Hotelling games on networks

Gaëtan FOURNIER Marco SCARSINI

Tel Aviv University

LUISS, Rome

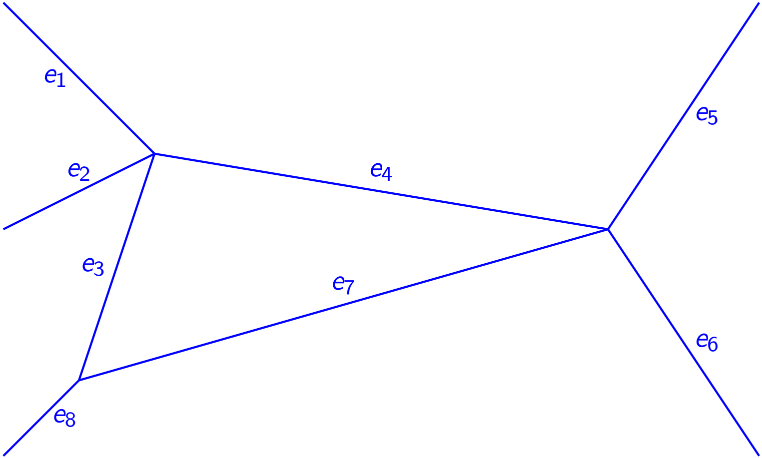
NUS December 2015

Hypothesis on buyers

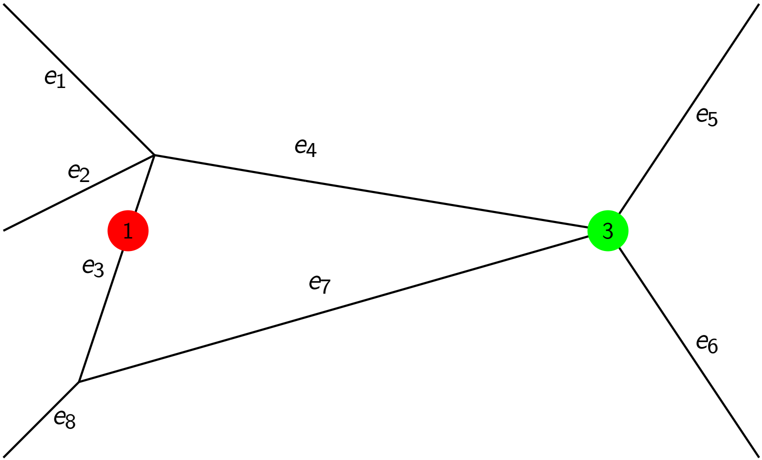
1. Infinite number of buyers, distributed on the network.
2. They want to buy one share of a particular good whose price is fixed: they shop to the closest location.

