Motivational Ratings*

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Abstract

Rating systems do not only provide information to users, they also motivate the rated agent. This paper solves for the optimal rating system within the standard career concerns framework. We show how the optimal rating system combines information of different kinds and different vintages. While the parameters of the optimal system depends on the environment—in particular, whether the market has access to previous ratings, or to alternative sources of information– it is always a mixture (two-state) rating system, with a state that reflects the information of the rating system, and the other the preferences of the rated agent.

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1 Introduction

Helping users make informed decisions is only one of the goals of ratings. Another is to motivate the rated firm or agent. These two goals are not necessarily aligned.

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Too much information depresses career concerns, and distorts the agent's choices.[1] The purpose of this paper is to examine this trade-off. In particular, we ask: how should different sources of information be combined? At what rate, if any, should past observations be discounted? And how do standard rating mechanisms compare?

We prove that the optimal rating system *always* confounds the different signals, yet *never* adds any irrelevant noise. To maximize incentives for effort, the rater combines the entire history of signals in a one-dimensional statistic, which is neither a simple function of the rater's current belief (about the agent's type), nor enables the market to back out this belief from the rating history.

Yet, the optimal rating system has a remarkably simple structure: it is a linear combination of two processes, namely, the rater's underlying belief, and an *incentive state* that reflects both the agent's preferences and the determinants of the signal processes. That is, the optimal rating process admits a simple decomposition as a two-dimensional Markov mixture model. But it is neither a function of the rater's actual (hidden) belief only, nor of her latest rating and signal.[2]

The agent's preferences determine the *impulse response* of the incentive state, via his impatience. More precisely, the agent's discount rate pins down the rate at which past observations get discounted in the overall rating. Instead, the characteristics of the signal processes determine the *weights* of the signal innovations in the incentive state; that is, these characteristics determine the relative importance of the signals in the overall ratings (including some that may affect the rating negatively).

How the discount rate affects the impulse response of the rating depends on the environment in which ratings take place: if past ratings are no longer available to the market (so that ratings are private), then the impulse response is precisely equal to the discount rate (relative to the rate of mean reversion); if they are available at all future times, then the impulse response is equal to the square root of the discount rate.

While the remarkably simple structure of the optimal rating policy owes to the

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[1] In the case of health care, Dranove, Kessler, McClellan and Satterthwaite (2003) find that, at least in the short run, report cards decreased patient and social welfare. In the case of education, Chetty, Friedman and Rockoff (2014a,b) argue that the benefits of value-added measures of performance outweigh the counterproductive behavior that it encourages—but gaming is widely documented as well (see, among many others, Jacob and Lefgren, 2005).

[2] This contrasts with several algorithms built on the principle that the new rating is a function of the old ratings and the most recent review(s) (Jøsang, Ismail, and Boyd 2007). On the other hand, there is also significant evidence that, in many cases, observed ratings (based on proprietary rules) cannot be explained by a simple (time-homogenous) Markov model. See, *e.g.*, Frydman and Schuermann (2008), who precisely argue that two-dimensional Markov models provide a better explanation for actual credit risk dynamics. Such two-state systems are already well-studied under the name of mixture (multinomial) models. See, *e.g.*, Adomavicius and Tuzhilin (2005).

Gaussian framework that we adopt, we believe that some of its features are compelling and robust: the optimal rating policy should balance the rater's information, as summarized by its belief, with some short-termism that is in proportion to the agent's impatience.[3] Signals that boost career concerns should see their weight amplified, while those that stifle career concerns should be muted. Our analysis clarifies what signal characteristics strengthen incentives.

Our analysis builds on the seminal model of Holmström (1999/82).[4] An agent exerts effort unbeknownst to the market, that pays him a competitive wage at all times. This wage is based on the market's expectation of the agent's productivity, which depends on instantaneous effort and his ability, a mean-reverting process. This expectation is based on the market's information. Rather than observing directly a noisy signal reflecting ability and effort, the market obtains its information via the rating set by some intermediary. The intermediary has potentially many sources of information about the agent, and freely chooses how to convert these signals into the rating. In brief, we view a rating system as an information channel that must be optimally designed. We focus on a simple objective which in our environment is equivalent to social surplus: to maximize the agent's incentive to exert effort, or equivalently, to solve for the range of effort levels that are implementable. (We also examine the trade-off between the level of effort and the precision of the market's information.)[5,6]

We allow for a broad range of mechanisms, imposing stationarity and (joint) normality only.[7] That is, a rating mechanism is equivalent to a time-invariant linear

[3]The ineffectiveness of irrelevant conditioning also resonates with standard principal-agent theory, see for instance Green and Stokey (1983).

