. Some words on regularity
6. A variant: capacity constraints instead of congestion costs

What are MFG?

The theory of Mean Field Games has been introduced by Lasry and Lions to describe the evolution of a population, where each agent has to choose the strategy (i.e., a path) which best fits his preferences, but is affected by the others through a global *mean field*.
It is a differential game, with a continuum of players, all indistinguishable and all negligible. It is a typical congestion game (agents pay a congestion price, hence they try to avoid the regions with high concentrations) and we look for a Nash equilibrium, which can be translated into a system of PDEs.

J.-M. Lasry, P.-L. Lions, Mean-Field Games, *Japan. J. Math.* 2007
P.-L. Lions, courses at Collège de France, 2006/12, videos available at
http://www.college-de-france.fr/site/pierre-louis-lions/_course.htm
P. Cardaliaguet, lecture notes, www.ceremade.dauphine.fr/~cardalia/

Limit of finite games

The goal behind the theory is to study the limit as $N \to \infty$ of games of N player, each one choosing a trajectory $x_i(t)$ and optimizaing a quantity

$$\int_0^T \left(\frac{|x_i'(t)|^2}{2} + g_i(x_1(t), \ldots, x_N(t)) \right) dt + \Psi_i(x_i(T)).$$

In particular, we are interested in the case where g_i penalizes points close to too many other players x_j, $j \neq i$.
Note that we consider here deterministic mean field games (no stochastic effects in the trajectories $x_i(t)$).

We will suppose that g_i only depends on the position x_i and on the distribution of the other player, and that all players have the same preferences. And **we will not study the discrete case** and pass to the limit, but **directly study the continuous case**.

MFG with density penalization- 1

Each agent in a population chooses his own trajectory in Ω, solving

$$\min \int_0^T \left(\frac{|x'(t)|^2}{2} + g(\rho_t(x(t))) \right) dt + \Psi(x(T)),$$

with given initial point $x(0)$; here g is a given increasing function of the density ρ_t at time t (we take $g(0) = 0$ and $g \geq 0$). The agent hence tries to avoid overcrowded regions.

Input: the evolution of the density ρ_t.

A crucial tool is the value function φ for this problem, defined as

$$\varphi(t_0, x_0) := \min \left\{ \int_{t_0}^T \left(\frac{|x'(t)|^2}{2} + g(\rho_t(x(t))) \right) dt + \Psi(x(T)), \; x(t_0) = x_0 \right\}.$$

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MFG with density penalization- 2

Optimal control theory tells us that φ solves

$$(HJ) \qquad -\partial_t\varphi(t,x) + \frac{1}{2}|\nabla\varphi(t,x)|^2 = g(\rho_t(x)), \quad \varphi(T,x) = \Psi(x).$$

Moreover, the optimal trajectories $x(t)$ follow $x'(t) = -\nabla\varphi(t,x(t))$.

Hence, given the initial ρ_0, we can find the density at time t by solving

$$(CE) \qquad \partial_t\rho - \nabla\cdot(\rho\nabla\varphi) = 0,$$

which give as **Output:** the evolution of the density ρ_t.
We have an equilibrium if **Input = Output**.
This requires to solve a coupled system (HJ)+(CE):

$$\begin{cases} -\partial_t\varphi + \frac{|\nabla\varphi|^2}{2} = g(\rho), \\ \partial_t\rho - \nabla\cdot(\rho\nabla\varphi) = 0, \\ \varphi(T,x) = \Psi(x), \quad \rho(0,x) = \rho_0(x). \end{cases}$$

Stochastic case : we can also insert random effects $dX = \alpha dt + dB$, obtaining $-\partial_t\varphi - \Delta\varphi + \frac{|\nabla\varphi|^2}{2} - g(\rho) = 0 \ ; \quad \partial_t\rho - \Delta\rho - \nabla\cdot(\rho\nabla\varphi) = 0$

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Variational principle

It happens that an equilibrium is found by minimizing the (global) energy

$$\mathcal{A}(\rho, v) := \int_0^T \int_\Omega \left(\frac{1}{2} \rho_t |v_t|^2 + G(\rho_t) \right) + \int_\Omega \Psi \rho_T$$

among pairs (ρ, v) such that $\partial_t \rho + \nabla \cdot (\rho v) = 0$, with given ρ_0, where G is the anti-derivative of g, i.e. $G' = g$ (in particular, G is convex).

Warning: as it often happens in congestion games, this is not the total cost for all the agents, as we put $G(\rho)$ instead of $\rho g(\rho)$.The equilibrium minimizes an overall energy (it's a *potential game*), but not the total cost: there is a *price of anarchy*.

