Internalization of Social Cost in Congestion Games

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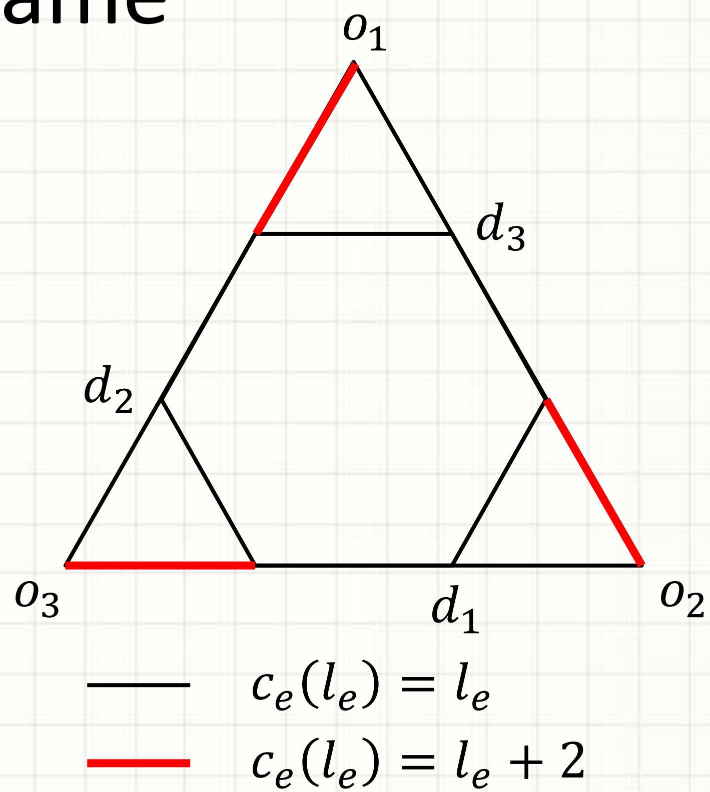
Internalization of Social Payoff

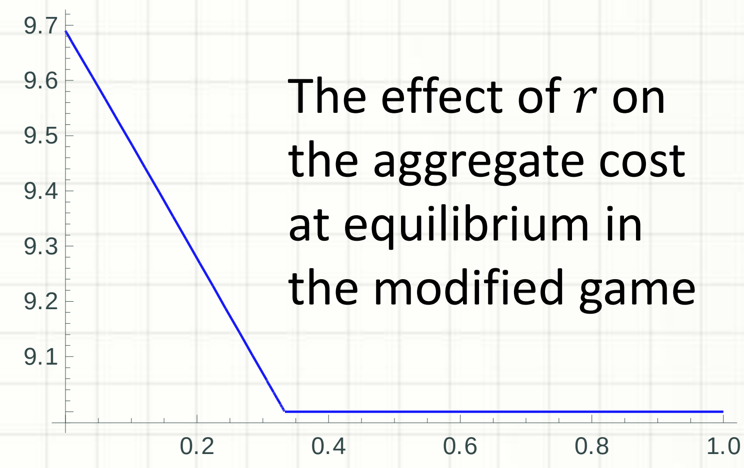
- A strategic game h has a (finite or infinite) set of players
- Each player i has a strategy set X_i and a payoff function h_i
- A social payoff is any function $f\colon X \stackrel{\text{def}}{=} \prod_i X_i \to \mathbb{R}$
- The altruism coefficient is an exogenous parameter $r \leq 1$
- In the modified game h^r, the (modified) payoff of player i is
$$h_i^r = (1-r)h_i + rf$$
- For the aggregate payoff $f = \sum_i h_i,\ h_i^r = h_i + r\sum_{j\neq i} h_j$
- A positive or negative r expresses altruism or spite
- Comparative statics concern the connection between r and the value of f at the Nash equilibria of the modified game h^r