[4]Modelling differences with Holmström (1999/82) include the continuous-time setting, the mean reversion in the type process, and the multidimensional signal structure. See Cisternas (2015) for a specification that is similar to ours in the first two respects.

[5]These two objectives feature prominently in economic analyses of ratings according to practitioners and theorists alike. For instance, as stated by Gonzalez *et al.* (2004), the rationale for ratings stems from their ability to gather and analyze information (information asymmetry), and affect the agents' actions (principal-agent). To quote Portes (2008), "Ratings agencies exist to deal with principal-agent problems and asymmetric information." To be sure, resolving information asymmetries and addressing moral hazard are not the only roles that ratings play. Credit ratings, for instance, play a role in the borrowing firms' default decision (Manso, 2013). Additionally, ratings provide information to the agent himself (*e.g.*, performance appraisal systems), see Hansen (2013). Also, whenever evaluating performance requires the input from the users, ratings must take into account their incentives to experiment and report (Kremer, Mansour and Perry, 2014; Che and Hörner, 2015).

[6]Throughout, we ignore the issues that rating agencies face in terms of possible conflict of interest, or their inability to commit, which motivates a broad literature.

[7]Our focus on such mechanisms abstracts away from some interesting questions nonetheless,

filter, mapping all the bits of information available to the intermediary into a (without loss) scalar rating. In general, such mechanisms are infinite-dimensional.

While patience unequivocally boosts incentives, it does not follow that the optimal rating system yields more precise information as the agent becomes more patient.[8] In fact, when ratings are public, precision is single-peaked in the discount rate, with information being perfectly precise when the agent is either arbitrarily patient or impatient. Roughly speaking, providing extra motivation to a very patient agent requires treating signals symmetrically, independently of their age. Under mean-reversion in the type, this means that such a treatment amounts to adding noise, as this information is largely irrelevant to the market. Hence, the rater gives up on the incentive state in that case. Similarly, if the agent is very impatient, providing extra motivation requires that the incentive state rely almost exclusively on the most recent signals, which is counterproductive, as it amounts once again to little more than noise. Surprisingly perhaps, this comparative statics is reversed with confidential ratings, in which case precision is U-shaped in the discount rate.

Perhaps it isn't too surprising that an optimally designed public rating system leads to lower incentives, but higher precision in the market belief than a confidential rating system. But this isn't as obvious as it sounds, as *given* a particular precision level (imposed as an exogenous constraint on the rating system), effort is strictly higher under confidential ratings, as we show. After all, a public rating system is a special case of a private one. Hence, it isn't *a priori* to be expected that incentives "peak" at a lower precision level under confidential ratings.

In Section 5.1, we extend our results to the case in which ratings are not exclusive. That is to say, the market has access to independent public information. We show how the optimal rating policy reflects the content of this free information. In Section 5.2, we discuss how our results extend to the case of multiple actions.[9] We show that it might be optimal for the optimal rating system to encourage effort production in dimensions that are unproductive, if this is the only way to encourage productive effort as well. Finally, in Section 6, we apply our techniques to compare existing methods, rather than derive the optimal ones.

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such as the granularity of the rating (the ratings scale), or their periodicity (*e.g.*, yearly vs. quarterly ratings), as well as the way ratings should be adjusted to account for the rated firm's age.

[8]Without rating system, the agent's impatience does not affect the precision of the market belief.

[9]If there are multiple dimensions to product quality, information disclosure on one dimension may encourage firms to cut back on their investments in others, leading to reduction in welfare (Bar-Isaac, Caruana, and Cuñat, 2008).

Related Literature. Foremost, our paper builds on Holmström (1999/1982). (See also Dewatripont, Jewitt and Tirole, 1999). His model elegantly illustrates why neither perfect monitoring nor lack of oversight cultivate incentives. His analysis prompts the question raised and answered in our model: what kind of feedback stimulates effort? Our interest in multifaceted information is reminiscent of Holmstrom and Milgrom (1991) who consider multidimensional effort and output to examine optimal compensation. Their model has neither incomplete information nor career concerns. Closer to us are the following themes.

