Acyclic Gambling Games

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Zero-sum stochastic games where each player controls his own state variable.

Player 1 Gambling House: compact state space X, possible transitions given by a continuous multifunction $\Gamma : X \rightrightarrows \Delta(X)$ with non empty convex compact values:

If the state of Player 1 is at x, he can select his new state according to any probability in $\Gamma(x)$.

Similarly for Player 2: state space Y, and transitions given by $\Lambda : Y \rightrightarrows \Delta(Y)$. Players only interact through a continuous running payoff $u : X \times Y \longrightarrow I\!R$ (payoff $-u$ for P2). States are perfectly observed.

Given $\lambda \in (0,1]$, the value of the stochastic game with discount λ is a continuous function of the initial positions and is characterized by:

$$\forall (x,y) \in X \times Y,\ v_\lambda(x,y) = \max_{p \in \Gamma(x)} \min_{q \in \Lambda(y)} \left(\lambda u(x,y) + (1-\lambda) v_\lambda(p,q)\right),$$
$$= \min_{q \in \Lambda(y)} \max_{p \in \Gamma(x)} \left(\lambda u(x,y) + (1-\lambda) v_\lambda(p,q)\right).$$

Question : $\lim_{\lambda \to 0} v_\lambda$?

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$$\begin{aligned} \forall (x,y) \in X \times Y,\ v_\lambda(x,y) &= \max_{p \in \Gamma(x)} \min_{q \in \Lambda(y)} \left(\lambda u(x,y) + (1-\lambda) v_\lambda(p,q)\right), \\ &= \min_{q \in \Lambda(y)} \max_{p \in \Gamma(x)} \left(\lambda u(x,y) + (1-\lambda) v_\lambda(p,q)\right). \end{aligned}$$

Question : $\lim_{\lambda \to 0} v_\lambda$?

Without further assumptions, CV of (v_λ) may fail (even in the 0-player case, i.e. when Γ and Λ are single-valued). Endow $\Delta(X)$ with the distance:

$$d_{KR}(p,p') = \sup_{f\, 1-Lip} \left| \int_{x\in X} f(x)dp(x) - \int_{x \in X} f(x)dp'(x) \right|.$$

From now on, we assume: Non expansive transitions.

$$\forall x \in X, \forall x' \in X, \forall p \in \Gamma(x), \exists p' \in \Gamma(x'),\ s.t.\ d_{KR}(p,p') \leq d(x,x').$$

- Always satisfied if X is finite.
- Satisfied if X is a simplex and $\Gamma(x)$ is the set of probabilities over X with mean x (*splitting games*)
- Implies that the family $(v_\lambda)_{\lambda\in(0,1]}$ is equicontinuous, so to prove uniform CV it is enough to show uniqueness of a uniform limit point.
- Uniform CV of (v_λ) is equivalent to Uniform CV of (v_n) (B. Ziliotto 2015). $v_n(x,y) = \frac{1}{n}\max_{p\in\Gamma(x)} \min_{q\in\Lambda(y)} \left(u(x,y) + (n-1)v_{n-1}(p,q)\right)$
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Reachable sets

Extend $\Gamma : X \rightrightarrows \Delta(X)$ to $\tilde{\Gamma} : \Delta(X) \rightrightarrows \Delta(X)$ by

$$\mathrm{Graph}\tilde{\Gamma} = \overline{\mathrm{conv}}\,\mathrm{Graph}\Gamma.$$

Define $\tilde{\Gamma}^0(p) = \{p\}$ for every p in $\Delta(X)$, and for $n \geq 0$, $\tilde{\Gamma}^{n+1} = \tilde{\Gamma}^n \circ \tilde{\Gamma}$.

$\tilde{\Gamma}^n(x)$ *represents the set of probabilities over states that Player 1 can reach in n stages from the initial state x in X.*

Define the reachable set $\Gamma^\infty(x)$ of P1 at x as the closure of $\bigcup_{n\geq 0} \tilde{\Gamma}^n(x)$.

Be careful that in general, for p in $\Delta(X)$ and q in $\Delta(Y)$:
$v_\lambda(p,q) \neq \max_{p'\in\tilde{\Gamma}(p)} \min_{q'\in\tilde{\Lambda}(q)} (\lambda u(p,q) + (1-\lambda) v_\lambda(p',q'))$.

