Workshop on Stochastic Games
IMS, National University of Singapore

# Symmetric Equilibria in Stochastic Timing Games

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# Motivation

Most timing games fall into two classes:

- First-mover advantage $\Rightarrow$ preemption
- Second-mover advantage $\Rightarrow$ war of attrition

➤ Equilibria for general games?

➤ Payoffs in different equilibria?

➤ When does preemption (have to) occur?

Examples

- Preemption: market entry
- War of attrition: market exit
- In general: strategic investment, real options

# Agenda

Stochastic stopping games in continuous time with non-zero-sum payoffs

- Subgame-perfect equilibria for general symmetric games
- "Efficiency" and degree of preemption in equilibrium

# Literature

- Fudenberg & Tirole (1985)
- Hendricks, Weiss & Wilson (1987, 88, 92)
- Laraki, Solan & Vieille (2005), Laraki & Solan (2013)
- Hamadène & Zhang (2010), Hamadène & Hassani (2011)
- zero-sum Dynkin games, applications ...

# Introduction: The setting

We consider a **stopping game** for the players $i \in \{1, 2\}$ in terms of

- a filtered probability space $\big(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, P\big)$ and
- the processes $(L_t)_{t\geq 0}$, $(F_t)_{t\geq 0}$ and $(M_t)_{t\geq 0}$,

where it is assumed that

- $(\mathscr{F}_t)_{t\geq 0}$ satisfies the "usual conditions",
- $L$, $F$, and $M$ are adapted, right-continuous (a.s.) and of class (D).

➤ Payoffs from *pure* strategies = stopping times:

$$V_i(\tau_i, \tau_j) = E\Big[\mathbf{1}_{\tau_i<\tau_j} L_{\tau_i} + \mathbf{1}_{\tau_i>\tau_j} F_{\tau_j} + \mathbf{1}_{\tau_i=\tau_j} M_{\tau_i}\Big].$$

# Mixed strategies Riedel & Steg (2014)

A **subgame** of the stopping game is *any* $(\mathscr{F}_t)$*-stopping time* $\vartheta$ with the connotation that no player has stopped before.

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An **extended mixed strategy** for player $i \in \{1, 2\}$ in the stopping game is a family of progressively measurable processes

$$\big(G_i, \alpha_i\big) = \big(G_i^\vartheta, \alpha_i^\vartheta\big)_{\vartheta \in \mathscr{T}}$$

where for any subgame/stopping time $\vartheta \in \mathscr{T}$:

- $G_i^\vartheta$ is non-decreasing, right-continuous, and $[0,1]$-valued with $G_i^\vartheta(\vartheta-) = 0$,
- $\alpha_i^\vartheta$ is $[0,1]$-valued and right-continuous where $\alpha_i^\vartheta < 1$,
- $\alpha_i^\vartheta(t) > 0 \Rightarrow G_i^\vartheta(t) = 1 \ \forall t \geq \vartheta$.

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[cf. Fudenberg & Tirole, 1985; Touzi & Vieille, 2002]

# Payoffs Riedel & Steg (2014)

The **continuation payoff** for player $i$ in the subgame at $\vartheta$ is

$$
\begin{aligned}
V_i^\vartheta\big(G_i^\vartheta, \alpha_i^\vartheta, G_j^\vartheta, \alpha_j^\vartheta\big) = E\bigg[ &\int_\vartheta^{\hat\tau^\vartheta} \big(1 - G_j^\vartheta(t)\big) L_t \, dG_i^\vartheta(t) \\
&+ \int_\vartheta^{\hat\tau^\vartheta} \big(1 - G_i^\vartheta(t)\big) F_t \, dG_j^\vartheta(t) \\
&+ \sum_{t<\hat\tau^\vartheta} \triangle G_i^\vartheta(t) \triangle G_j^\vartheta(t) M_t \\
&+ \lambda_{L,i}^\vartheta L_{\hat\tau^\vartheta} + \lambda_{L,j}^\vartheta F_{\hat\tau^\vartheta} + \lambda_M^\vartheta M_{\hat\tau^\vartheta} \;\bigg|\; \mathscr{F}_\vartheta \bigg]
\end{aligned}
$$

with "definite stopping" time

$$\hat\tau^\vartheta := \inf\{t \geq \vartheta \mid \alpha_1(t) + \alpha_2(t) > 0\}$$

and **outcome probabilities** $\lambda_{L,i}^\vartheta$, $\lambda_{F,i}^\vartheta$ and $\lambda_M^\vartheta$ determined by $\alpha_1, \alpha_2$.