Example 1: Congestion Game

- A three-player congestion game
- Player i ships unit weight $o_i \to d_i$
- Splittable among multiple routes
- Using only the <u>short</u> routes minimizes the aggregate cost
- Equilibrium of h^r for $1/3 \le r \le 1$
- For $0 \le r < 1/3$, at equilibrium players ship on their <u>long</u> routes a weight of $(1-3r)/(10-4r)$
- The aggregate cost is then

$$169\left(\frac{1-r}{10-4r}\right)^2 + 8$$





The effect of r on the aggregate cost at equilibrium in the modified game

The Cost of Anarchy

- Concerns a specified family of games $\mathcal{H}$
- The social payoff f is defined as the aggregate payoff
- It is assumed that $-f$, the <u>social cost</u>, is always positive
- With altruism coefficient $r \leq 1$, the <u>cost of anarchy</u> is

$$\mathrm{CoA}^r = \sup\left\{ \frac{f(y)}{f(x)} \;\middle|\; \begin{array}{l} h \in \mathcal{H}, x \text{ a strategy profile in } h, \\ y \text{ an equilibrium in } h^r \end{array} \right\}$$

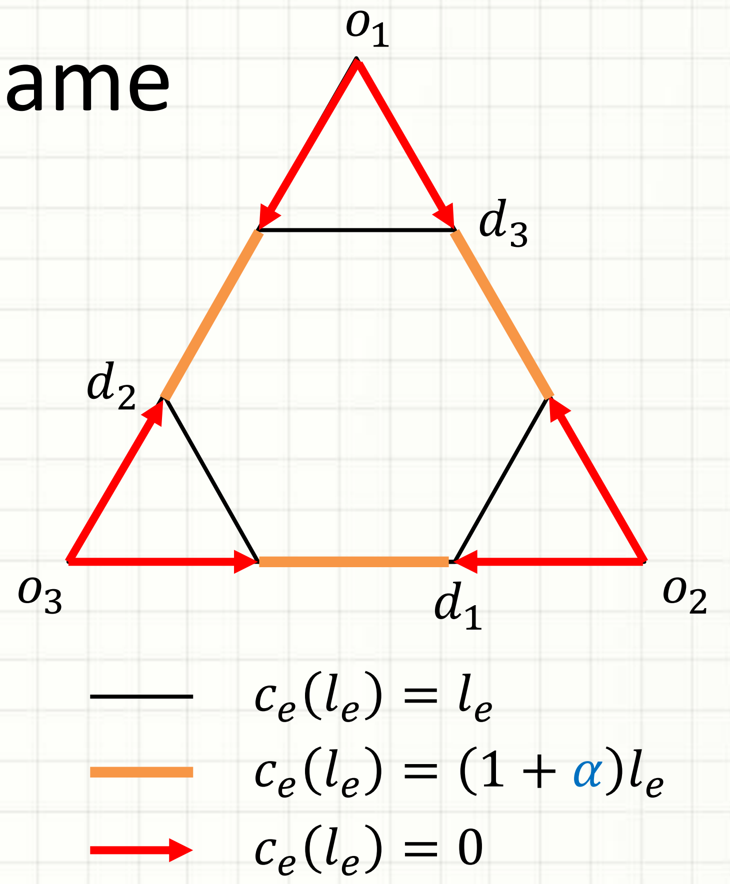
- This can only increase when $\mathcal{H}$ is enlarged

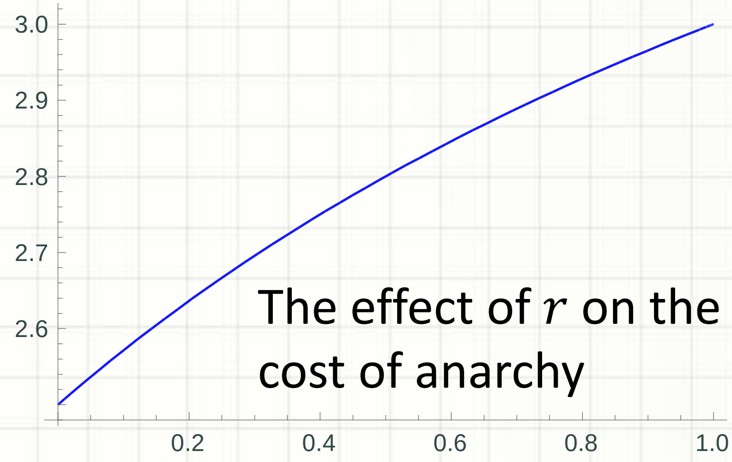
Example 2: Congestion Game

- Family of games $\mathcal{H} = \{h_\alpha\}_{0\le\alpha\le 1}$
- Player i ships unit weight $o_i \to d_i$
- This weight is unsplittable
- Using only the short routes minimizes the aggregate cost
- Using only the long routes is an equilibrium in h_α^r, for all $r \ge \alpha$
- The aggregate cost is then greater than the minimum by a factor of

$$\frac{5+4\alpha}{2+\alpha}$$

- For $r = \alpha$, this ratio gives CoA^r





The Cost of Anarchy in Congestion Games

Theorem (Caragiannis et al. 2010, Chen et al. 2014).