The 1-player case

Assume Y is a singleton.

Theorem (R. 2011, R. Venel 2013) : (v_λ) UCV to v such that $\forall x$ in X,

$$v(x) = \inf\{w(x), w : \Delta(X) \to [0,1] \text{ affine } C^0 \ s.t.$$

$$(1)\ \forall x' \in X, w(x') \geq \sup_{p \in F(x')} w(p)$$

$$(2)\ \forall r \in R, w(r) \geq u(r)\},$$

where $R = \{p \in \Delta(X), (p,p) \in Graph\ \tilde{\Gamma}\}$ (*invariant measures*).

Easy case: if the game is *leavable*, i.e. if $x \in \Gamma(x)$ for all x, then

$$v(x) = \min\{w(x), w \text{ excessive }, w \geq u\} = \sup_{p \in \Gamma^\infty(x)} u(p).$$

(Gambling Fundamental Theorem, Dubins Savage 1965)

This paper: 2 players

Say that the gambling game is:

- leavable if $\forall x \in X, \delta_x \in \Gamma(x)$ *and* $\forall y \in Y, \delta_y \in \Lambda(y)$.
- weakly acyclic if there exists potentials $\varphi : X \to I\!R$ l.s.c., and $\psi : Y \to I\!R$ u.s.c. such that:

$$\forall x \in X, \mathrm{Argmax}_{p \in \Gamma(x)} \varphi(p) = \{\delta_x\} \textit{ and } \forall y \in Y, \mathrm{Argmin}_{q \in \Lambda(y)} \psi(q) = \{\delta_y\}.$$

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strongly acyclic $\Longrightarrow$ weakly acyclic $\Longrightarrow$ leavable

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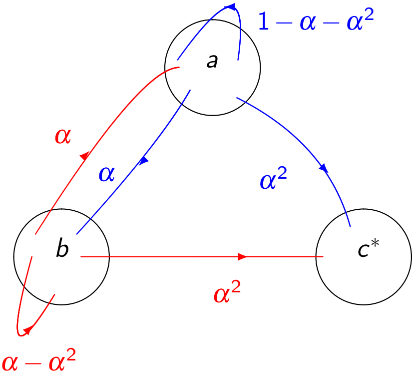
A weakly acyclic gambling house

$X = \{a, b, c\}$. c is absorbing, α and α can take any value in $[0, 1/2]$.



Weak potential: $\varphi(a) = \varphi(b) = 1, \varphi(c) = 0$.
Not strongly acyclic since $b \in \Gamma^\infty(a)$ and $a \in \Gamma^\infty(b)$.

Excessive, Depressive, Balanced

Given $v : X \times Y \longrightarrow I\!R$, say that:

1) v is excessive if: $\forall (x,y)$, $v(x,y) = \max_{p \in \Gamma(x)} v(p,y)$.

2) v is depressive if: $\forall (x,y)$, $v(x,y) = \min_{q \in \Lambda(y)} v(x,q)$.

3) v is balanced if $\forall (x,y)$,
$v(x,y) = \max_{p \in \Gamma(x)} \min_{q \in \Lambda(y)} v(p,q) = \min_{q \in \Lambda(y)} \max_{p \in \Gamma(x)} v(p,q)$.

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Theorem : Consider a compact non expansive gambling game.

If the game is strongly acyclic, then (v_λ) uniformly converges to the unique continuous function $v : X \times Y \longrightarrow I\!R$ satisfying:

1) v is excessive, i.e. $\forall (x,y) \in X \times Y,\ v(x,y) = \max_{p \in \Gamma(x)} v(p,y)$
2) v is depressive, i.e $\forall (x,y) \in X \times Y,\ v(x,y) = \min_{q \in \Lambda(y)} v(x,q)$.
3) $\forall (x,y) \in X \times Y, \exists p \in \Gamma^\infty(x), v(x,y) = v(p,y) \leq u(p,y)$,
4) $\forall (x,y) \in X \times Y, \exists q \in \Lambda^\infty(y), v(x,y) = v(x,q) \geq u(x,q)$.