# Equilibrium concept Riedel & Steg (2014)

A mixed strategy is **time consistent** if for all stopping times $\vartheta \leq \tau$

- $t \geq \tau \Rightarrow G_i^\vartheta(t) = G_i^\vartheta(\tau-) + \big(1 - G_i^\vartheta(\tau-)\big) G_i^\tau(t)$ (Bayes),
- $\alpha_i^\vartheta(t) = \alpha_i^\tau(t)$ (conditional stopping probabilities).

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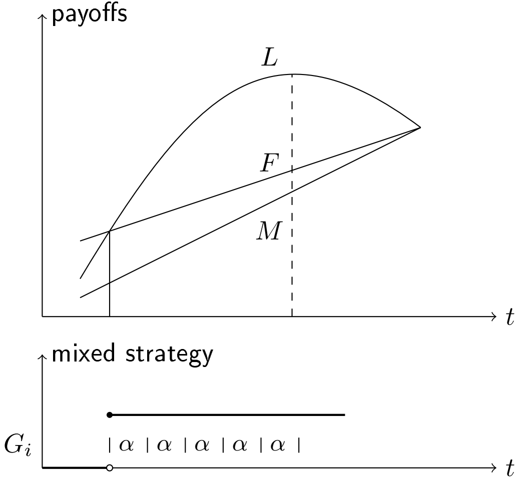
A **subgame-perfect equilibrium** for the stopping game is a pair of time consistent extended mixed strategies such that for all stopping times $\vartheta$, $i, j \in \{1, 2\}$, and extended mixed strategies $\big(G_a^\vartheta, \alpha_a^\vartheta\big)$

$$V_i^\vartheta(G_i^\vartheta, \alpha_i^\vartheta, G_j^\vartheta, \alpha_j^\vartheta) \ge V_i^\vartheta(G_a^\vartheta, \alpha_a^\vartheta, G_j^\vartheta, \alpha_j^\vartheta) \quad \text{a.s.},$$

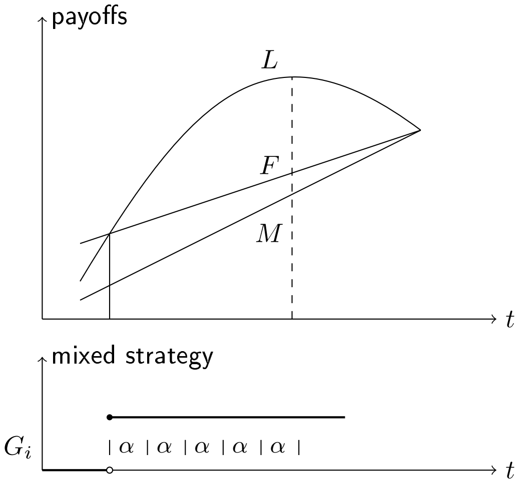
i.e., such that every pair $\big(G_1^\vartheta, \alpha_1^\vartheta\big)$, $\big(G_2^\vartheta, \alpha_2^\vartheta\big)$ is an *equilibrium* in the subgame at $\vartheta$, respectively.

# Preemption for $L > F$ [cf. Fudenberg & Tirole, 1985]



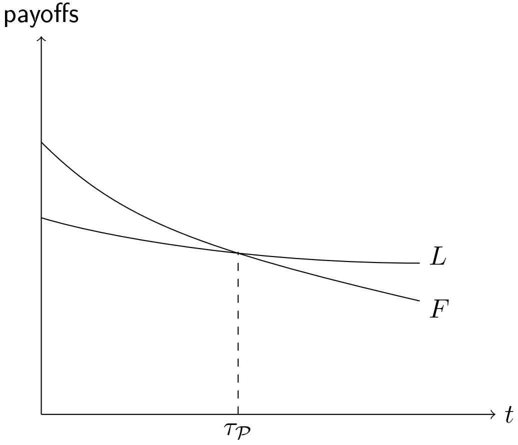

extended strategies $\Rightarrow$ partial coordination $\Rightarrow$ immediate stopping,
symmetric payoffs $F_\vartheta$