Important: this problem is convex in the variables $(\rho, w := \rho v)$ and it recalls Benamou-Brenier formulation for optimal transport.
This formulation can be used to do numerics!!

J.-D. Benamou, Y. Brenier A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem, *Numer. Math.*, 2000.

J.-D. Benamou, G. Carlier Augmented Lagrangian methods for transport optimization, Mean-Field Games and degenerate PDEs, preprint.

Duality

As all convex minimization problem, $\min \mathcal{A}$ admits a dual problem, obtained from

$$\min_{\rho,v} \mathcal{A}(\rho, v) + \sup_{\phi} \int_0^T \int_\Omega (\rho \partial_t \phi + \nabla\phi \cdot \rho v) + \int_\Omega \phi_0 \rho_0 - \int_\Omega \phi_T \rho_T,$$

interchanging inf and sup. We get

$$\sup \left\{ -\mathcal{B}(\phi, p) := \int_\Omega \phi_0 \rho_0 - \int_0^T \int_\Omega G^*(p_+) \ : \ \phi_T \le \Psi, \, -\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 = p \right\},$$

where G^* is the Legendre transform of G, i.e. $G^*(p) = \sup_q \ pq - G(q)$.

For optimal (ρ, v, ϕ, p) we have (ρ-a.e.) $v = -\nabla\phi$, $p = g(\rho)$ and $\phi_T = \Psi$ i.e., a solution to the MFG system (up to some technicalities).

Measures on possible trajectories

The same variational problem can also be written in the following way: let $\mathcal{C} = H^1([0,T];\Omega)$ be the space of curves valued in Ω and $e_t : \mathcal{C} \to \Omega$ the evaluation map, $e_t(\gamma) = \gamma(t)$. Solve

$$\min \left\{ \int_{\mathcal{C}} K dQ + \int_0^T \mathcal{G}((e_t)_\# Q) + \int_\Omega \Psi d(e_T)_\# Q,\ Q \in \mathcal{P}(\mathcal{C}), (e_0)_\# Q = \rho_0 \right\},$$

where $K : \mathcal{C} \to \mathbb{R}$ and $\mathcal{G} : \mathcal{P}(\Omega) \to \overline{\mathbb{R}}$ are given by $K(\gamma) = \frac{1}{2}\int_0^T |\gamma'|^2$ and $\mathcal{G}(\rho) = \int G(\rho(x))dx$. ($\#$ denotes image measure, or push-forward).

Existence: by semicontinuity in the space $\mathcal{P}(\mathcal{C})$.

Optimality conditions: take $\overline{Q}$ optimal, $\widetilde{Q}$ another competitor, and $Q_\varepsilon = (1-\varepsilon)\overline{Q} + \varepsilon\widetilde{Q}$. Setting $\rho_t = (e_t)_\#\overline{Q}$ and $h(t,x) = g(\rho_t(x))$, differentiating w.r.t. ε gives

$$J_h(\widetilde{Q}) \geq J_h(\overline{Q}),$$

where J_h is the linear functional

$$J_h(Q) = \int K dQ + \int_0^T \int_\Omega h(t,x)(e_t)_\# Q + \int_\Omega \Psi d(e_T)_\# Q.$$

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Back to an equilibrium

Look at J_h. It is well-defined for $h \geq 0$ measurable.
But if $h \in C^0$ we can also write $\int_0^T \int_\Omega h(t,x)(e_t)_\# Q = \int_C dQ \int_0^T h(t,\gamma(t))dt$ and hence we get that

$$Q \mapsto \int_C dQ(\gamma)\left(K(\gamma) + \int_0^T h(t,\gamma(t))dt + \Psi(\gamma(T))\right)$$

is minimal for $Q = \overline{Q}$. Hence $\overline{Q}$ is concentrated on curves minimizing $K(\gamma) + \int_0^T h(t,\gamma(t))dt + \Psi(\gamma(T))$. This means **Input=Output**.

A rigorous proof can also be done even for $h \notin C^0$ but one has to choose a precise representative. Techniques from incompressible fluid mechanics **(incompressible Euler** *à la* **Brenier)** allow to handle the case $G(\rho) \approx \rho^q$, $h \in L^{q'}$, $q, q' > 1$ using $\hat{h}(x) := \limsup_{r \to 0} \fint_{B(x,r)} h(t,y)dy$ (use of the *maximal function* needed to justify some convergences, which requires h to be better than L^1).

L. Ambrosio, A. Figalli, On the regularity of the pressure field of Brenier's weak solutions to incompressible Euler equations, *Calc. Var. PDE*, 2008.