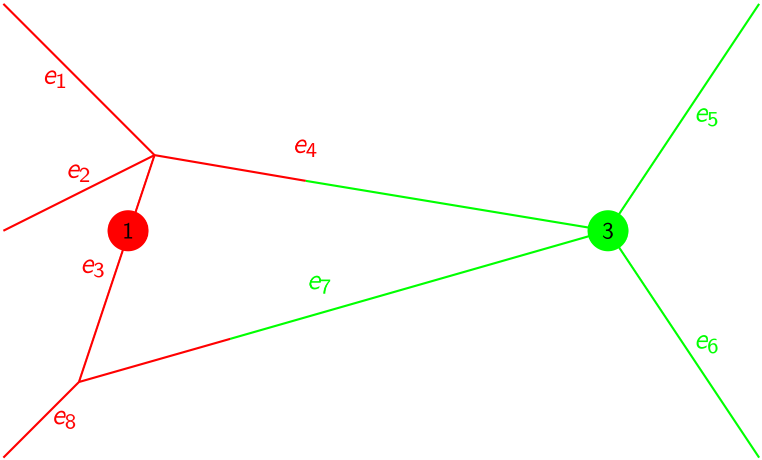
Hypothesis on sellers

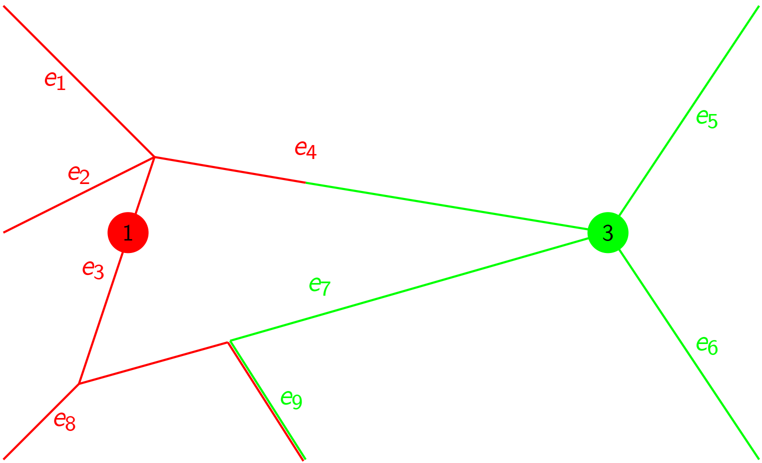
1. A fixed number of sellers cover the demand on this network.
2. They simultaneously choose their locations.
3. They want to sell as much as possible.



$$G = (X, E), \quad \lambda : E \to \mathbb{R}_\star^+$$

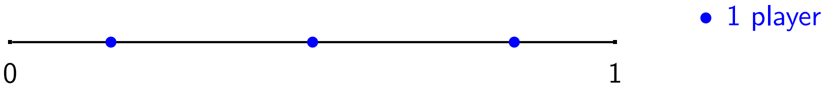

e_1
e_2
e_3
e_4
e_5
e_6
e_7
e_8
1
3


e_1
e_2
e_3
e_4
e_5
e_6
e_7
e_8
1
3


e_1
e_2
e_3
e_4
e_5
e_6
e_7
e_8
e_9
1
3

Not a potential game:

- There is no pure equilibrium for 3 players in the unit interval.



Finite number k of possible locations:

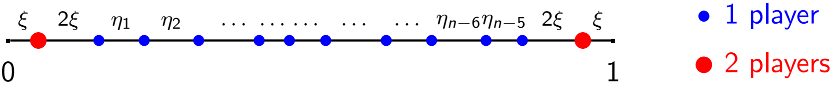
- At equilibrium with a large number of players, every location is occupied.
- The network is dived into k part of lengths $L_1, \ldots, L_k$.
- Such an equilibrium is an equilibrium in the congestion game with parallel edges with cost $\frac{L_i}{n}$ when n users choose the edge i.

Results with uniform density

- Existence of pure Nash equilibrium for any graph when the number of player is large enough.
- Efficiency of these equilibria in terms of distance consumers have to travel: asymptotic convergence.

The unit interval

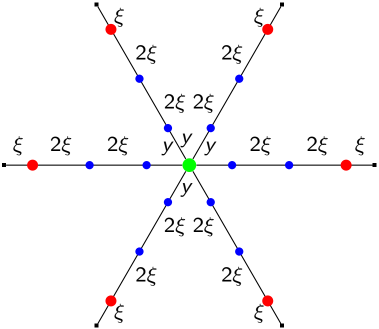
1. For $n = 2$, there exists a pure Nash equilibrium.
2. For $n = 3$, there is no pure Nash equilibrium.
3. For $n \geq 4$,there exists a pure Nash equilibrium.



$$\begin{cases} \forall i \in [1, n-5], & 0 \leq \eta_i \leq 2\xi, \\ \forall i \in [1, n-6], & \frac{\eta_i + \eta_{i+1}}{2} \geq \xi \\ \sum_{i=1}^{n-5} \eta_i + 6\xi = 1 \end{cases}$$

The star $S_k(r)$

1. For $n \leq k$, there exists a pure Nash equilibrium.
2. For $n \in \ ]k, 3k-1[$, there is no pure Nash equilibrium.
3. For $n \geq 3k-1$, there exists a pure Nash equilibrium.



- 1 player
- 2 players
- r players

Equilibrium with $4k + r$ players
$(2r\xi/k \leq y \leq 2(r+1)\xi/k)$

Asymptotic existence of pure Nash equilibrium

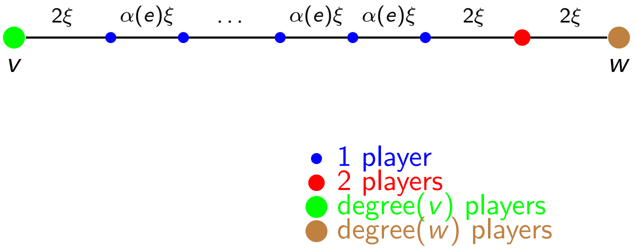
On any finite graph Hotelling games always have pure Nash equilibrium, provided the number of players is larger than $N := 3\,\mathrm{card}(E) + \sum_{e \in E} \left\lceil \frac{5\lambda(e)}{\lambda^\star} \right\rceil$.

$\lambda^\star = \min_E \lambda$ (the length of the shortest edge).