# Preemption for $L > F$ [cf. Fudenberg & Tirole, 1985]



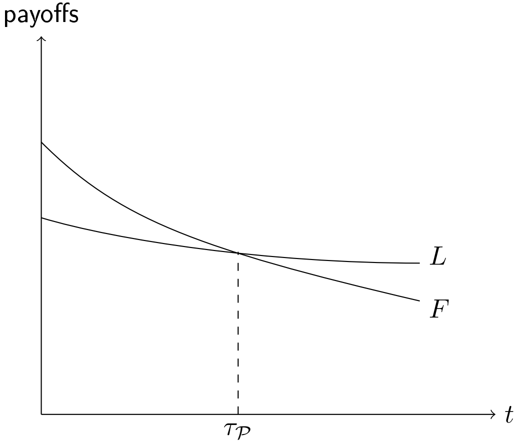

Riedel & Steg (2014): extension to stochastic games (also asymmetric)

# War of attrition for $F \geq L$




- $\tau_{\mathcal{P}}$ "preemption point"
- $\sup_{\tau \leq \tau_{\mathcal{P}}} L_\tau \Rightarrow \tau^* = 0$

# War of attrition for $F \geq L$




- $\tau_{\mathcal{P}}$ "preemption point"
- $\sup_{\tau \leq \tau_{\mathcal{P}}} L_\tau \Rightarrow \tau^* = 0$
- **waiting cost** $dL < 0$

# War of attrition for $F \geq L$

Define preemption point $\tau_{\mathcal{P}}^{\vartheta} := \inf\{t \geq \vartheta \mid L_t > F_t\}$

Constrained leader payoff $\tilde{L}^{\vartheta} := \begin{cases} L & \text{if } t < \tau_{\mathcal{P}}^{\vartheta} \\ F_{\tau_{\mathcal{P}}^{\vartheta}} & \text{if } t \geq \tau_{\mathcal{P}}^{\vartheta} \end{cases}$

$\Rightarrow$ quit if optimal to stop $\tilde{L}^{\vartheta}$

$\Rightarrow$ value given by Snell envelope $U_{\tilde{L}}^{\vartheta} = M_{\tilde{L}}^{\vartheta} - D_{\tilde{L}}^{\vartheta}$

$\Rightarrow$ **expected loss of waiting** given by monotone compensator $D_{\tilde{L}}^{\vartheta}$

# General symmetric equilibria

## Theorem

*Assume that* $\min(L, F)$ *is upper-semicontinuous in expectation. The mixed strategies* $G_1$ *and* $G_2$ *given by*

$$G_1^\vartheta(t) := 1 - \mathbf{1}_{t<\tau_\mathcal{P}^\vartheta} \exp\Big\{-\int_\vartheta^t \frac{dD_{\tilde{L}}^\vartheta(s)}{F_s - L_s}\Big\}$$

$$G_2^\vartheta(t) := 1 - \mathbf{1}_{t<\tau_\mathcal{P}^\vartheta} \exp\Big\{-\int_\vartheta^t \mathbf{1}_{F>L} \frac{dD_{\tilde{L}}^\vartheta(s)}{F_s - L_s}\Big\}$$

*are part of a subgame perfect equilibrium with symmetric payoffs.*

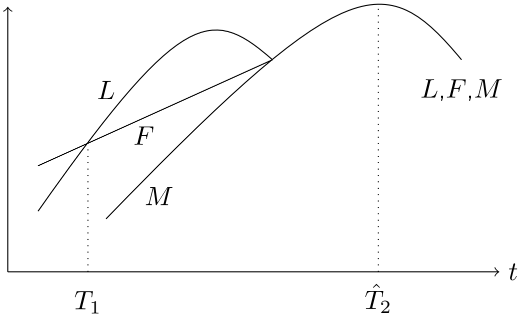
- $dD_{\tilde{L}}^\vartheta$ expected loss from late stopping $\to$ teased by payoff $F$
- time consistency over subgames $\vartheta$: $D_{\tilde{L}}^\vartheta$ depends on $\tau_\mathcal{P}^\vartheta$

# Avoiding preemption

So far: game ends wherever $L > F$

High future continuation values ⇒ abstain from preemption

⇒ higher equilibrium payoffs




**Collusion:** $M \geq L, F$

at global Maximum

Case B, Fudenberg and Tirole (1985)

# Payoff bounds

## Proposition

*Suppose $M \leq \min(L, F)$. Then, in any* ***payoff-symmetric*** *equilibrium and for any $\vartheta \in \mathscr{T}$,*