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Continuous Wardrop equilibria

A very much related problem is the following continuous version of Wardrop equilibria: find $Q \in \mathcal{P}(C)$ such that $Q-$a.e. curve is a geodesic for the distance

$$d_k(x,y) := \inf\left\{\int_0^1 k(\gamma)|\gamma'| \,:\, \gamma(0)=x, \gamma(1)=y\right\}$$

with $k = g(i_Q)$, where i_Q is the *traffic intensity* defined (as a measure on Ω) through

$$< i_Q, \phi > := \int_C dQ(\gamma) \int_0^1 \phi(\gamma(t))|\gamma'(t)|dt.$$

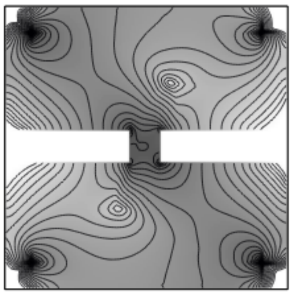
Also this equilibrium problem is a potential game, and solutions can be found by solving

$$\min\left\{\int_\Omega G(i_Q(x))dx \,:\, Q \text{ admissible}\right\}.$$

J. G. Wardrop, Some theoretical aspects of road traffic research, *Proc. Inst. Civ. Eng.*, 1952.

G. Carlier, C. Jimenez and F. Santambrogio, Optimal transportation with traffic congestion and Wardrop equilibria, *SIAM J. Control Optim.* , 2008.

An example



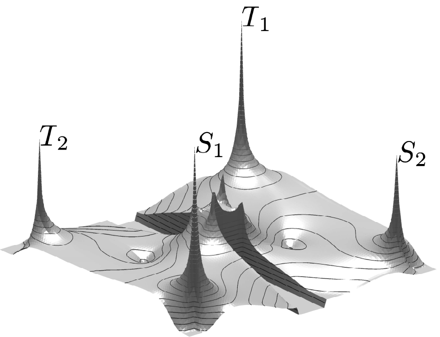


Figure: Traffic intensity i_Q at equilibrium in a city with a river and a bridge, with two sources S_1 and S_2, and two targets T_1 and T_2. Traffic concentrates close to origins, destinations, and concave corners of the domain.

F. Benmansour, G. Carlier, G. Peyré and F. Santambrogio, Numerical Approximation of Continuous Traffic Congestion Equilibria, *Net. Het. Media*, 2009.
F. Benmansour, G. Carlier, G. Peyré and F. Santambrogio, Fast Marching Derivatives with Respect to Metrics and Applications, *Numerische Mathematik*, 2010.

Wardrop vs MFG – difference and similarities

- Wardrop equilibrium is a *statical* problem, while MFG are *dynamical* (more refined modeling).
- Additive costs versus multiplicative ones (i.e. *conformal Riemannian distances*): different mathematical techniques.
- Prescribing the final density ρ_1 or a final cost Ψ is just a matter of taste (but the former is impossible in the stochastic case).
- In Wardrop we usually prescribe $(e_0, e_1)_\# Q = \pi \in \mathcal{P}(\Omega \times \Omega)$ (*who-goes-where* problem: agents are not indistinguishable), while in MFG we usually give ρ_0 and Ψ.
- In the indistinguishable case for the Wardrop problem (i.e. prescribing $(e_0)_\# Q = \rho_0$, $(e_1)_\# Q = \rho_1$), then there is a divergence-constrained formulation

$$\min \left\{ \int G(v(x))dx \; : \; \nabla \cdot v = \rho_0 - \rho_1 \right\}$$

(MFG, instead, contains a space-time divergence constraints).

- In both cases, we should give a meaning to the integral of h $(= g(\rho_t)$ or $g(i_Q))$ on curves, which requires regularity, or at least summability.

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Back to MFG – the problem of regularity

Obtaining classical solutions to the MFG system is a hard question. One possible strategy, suggested by P-L Lions, is to reduce everything to a (non-linear and degenerate) elliptic equation in φ. For instance, if $g(\rho) = \rho$, we can replace ρ with $-\partial_t\varphi + \frac{1}{2}|\nabla\varphi|^2$ and obtain

$$\partial_{tt}\varphi + \frac{1}{2}\Delta_4\varphi - 2\partial_t\nabla\varphi\cdot\nabla\varphi - \partial_t\Delta\varphi = 0.$$

This PDE is degenerate elliptic and corresponds to the minimization of $\iint(\partial_t\varphi - \frac{1}{2}|\nabla\varphi|^2)^2$ (with suitable boundary conditions; actually, this is just the dual problem).