Sketch of the proof

1/ The graph $G = (X, E)$ and n are fixed. We want to construct a pure Nash equilibrium with n players on G. We fix a general dilatation parameter $\xi > 0$.

2/ On each edge, we put a number of players $n(e)$ that only depends on the length $\lambda(e)$ of the edge and on ξ.



Where α is such that the number of players on e is $n(e)$.

3/ We prove that if ξ is small enough this profile of location is an equilibrium, with a number of player equal to

$$\sum_{e} n(e) = 3\,\text{card}(E) + \sum_{e\in E} \left\lceil \frac{\lambda(e)}{2\xi} \right\rceil$$

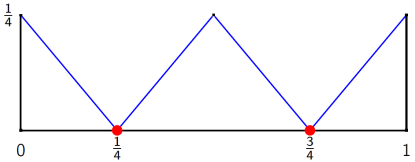
4/ Can we find ξ such that $f(\xi) = n$?

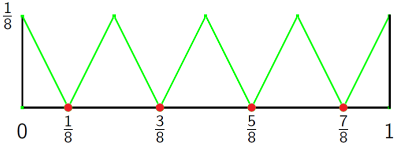
5/ No but we can find n' such that there exists ξ such that $f(\xi) = n'$, $n' \geq n$, and $n' - n \leq card(E)$.

6/ We select the equilibrium with n' players. We can remove up to one unnecessary player on each edge to have an equilibrium with n player.

Results with uniform density

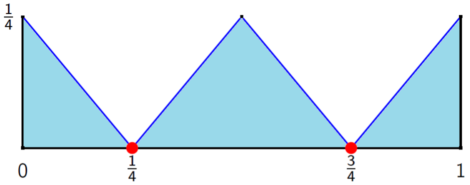
- Existence of pure Nash equilibrium for any graph when the number of player is large enough.
- Efficiency of these equilibria in terms of distance consumers have to travel: asymptotic convergence.

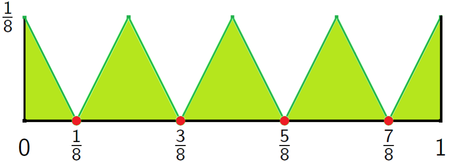




Travelling distances of consummers, in equilibrium and in social optimum.

Equilibrium social cost: ?
Optimum social cost: ?





Social costs in equilibrium and in social optimum.

Equilibrium social cost: $\frac{1}{8}$

Optimum social cost: $\frac{1}{16}$

- For $\boldsymbol{x} \in S^n$, the **social cost** $\sigma(\boldsymbol{x})$ is given by:

$$\sigma(\boldsymbol{x}) := \int_S \min_{i \in \{1,\dots,n\}} d(x_i, y) dy$$

- **The price of anarchy** is given by:

$$\operatorname{IPoA}(n) := \frac{\max_{\boldsymbol{x} \in \mathcal{E}_n(\mathcal{H})} \sigma(\boldsymbol{x})}{\min_{\boldsymbol{x} \in S^n} \sigma(\boldsymbol{x})},$$

- **The price of stability** is given by:

$$\operatorname{IPoA}(n) := \frac{\min_{\boldsymbol{x} \in \mathcal{E}_n(\mathcal{H})} \sigma(\boldsymbol{x})}{\min_{\boldsymbol{x} \in S^n} \sigma(\boldsymbol{x})},$$

where $\mathcal{E}_n(\mathcal{H})$ is the set of equilibrium with n players.

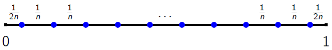


Figure: Social optimum $\bar{x}$ with n players.

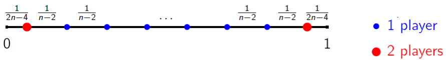


Figure: Best equilibrium $\check{x}$ with n players.

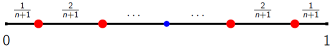


Figure: Worst equilibrium $\hat{x}$ with n players (n odd)

Figure: Worst equilibrium $\hat{x}$ with n players (n even).

On the unit interval, we have:

$$\mathrm{IPoA}(n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 2\left(\dfrac{n}{n+1}\right) & \text{if } n > 3 \text{ is odd.} \end{cases}$$

For $n \geq 4$

$$\mathrm{IPoS}(n) = \frac{n}{n-2}$$

Theorem

Suppose that the game $\mathcal{H}(n, S)$ has an equilibrium. Then

(a)

$$\mathsf{IPoA}(n) \to 2 \text{ as } n \to \infty$$

(b)

$$\mathsf{IPoS}(n) \to 1 \text{ as } n \to \infty$$

Stochastic dominance / Majorization

For a vector $\boldsymbol{z} = (z_1, \dots, z_n)$, we denote $z_{[1]} \geq \dots \geq z_{[n]}$ its decreasing rearrangement.