Interpretation:
1) and 2) *It is always safe not to move.*
3) and 4) *Each player can reach the zone when the current payoff is at least as good than the limit value, without degrading the limit value.*

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The positive result relies on the 4 following propositions:

Prop 1: The family $(v_\lambda)_{\lambda\in(0,1]}$ is equicontinuous.

Prop 2: Assume the game is leavable, and let v be a limit point of $(v_\lambda)_{\lambda\in(0,1]}$ for the uniform convergence. Then for each (x,y):

$$\exists p \in \Gamma^\infty(x), v(x,y) \leq v(p,y) \leq u(p,y),$$
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Prop 3: Assume the game is weakly acyclic.
If v in $C(X\times Y)$ is balanced, then v is excessive and depressive.

Prop 4: Assume the game is strongly acyclic. There exists at most one excessive depressive function v in $C(X\times Y)$ satisfying for each (x,y):

$$\exists p \in \Gamma^\infty(x), v(x,y) = v(p,y) \leq u(p,y),$$
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An example with a countable state space

$X = \{1, 2, ..., n, ...\} \cup \{+\infty\}$, compact with $d(n, m) = |\frac{1}{n} - \frac{1}{m}|$.

P1 can stay or move +1: $\Gamma(n) = \{\alpha\delta_n + (1-\alpha)\delta_{n+1}, \alpha \in [0,1]\}$, state $+\infty$ is absorbing.
The gambling house of P2 is a copy. Strong acyclicity.
Running payoff $u : X \times Y \longrightarrow I\!R$ continuous.

Consider $v : X \times Y \longrightarrow I\!R$.
v excessive means that $v(n, m)$ is weakly decreasing in n,
v depressive means that $v(n, m)$ is weakly increasing in m.
v balanced: for each n and m, the value of the matrix game ("local game" at (n, m)): $\begin{pmatrix} v(n+1, m) & v(n+1, m+1) \\ v(n, m) & v(n, m+1) \end{pmatrix}$ is $v(n, m)$.

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Suppose that v is balanced, but not excessive.
There exists (n,m) such that $v(n+1,m) > v(n,m)$.
Since $v(n,m)$ is the value of $\begin{pmatrix} v(n+1,m) & v(n+1,m+1) \\ v(n,m) & v(n,m+1) \end{pmatrix}$, then $v(n,m) \geq v(n+1,m+1)$.
Since $v(n+1,m)$ is the value of $\begin{pmatrix} v(n+2,m) & v(n+2,m+1) \\ v(n+1,m) & v(n+1,m+1) \end{pmatrix}$, then $v(n+2,m+1) \geq v(n+1,m)$.
We obtain $v(n+2,m+1) - v(n+1,m+1) \geq v(n+1,m) - v(n,m)$.

Hence v can not be continuous at infinity.

Prop 3: If the game is weakly acyclic, a continuous balanced function is also excessive and depressive.

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Uniqueness

Prop 4: Assume the game is strongly acyclic. There exists at most one excessive depressive function v in $C(X \times Y)$ satisfying for each (x, y):

$$\exists p \in \Gamma^\infty(x), v(x,y) = v(p,y) \leq u(p,y),$$
$$\exists q \in \Lambda^\infty(y), v(x,y) = v(x,q) \geq u(x,q).$$

Proof: Suppose v and w satisfy this. Consider:

$$Z = \mathrm{Argmax}_{X \times Y}\, v - w, \text{ and } (x_0, y_0) \in \mathrm{Argmin}_{(x,y)\in Z}\varphi(x) - \psi(y).$$

By assumption, one can find p in $\Gamma^\infty(x_0)$ s.t. $v(x_0, y_0) = v(p, y_0) \leq u(p, y_0)$. w being excessive, one can show that $w(p, y_0) \leq w(x_0, y_0)$. So $\mathrm{Supp}(p) \times \{y_0\} \subset Z$.

By definition of (x_0, y_0), this implies: $\varphi(p) - \psi(y_0) \geq \varphi(x_0) - \psi(y_0)$. By strong acyclicity, $p = x_0$, and we get $v(x_0, y_0) \leq u(x_0, y_0)$.