With altruism coefficient $0 \leq r \leq 1$, the price of anarchy CoA^r for atomic linear congestion games is $(5 + 4r)/(2 + r)$.

Thus, an increasing concern for the social cost paradoxically result in a greater cost of anarchy.

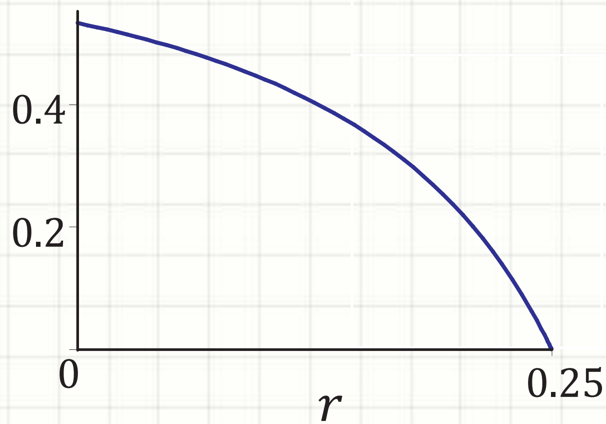
This refers to the class of all congestion games with

- Unsplittable unit weights, and
- cost functions $c_e(l_e) = a_e l_e + b_e$, with $a_e, b_e \geq 0$.

“Paradoxical” Comparative Statics

- Negative comparative statics, whereby the social payoff decreases with increasing r, do occur
- Generalized rock–scissors–paper game h:

Social payoff at equilibrium in h^r



	R	S	P
R	0	4	−4
S	−1	0	4
P	1	−1	0

- Why do the payoffs decrease?
- This can be linked to the instability of the equilibria
- The unique equilibrium in h^r is not an ESS
- A similar link holds very generally

Static Stability in Symmetric Games

Definition. In a symmetric N-player game with payoff function g, a strategy y is stable, weakly stable or definitely unstable if

$$\frac{1}{N}\sum_{j=1}^{N}\left(g\left(x,\underbrace{x,\dots,x}_{j-1\text{ times}},y,\dots,y\right)-g\left(y,\underbrace{x,\dots,x}_{j-1\text{ times}},y,\dots,y\right)\right)$$

is negative, nonpositive or positive, respectively, for all $x \neq y$ in some neighborhood of y.

- Stability means that moving the players one-by-one from y to any nearby alternative strategy x on average harms them
- "Neighborhood" refers to a specified topology on strategies
- Taking this to be the trivial topology defines global stability
- ESS and some other notions of stability are special cases
- A stable strategy is not necessarily an equilibrium strategy

Symmetrization of Asymmetric Games

- An asymmetric N-player game h can be <u>symmetrized</u>
- Players switch roles, with all possible permutations
- Their common strategy space is $X = X_1 \times X_2 \times \cdots \times X_N$
- A strategy $x = (x_1, x_2, \dots, x_N)$ specifies the strategy x_i the player will use when assuming the role of any player i in h
- His payoff in the symmetric game g is obtained by averaging
- For any N strategies $x^1 = (x_1^1, \dots, x_N^1), \dots, x^N = (x_1^N, \dots, x_N^N)$,

$$g(x^1, x^2, \dots, x^N) = \frac{1}{N!} \sum_{\rho \in \Pi} h_{\rho^{-1}(1)}\left(x_1^{\rho(1)}, x_2^{\rho(2)}, \dots, x_N^{\rho(N)}\right)$$

- Π is the set of all permutation of $(1,2, \dots, N)$
- For $\rho \in \Pi$, $\rho(i)$ is the player assigned to the role i
- Superscripts index players' strategies in the symmetric game

Symmetrization of Asymmetric Games

Lemma. In an asymmetric N-player game h, a strategy profile y is stable as a strategy in the symmetrized game g if and only if

$$\sum_{S \neq \emptyset} \frac{1}{\binom{N-1}{|S|-1}} \left(h_S(\, y \mid x_S \,) - h_S(\, x \mid y_S \,) \right) < 0$$

for all strategy profiles $x \neq y$ in some neighborhood of y.