Definition

Let $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$ be such

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$$

if, for all $k \in \{1, \dots, n\}$

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}.$$

then we say that $\boldsymbol{x}$ is *majorized* by $\boldsymbol{y}$ ($\boldsymbol{x} \prec \boldsymbol{y}$).

Definition

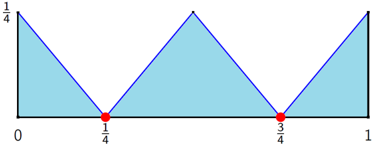
A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is said **Schur-convex** if $\mathbf{x} \prec \mathbf{y}$ implies $\phi(\boldsymbol{x}) \leq \phi(\boldsymbol{y})$.

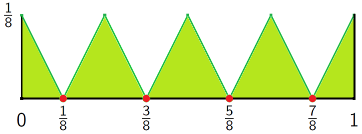
Proposition

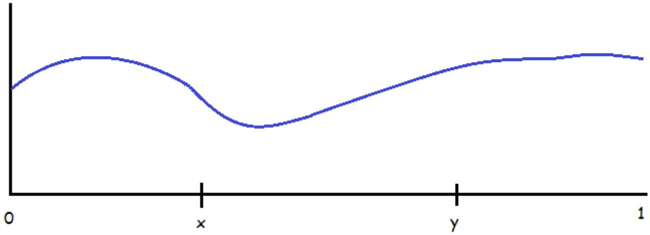
If $\psi : \mathbb{R} \to \mathbb{R}$ is a convex function,

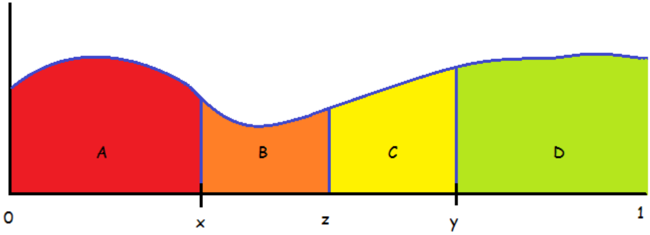
$$\phi(x_1, \ldots, x_n) = \sum_{i=1}^{n} \psi(x_i),$$

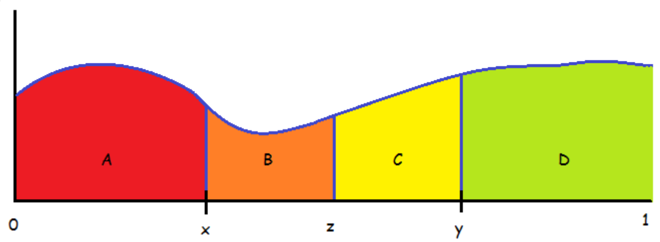
then ϕ is Schur-convex.






0
x
y
1


A
B
C
D
0
x
z
y
1



$$A = B = C = D$$

$$x = Q_{\frac{1}{4}}, \quad z = Q_{\frac{1}{2}}, \quad y = Q_{\frac{3}{4}}$$

No general equilibrium with 4 players

There exists a pure Nash equilibrium on the unit interval with 4 players and with density f if and only if f satisfies $Q_{\frac{1}{2}} = \frac{Q_{\frac{1}{4}} + Q_{\frac{3}{4}}}{2}$

Asymptotic existence of ϵ-equilibrium.

Suppose that:

1. f is K-Lipschitz
2. There exist m and M such that for all x, $0 < m \leq f(x) \leq M$

Then:

$$\forall \epsilon > 0,\ \exists N(\epsilon) \in \mathbb{N},\ \forall n \geq N(\epsilon),$$

there exists an $\epsilon-$ pure equilibrium in the game with n players and density distribution f.

Sketch of the proof:

1/ Fix an $\epsilon > 0$.

2/ Approximate f by a step function g with precision ϵ_2

3/ Construct an exact equilibrium on the game with density distribution g. It exists if the number of player is larger that of bound $N(\epsilon_1)$.

4/ Prove that if ϵ_1 is small enough, the equilibrium is an ϵ-equilibrium in the original game, with density distribution f.

During this constructive proof, we found that

$$N(\epsilon) := 4card(E) + \frac{2L(M+\epsilon)}{(m-\epsilon)}(\frac{K}{\epsilon} + \frac{2}{\min \lambda_e}) + \frac{3LK}{2\epsilon}$$

Thank you