Similarly, one can show $w(x_0, y_0) \geq u(x_0, y_0)$.
Hence $\mathrm{Max}_{X \times Y}\, v - w = v(x_0, y_0) - w(x_0, y_0) \leq 0$, and $v \leq w$.
By symmetry, $v = w$.

Prop 2: Assume the game is leavable, and let v be a limit point of $(v_\lambda)_{\lambda\in(0,1]}$ for the uniform convergence. Then for each (x,y):

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Proof: Let $(\lambda_n)_n$ such that $\|v_{\lambda_n} - v\| \to_{n\to\infty} 0$. We prove the first condition. Fix (x,y), if $v(x,y) \leq u(x,y)$ it is OK with $p = x$.

Assume $v_{\lambda_n}(x,y) > u(x,y) + \lambda_n$ for n large.

For each n, define inductively $(p^n_t)_{t=0,\dots,T_n}$ in $\Delta(X)$ by:

- $p^n_0 = x$,
- As long as $v_{\lambda_n}(p^n_t, y) > u(p^n_t, y) + \lambda_n$, let p^n_{t+1} in $\tilde{\Gamma}(p^n_t)$ be s.t.

$$\lambda_n u(p^n_t, y) + (1-\lambda_n) v_{\lambda_n}(p^n_{t+1}, y) \geq v_{\lambda_n}(p^n_t, y).$$

  Then:

$$v_{\lambda_n}(p^n_{t+1}, y) \geq v_{\lambda_n}(p^n_t, y) + \frac{\lambda_n^2}{1-\lambda_n} > v_{\lambda_n}(p^n_t, y).$$

- Consequently, $\exists T_n$ such that $v_{\lambda_n}(p^n_{T_n}, y) \leq u(p^n_{T_n}, y) + \lambda_n$.
- Define $p^n = p^n_{T_n}$ and consider a limit point $p^* \in \Gamma^\infty(x)$. Then:

$$v(x,y) \leq v(p^*, y) \leq u(p^*, y).$$

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- Define $p^n = p^n_{T_n}$ and consider a limit point $p^* \in \Gamma^\infty(x)$. Then:

$$v(x,y) \leq v(p^*, y) \leq u(p^*, y).$$

Prop 2: Assume the game is leavable, and let v be a limit point of $(v_\lambda)_{\lambda\in(0,1]}$ for the uniform convergence. Then for each (x,y):

$$\exists p \in \Gamma^\infty(x), v(x,y) \le v(p,y) \le u(p,y),$$
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Proof: Let $(\lambda_n)_n$ such that $\|v_{\lambda_n} - v\| \to_{n\to\infty} 0$. We prove the first condition. Fix (x,y), if $v(x,y) \le u(x,y)$ it is OK with $p = x$.
Assume $v_{\lambda_n}(x,y) > u(x,y) + \lambda_n$ for n large.
For each n, define inductively $(p^n_t)_{t=0,\ldots,T_n}$ in $\Delta(X)$ by:

- $p^n_0 = x$,
- As long as $v_{\lambda_n}(p^n_t,y) > u(p^n_t,y) + \lambda_n$, let p^n_{t+1} in $\tilde{\Gamma}(p^n_t)$ be s.t.

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Back to the example with countable state spaces

Consider a continuous running payoff u.
(v_λ) uniformly converges to the unique continuous function v such that:
1) v is weakly decreasing in n, weakly increasing in m.

It is always safe not to move.

2) for each (n, m), there exists $n' \geq n$ and $m' \geq m$ s.t.

$$v(n,m) = v(n',m) \leq u(n',m), \text{ and } v(n,m) = v(n,m') \geq u(n,m').$$

Each player can reach the zone when the current payoff is at least as good than the limit value, without degrading the limit value.

Example: $u(n,m) = |\frac{1}{n} - \frac{1}{m}|$. Player 1 wants to be far from Player 2.
Then $v(n,m) = u(n,m)$ if $n < m$, and $v(n,m) = 0$ if $n \geq m$.

Link with the Mertens-Zamir system

X is a simplex $\Delta(K)$, and for each x, $\Gamma(x)$ is the set of splittings of x (proba on $\Delta(X)$ with mean x). Similarly, $Y = \Delta(L)$. The running payoff $u : X \times Y \longrightarrow I\!R$ is Lipschitz continuous.