- For a set of players S, h_S denotes the sum $\sum_{i \in S} h_i$
- The profile $y \mid x_S$ agrees with x inside S, with y outside it
- Weak stability, definite instability are characterized similarly

Potential Games

- A game h is a potential game if it has an exact potential P
- $P: X \to \mathbb{R}$ satisfies, for every player i and strategy profile y,
$$h_i(\, y \mid x_i \,) - h_i(y) = P(\, y \mid x_i \,) - P(y), \qquad x_i \in X_i$$
- That is, the change in i's payoff is equal to the change in P

Proposition. A strategy profile y in a potential game h is stable, weakly stable or definitely unstable in the symmetrized game g if and only if it is a strict local maximum, local maximum or strict local minimum point of P, respectively.

- "Local" refers to the topology of the set of strategy profiles
- A unique global maximum point of P is necessary stable in g
- It is of course also an equilibrium in h
- For all $r \leq 1$, the modified game h^r has the exact potential
$$P^r = (1 - r)P + rf$$

Comparative Statics Result

Theorem 1. Consider a game h, a social payoff function f, and an assignment of a strategy profile y^r to each $r_0 \leq r \leq r_1 \leq 1$ such that

- the players' payoff functions and f are Borel measurable,
- the function $r \mapsto y^r$ is continuous and finitely-many-to-one in $[r_0, r_1]$ and $\pi: r \mapsto f(y^r)$ is absolutely continuous.

If, for all $r_0 < r < r_1$, y^r is stable, weakly stable or definitely unstable as a strategy in the game obatined by symmetrizing h^r, then π is strictly increasing, nondecreasing or strictly decreasing, respectively.

- May be applied to equilibria y^r in the modified games h^r
- The function π specifies the corresponding social payoff

Splittable Linear Congestion Games

- N players share a finite set E of resources
- Player i has <u>weight</u> $w_i > 0$, and a set $\check{X}_i$ of "pure" strategies, which are vectors of the form $\sigma = (\sigma_e)_{e \in E}$ with $\sigma_e \in \{0, w_i\}$
- His set of ("mixed") strategies X_i is the convex hull of $\check{X}_i$
- Each $x_i = (x_{ie})_{e \in E} \in X_i$ describes i's use of the resources
- For $x = (x_1, x_2, \ldots, x_N) \in X$, the <u>load</u> on e is $l_e = \sum_{i=1}^{N} x_{ie}$
- The <u>cost</u> of e is given by $c_e(l_e) = a_e l_e + b_e$, with $a_e > 0$
- Player i's payoff is the negative of his total cost:

$$h_i(x) = -\sum_{e \in E} x_{ie}\, c_e(l_e)$$

- The social payoff is the negative of the aggregate cost:

$$f(x) = -\sum_{e \in E} l_e\, c_e(l_e)$$

Splittable Linear Congestion Games

- With altruism coefficient $r \le 1$, the modified payoff is

$$h_i^r(x) = -\sum_{e \in E} \left(x_{ie} + r \sum\nolimits_{j \ne i} x_{je} \right) c_e(l_e)$$

- The modified game h^r has the exact potential

$$P^r(x) = -\sum_{e \in E} \left(a_e \frac{(1+r) l_e^2 + (1-r) \sum_{i=1}^{N} x_{ie}^2}{2} + b_e l_e \right)$$

- For $-1 \le r < 1$, the function P^r is strictly concave
- Its maximum point y^r is the unique equilibrium in h^r
- It is also stable as a strategy in the symmetrized game g^r

Splittable Linear Congestion Games

Theorem 2. For a linear congestion game with splittable flow h, and the negative of the aggregate cost as the social payoff f, for every $-1 \leq r < 1$ the unique equilibrium y^r in h^r satisfies

$$f(y^r) = \pi(r),$$

where π is a continuous and piecewise continuously differentiable function on $[-1,1]$ with $\pi(1) = \max_x f(x)$.

Moreover, there is a partition of $[-1,1]$ into finitely many intervals within which π is either constant or strictly increasing.