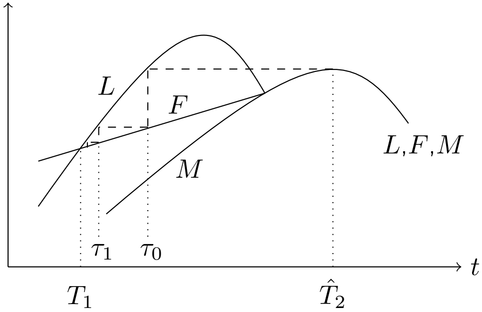
$$V_i^\vartheta\big(G_i^\vartheta, \alpha_i^\vartheta, G_j^\vartheta, \alpha_j^\vartheta\big) \leq U_{L\wedge F}(\vartheta) := \operatorname*{ess\,sup}_{\tau \geq \vartheta} E\big[L_\tau \wedge F_\tau \;\big|\; \mathscr{F}_\vartheta\big].$$

$\Rightarrow$ definite preemption at $\tau_0^\vartheta := \inf\{t \geq \vartheta \mid L > U_{L\wedge F}\}$

$\Rightarrow$ equilibrium payoffs bounded by $\operatorname{ess\,sup}_{\tau \in [\vartheta, \tau_0^\vartheta]} E\big[L_\tau \wedge F_\tau \;\big|\; \mathscr{F}_\vartheta\big]$

# Preemption points

**Preemption:** $L > F, M$ at global Maximum




Case A, Fudenberg and Tirole (1985)

# Efficient equilibria

## Theorem

*Assume $M \leq \min(L, F)$ and that the latter is upper-semicontinuous in expectation. Then there exists a maximal payoff-symmetric equilibrium with value*

$$V_1^\vartheta = V_2^\vartheta = \operatorname*{ess\,sup}_{\tau \in [\vartheta, \tilde{\tau}(\vartheta)]} E\big[L_\tau \wedge F_\tau \;\big|\; \mathscr{F}_\vartheta\big]$$

*for any $\vartheta \in \mathscr{T}$, where $\tilde{\tau}(\vartheta)$ is the latest sustainable preemption point after $\vartheta$ given by the limit*

$$\tilde{\tau}(\vartheta) := \lim_{n \to \infty} \tau_n(\vartheta)$$

*of iterating the previous scheme.*

(Problem: Make $\alpha_i^\vartheta$ measurable.)

# Conclusion and Outlook

- Construction of subgame perfect equilibria for symmetric games with arbitrary **local** first or second mover advantages
- Algorithm to determine latest sustainable preemption points
- Least preemption yields payoff-maximal equilibrium
- Application to strategic real option models with varying incentives:

  e.g. Steg & Thijssen (2015) $\rightarrow$ Markovian stopping rates

Thank you!

# Application Steg & Thijssen (2015)

**Example:**

- Two firms operating in the same market, not much duopoly profit
- Profit that monopolist could make given by process $X$
- Both have the option to switch to a new market at sunk cost $I$
- There, monopolist could earn profit given by process $Y$

$$L_t = E\left[\int_t^\infty e^{-rs} Y_s \, ds \,\middle|\, \mathscr{F}_t\right] - e^{-rt} I$$

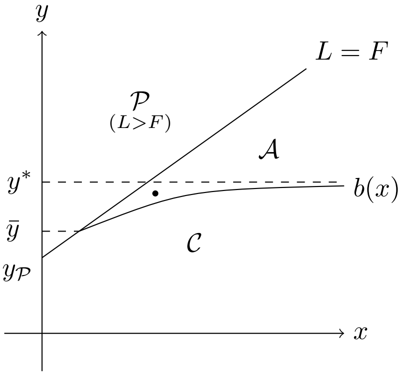
$$F_t = E\left[\int_t^\infty e^{-rs} X_s \, ds \,\middle|\, \mathscr{F}_t\right]$$

$$M_t = -\, e^{-rt} I$$

$X$ and $Y$ geometric Brownian motions

# Application Steg & Thijssen (2015)

## Equilibrium:




Attrition: $\frac{dG(t)}{1-G(t)} = e^{-rt}\frac{Y_t - rI}{F_t - L_t}\,dt$

Preemption: $\alpha(t) = \frac{L_t - F_t}{L_t - M_t} \to 0$ on boundary