Theorem (Mertens-Zamir 1971, Oliu-Barton 2015): (v_λ) uniformly converges to the unique continuous function v such that:

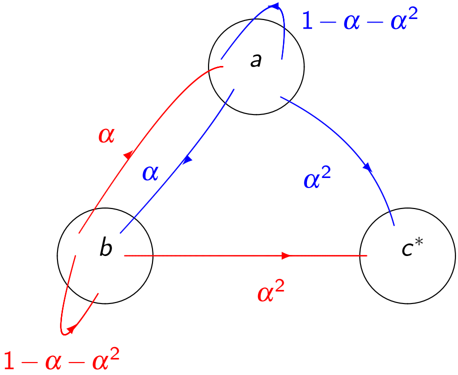
$$\left\{ \begin{array}{lcl} v & = & \mathrm{vex}_{\mathrm{II}} \max\{u, v\} \\ v & = & \mathrm{cav}_{\mathrm{I}} \min\{u, v\} \end{array} \right.$$

Here $\Gamma^\infty = \Gamma$. Our result gives:
(v_λ) uniformly converges to the unique continuous concave-convex function v satisfying: for all (x, y) in $\Delta(K) \times \Delta(L)$ there exists a splitting p of x and a splitting q of y s.t.

$$v(x,y) = v(p,y) \leq u(p,y) \text{ and } v(x,y) = v(x,q) \geq u(x,q).$$

A counter-example with weak acyclicity

$X = \{a, b, c\}$. c is absorbing, α and α can take any value in a fixed set $I \subset [0, 1/2]$ such that 0 is an accumulation point of I.



The Gambling house of P2 is a copy, $Y = \{a', b', c'\}$ with choice of α' and α' in $I' \subset [0, 1/2]$ such that 0 is an accumulation point of I'.

The payoff function u is written as follows :

	a'	b'	c'
a	0	1	1
b	1	0	1
c	1	1	0

Interpretation: player 1 and player 2 both move on a space with 3 points, player 2 want to be at the same location as player 1, and player 1 wants the opposite.

Proposition 4 fails here:

For any x in [0,1], the function

	a'	b'	c'
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is excessive depressive, and for all (x,y) there exists $p \in \Gamma^\infty(x)$ and $q \in \Lambda^\infty(y)$ such that: $v(x,y) = v(p,y) \leq u(p,y)$, and $v(x,y) = v(x,q) \geq u(x,q)$.

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Assume the choice set I' of player 2 is $[0, 1/4]$, $\min I = 0$, $\max I = 1/4$.
For λ small enough, the game with discount λ has a value in pure strategies and it is optimal :
- for player 2: at (a,a) and (b,b), to stay there, and at (a,b') and (b,a') to move with a choice of $\alpha'_\lambda = \sqrt{\lambda/(1-\lambda)} \sim \sqrt{\lambda}$,
- for player 1 at (a,b') and (a',b): to stay there.

Lemma: Let λ_n be a vanishing sequence of discount factors such that $\sqrt{\lambda_n} \in I$ for each n.

Then $(v_{\lambda_n})_n$ converges to

	a'	b'	c'
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Lemma: Let λ_n be a vanishing sequence of discount factors such that for each n, the interval $(\frac{1}{2}\sqrt{\lambda_n}, 2\sqrt{\lambda_n})$ does not intersect I. Then $\limsup_n v_{\lambda_n}(a,a') \leq 4/9$.

Corollary: if $I = \{\frac{1}{2^{2n}}, n \geq 1\} \cup \{0\}$, v_λ does not converge.

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2) Our counter-example is close to an adaptation of a counterexample of Ziliotto (2013) mentioned in Sorin Vigeral (2015). The difference is that here we have a *product* stochastic game.

3) A few open questions :
Uniform value ? (Oliu-Barton 2015 for the Mertens-Zamir setup).
Description of ε- optimal strategies ?
Other sets of conditions giving convergence (e.g., semi-algebraicity... only 1 player is strongly acyclic) ?
What if the gambling game is not leavable ?

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