- Thus, such games always have "normal" comparative statics

Splittable Linear Congestion Games

Proof of the theorem (an outline):

- Every equilibrium in h^1 maximizes P^1 $(= f)$
- Consider the projection on the first two coordinates of
$\{\, (r, \alpha, y) \mid -1 \leq r \leq 1,\ \alpha = f(y),\ y$ is an equilibrium in $h^r \,\}$
- It is the graph of a continuous function, $\pi: [-1,1] \to \mathbb{R}$
- The set is semialgebraic
- By the Tarski–Seidenberg theorem, so is π
- There are points $-1 = r_0 < r_1 < \cdots < r_K = 1$ such that π is analytic and monotone in (r_{i-1}, r_i), $i = 1,2,\ldots,K$
- It is either strictly monotone or constant there
- By Theorem 1, in the first case $\pi|_{(r_{i-1}, r_i)}$ is strictly increasing

Nonatomic Congestion Games

- A continuum of identical players: the unit interval $[0,1]$
- A set $\check{X}$ of "pure" strategies: binary vectors $\sigma = (\sigma_e)_{e \in E}$
- A strategy profile $i \mapsto \sigma(i)$ defines a population strategy:

$$y = \int_0^1 \sigma(t)\, dt$$

- The vector $y = (y_e)_{e \in E}$ lies in the convex hull X of $\check{X}$
- A strictly increasing, continuously differentiable cost function $c_e\colon [0,\infty) \to \mathbb{R}$ determines the cost of resource e as $c_e(y_e)$
- Defines a population game g and a social (mean) payoff ϕ:

$$g(x,y) = -\sum_{e \in E} x_e\, c_e(y_e) \qquad \phi(y) = -\sum_{e \in E} y_e\, c_e(y_e)$$

- Meaningful for all x and y in X, and even in its cone $\hat{X}$

Population Games

- Represent very many players who are "playing the field"
- A convex strategy set X in a linear topological space
- The payoff $g(x, y)$ depends on a player's own strategy x and the population strategy y, and is continuous in the latter
- A social payoff is any continuous function $\phi: \hat{X} \to \mathbb{R}$ such that the differential $d\phi: \hat{X}^2 \to \mathbb{R}$ exists and is continuous in y

$$d\phi(x, y) = \left.\frac{d}{dt}\right|_{t=0^+} \phi(tx + y)$$

- With altruism coefficient $r \leq 1$, the modified game is

$$g^r(x, y) = (1 - r)g(x, y) + r\, d\phi(x, y)$$

- For a nonatomic congestion game and for the mean payoff

$$g^r(x, y) = -\sum_{e \in E} (c_e(y_e) + r\, y_e\, c_e'(y_e))$$

Population Games

- An equilibrium strategy y in g^r satisfies
$$g^r(y,y) \geq g^r(x,y), \qquad x \in X$$
- A population game g may have an exact potential Φ
- This is so for Φ satsifying a continuity condition and
$$d\Phi(x,y) = g(x,y), \qquad x,y \in X$$
- The modified game g^r then has the exact potential
$$\Phi^r = (1-r)\Phi + r\,\phi$$
- An exact potential for a nonatomic congestion game is
$$\Phi(x) = -\sum_{e\in E} \int_0^{x_e} c_e(t)\,dt$$
- As cost functions are strictly increasing, Φ is strictly concave
- If the marginal social costs $MC_e(t) = \; d/\,dt\,\big(tc_e(t)\big)$ are strictly increasing, the social payff ϕ is also strictly concave

Nonatomic Congestion Games

Proposition. For g describing a nonatomic congestion game where the marginal social costs are strictly increasing in $[0,1]$, and for the mean payoff as the social payoff ϕ,

- for $0 \leq r \leq 1$, g^r has a unique equilibrium strategy y^r, and
- the strategy y^1 maximizes the mean payoff.

Theorem 3. For g and ϕ as above, if in addition each of the cost functions is a polynomial, then

- the mapping $r \mapsto y^r$ is continuous,
- the function $\pi: [0,1] \to \mathbb{R}$ defined by $\pi(r) = \phi(y^r)$ is piecewise continuously differentiable, and
- there is a partition of $[0,1]$ into finitely many intervals within which π is either constant or strictly increasing.