It is easier when $g(\rho) = \log\rho$, which reduces degeneracy

$$\Delta_{t,x}\varphi + \nabla\varphi\cdot D^2\varphi\cdot\nabla\varphi - 2\partial_t\nabla\varphi\cdot\nabla\varphi = 0$$

(it is actually non-degenerate as soon as $|\nabla\varphi|$ is bounded). This corresponds to $\min\iint e^{(\partial_t\varphi - \frac{1}{2}|\nabla\varphi|^2)}$.

Yet, let us see **a different technique, based on duality** (originating again from Brenier's works on incompressible Euler).

Using duality

Take arbitrary (ρ, v) and (ϕ, p) admissible in the primal and dual problem. Compute

$$\mathcal{A}(\rho, v) + \mathcal{B}(\phi, p)$$
$$= \int_\Omega (\Psi - \phi_T)\rho_T + \int_0^T \int_\Omega (G(\rho) + G^*(p_+) - p\rho) + \frac{1}{2}\int_0^T \int_\Omega \rho|v + \nabla\phi|^2.$$

Notice $(G(\rho) + G^*(p_+) - p\rho) \geq \frac{\lambda}{2}|\rho - g^{-1}(p_+)|^2$ where $\lambda = \inf g'$. Suppose $\lambda > 0$.

We know $\min \mathcal{A} + \min \mathcal{B} = 0$. Take $(\rho, v), (\phi, p)$ optimal. We get

$$\begin{aligned} \rho &= g^{-1}(p_+) & \\ \Psi &= \phi_T & \text{on } \{\rho_T > 0\} \\ v &= -\nabla\phi & \text{on } \{\rho > 0\}, \end{aligned}$$

i.e. (again) a solution of the MFG system, in a suitable sense.

H^1 regularity from duality

Suppose for simplicity $\Omega = \mathbb{T}^d$ to be the flat torus. We go on from

$$\mathcal{A}(\rho, v) + \mathcal{B}(\phi, p) \geq c \int_0^T \int_\Omega |\rho - g^{-1}(p_+)|^2.$$

Again, take $(\rho, v), (\phi, p)$ optimal. Take (ρ^δ, v^δ) translation of (ρ, v) (i.e. $\rho^\delta(t,x) = \rho(t, x+\delta)$, up to some cut-off functions to correct at $t = 0$ and $t = T$).

From the fact that $\delta \mapsto \mathcal{A}(\rho^\delta, v^\delta)$ is smooth and minimal for $\delta = 0$, we can prove $\mathcal{A}(\rho^\delta, v^\delta) \leq \mathcal{A}(\rho, v) + C|\delta^2|$. We get

$$\int_0^T \int_\Omega |\rho^\delta - \rho|^2 = \int_0^T \int_\Omega |\rho^\delta - g^{-1}(p_+)|^2 \leq \mathcal{A}(\rho^\delta, v^\delta) + \mathcal{B}(\phi, p) \leq C|\delta|^2,$$

which means $\rho \in L^2_{loc}((0,T); H^1(\Omega))$. We can also adapt to time translation and obtain $\rho \in H^1_{loc(t,x)}$. We can also get $\iint \rho |D^2\phi|^2 < \infty$.

MFG with density constraints - 1

How to define a mean field game if we want to replace the penalization $+g(\rho)$ with the (capacity) constraint $\rho \le 1$?

Naive idea: when $(\rho_t)_t$ is given, every agent minimizes his own cost paying attention to the constraint $\rho_t(x(t)) \le 1$. But if ρ already satisfies $\rho \le 1$, one extra agent will not violate the constraint (it's a *non-atomic game*). Hence the constraint becomes empty.

Instead, let's look at the variational problem

$$\min \left\{ \int_0^T \int_\Omega \frac{1}{2}\rho_t |v_t|^2 + \int_\Omega \Psi \rho_T \ : \ \rho \le 1 \right\}.$$

It means $G(\rho) = 0$ for $\rho \in [0,1]$ and $+\infty$ otherwise. There is a dual

$$\sup \left\{ \int_\Omega \phi_0 \rho_0 - \int_0^T \int_\Omega p_+ \ : \ \phi_T \le \Psi, \ -\partial_t \phi + \frac{1}{2}|\nabla \phi|^2 = p \right\}.$$

This problem is also obtained as the limit $m \to \infty$ of $g(\rho) = \rho^m$. Indeed the functional $\frac{1}{m+1}\int \rho^{m+1}$ Γ-converges to the constraint $\rho \le 1$.

F. Santambrogio, A Modest Proposal for MFG with Density Constraints, *NHM* 2012.

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It means $G(\rho) = 0$ for $\rho \in [0,1]$ and $+\infty$ otherwise. There is a dual

$$\sup \left\{ \int_\Omega \phi_0 \rho_0 - \int_0^T \int_\Omega p_+ \ : \ \phi_T \le \Psi, \ -\partial_t \phi + \frac{1}{2}|\nabla \phi|^2 = p \right\}.$$

This problem is also obtained as the limit $m \to \infty$ of $g(\rho) = \rho^m$. Indeed the functional $\frac{1}{m+1} \int \rho^{m+1}$ Γ-converges to the constraint $\rho \le 1$.

F. SANTAMBROGIO, A Modest Proposal for MFG with Density Constraints, *NHM* 2012.

MFG with density constraints - 1

How to define a mean field game if we want to replace the penalization $+g(\rho)$ with the (capacity) constraint $\rho \leq 1$?

Naive idea: when $(\rho_t)_t$ is given, every agent minimizes his own cost paying attention to the constraint $\rho_t(x(t)) \leq 1$. But if ρ already satisfies $\rho \leq 1$, one extra agent will not violate the constraint (it's a *non-atomic game*). Hence the constraint becomes empty.

Instead, let's look at the variational problem

$$\min \left\{ \int_0^T \int_\Omega \frac{1}{2} \rho_t |v_t|^2 + \int_\Omega \Psi \rho_T \ : \ \rho \leq 1 \right\}.$$

It means $G(\rho) = 0$ for $\rho \in [0,1]$ and $+\infty$ otherwise. There is a dual

$$\sup \left\{ \int_\Omega \phi_0 \rho_0 - \int_0^T \int_\Omega p_+ \ : \ \phi_T \leq \Psi, \ -\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 = p \right\}.$$

This problem is also obtained as the limit $m \to \infty$ of $g(\rho) = \rho^m$. Indeed the functional $\frac{1}{m+1} \int \rho^{m+1}$ Γ-converges to the constraint $\rho \leq 1$.

F. Santambrogio, A Modest Proposal for MFG with Density Constraints, *NHM*, 2012.

MFG with density constraints - 2

The system we get is

$$\begin{cases} -\partial_t\varphi + \frac{|\nabla\varphi|^2}{2} = p, \\ \partial_t\rho - \nabla\cdot(\rho\nabla\varphi) = 0, \\ p \geq 0,\ \rho \leq 1,\ p(1-\rho) = 0, \\ \varphi(T,x) = \Psi(x), \quad \rho(0,x) = \rho_0(x). \end{cases}$$

Each agent solves min $\int_0^T \left(\frac{|x'(t)|^2}{2} + p(t,x(t))\right)dt + \Psi(x(T))$.

Here p is a **pressure** arising from the incompressibility constraint $\rho \leq 1$ but finally acts as a **price.** In order to give a meaning to the above problem we need a bit of regularity. The same kind of duality argument, as in the works by Brenier and Ambrosio-Figalli, allow to get

$$p \in L^2_{loc}((0,T); BV(\Omega)).$$

P. Cardaliaguet, A. Mészáros, F. Santambrogio, First order Mean Field Games with density constraints: Pressure equals Price, preprint

Y. Brenier, Minimal geodesics on groups of volume-preserving maps and generalized solutions of the Euler equations, *Comm. Pure Appl. Math.*, 1999.

Continuous Wardrop equilibria with capacity constraints

Open Problem Given $\pi \in \mathcal{P}(\Omega \times \Omega)$ (or - which is easier - given $\rho_0, \rho_1 \in \mathcal{P}(\Omega)$), find $Q \in \mathcal{P}(C)$ and p smooth enough such that

- $p \geq 0,\ i_Q \leq 1,\ p(1 - i_Q) = 0$
- Q-a.e. curve γ is geodesic for the distance d_k with $k = i_Q + p$
- $(e_0, e_1)_\# Q = \pi$ (or $(e_0)_\# Q = \rho_0, (e_1)_\# Q = \rho_1$)

The corresponding variational problem are

$$\min \left\{ \int |i_Q| dx \ :\ i_Q \leq 1 \right\}$$

and

$$\min \left\{ \int |v(x)| dx \ :\ \nabla \cdot v = \rho_0 - \rho_1,\ |v| \leq 1 \right\}.$$

The difficult issue is the regularity of the pressure/price p, which is a priori a measure (since $p \geq 0$) but must be integrated on each trajectory.

The End

Thank you for your attention