- *Reputation.* The eventual disappearance of reputation in standard discounted models (as in Holmström, 1999/1982) motivates study of reputation effects when players' memory is limited. There are many ways to model such limitations. One is to simply assume that the market can only observe the last K periods (in discrete time), as is done in Liu and Skrzypacz (2014). This allows reputation to be rebuilt. Even closer to us is Ekmekci (2011) who interprets the map from signals to reports as ratings, as we do. His model features an *informed* agent. Ekmekci shows that, absent reputation effects, information censoring cannot improve attainable payoffs. However, if there is an initial probability that the seller is a commitment type that plays a particular strategy every period, then there exists a finite rating system and an equilibrium of the resulting game such that, the expected present discounted payoff of the seller is almost his Stackelberg payoff after every history. As in our paper, Pei (2015) introduces an intermediary in a model with moral hazard and adverse selection. The motivation is very close to ours, but the modeling and the assumptions markedly differ. In particular, the agent knows his own type, and the intermediary can only choose between disclosing and withholding the signal, while having no ability to distort its content. Most importantly, perhaps, the intermediary isn't disinterested, but rather a strategic player with her own payoff that she maximizes in the Markov perfect equilibrium of the game.

- *Design of reputation systems.* The literature on information systems has explored the design of rating and recommendation mechanisms. See, in particular, Dellarocas (2006) for a study of the impact of the frequency of reputation profile updates on cooperation and efficiency in settings with pure moral hazard and noisy ratings. This literature abstracts away from career concerns, the key driver of our analysis.

- *Design of information channels.* There is a vast literature in information theory on how to design information channels. Restrictions on the channel's quality are derived from physical (limited bandwith, for instance) rather than strategic considerations. It is impossible to do justice to this literature. See, among many others, Chu (1972), Ho and Chu (1972) and, closer to economics, Radner (1961). Design under incentive constraints has been considered recently by Ely (2015) and Renault, Solan and Vieille (2015). However, these are models in which information disclosure is distorted because of the incentives of the users of information; the underlying information process is exogenous.

We complement this cursory review with further references as we proceed. When we do so simply reflects how our exposition is arranged, and neither the relevance nor the importance of the papers quoted.

2 The Baseline Model

Here, we briefly review a version of Holmström's career concerns model that serves as a building block to the analysis. The set-up differs somewhat from Holmström's model (it is cast in continuous time and the type process is reverting) as well as from Cisternas (2015) (information about the agent need not be one-dimensional).[10]

The relationship involves an infinitely-lived agent (he) and a competitive market (it). At every moment in time $t \geq 0$, the agent exerts effort a_t unbeknownst to the market, at a flow cost $c(a_t)$, where c is twice differentiable, $c(0) = c'(0) = 0$, and $c'' > 0$. The agent is characterized by his ability, or *type*, θ_t, which alongside effort determines his flow output. Specifically, cumulative output X_t solves

$$\mathrm{d}X_t = (a_t + \theta_t)\,\mathrm{d}t + \sigma_1\,\mathrm{d}W_{1,t}, \tag{1}$$

where W_1 is an independent standard Brownian motion, and $\sigma_1 > 0$.[11] Without loss of generality, we normalize the output X_0 at time 0 to zero.[12]

[10] Mean-reversion allows us to define stationary equilibria for all (stationary) rating policies; consider for instance the policy in which no information about the agent gets ever disclosed, a policy that cannot be ruled out *a priori*. Without mean-reversion, the stationary conditional belief of the market is not well-defined, as its variance "blows up." Allowing for multiple signals about the agent is essential to understand their relative importance in the rating, and how the rating combines them optimally.

[11] Throughout, when we refer to an independent standard Brownian motion, we mean a standard Brownian motion independent of all the other random variables and random processes of the model.

[12] Plainly, both actions and types are one-dimensional variables. The former assumption is relaxed in Section 5.2, where we show that (under some assumptions on the cost structure) it can be embedded in the one-dimensional case. Because the agent's type matters to the market to the extent that it affects output, nothing would change either if the state were multi-dimensional, but only

We assume that θ_0 has a Gaussian distribution. It is drawn from $\mathcal{N}(0, \gamma^2/2)$. The law of motion of θ is mean-reverting, with increments

$$\mathrm{d}\theta_t = -\theta_t \,\mathrm{d}t + \gamma \,\mathrm{d}Z_t, \tag{2}$$

where Z is an independent standard Brownian motion and $\gamma > 0$. The unit rate of mean-reversion of the process is a mere normalization, as is its zero mean. The specification of variance for θ_0 ensures that the process θ is stationary.

As is standard, we interpret this stochastic differential equation in (2) in the weak sense, and solutions refer to weak solutions throughout.

Output (or equivalently its value, as its price is normalized to one) need not be the only information available to the market.[13] We model such sources of information as processes $\{S_{k,t}\}$, $k = 2, \ldots, K$, $K \geq 1$,[14] which are solutions to

$$\mathrm{d}S_{k,t} = (\alpha_k a_t + \beta_k \theta_t) \,\mathrm{d}t + \sigma_k \,\mathrm{d}W_{k,t}, \tag{3}$$

for some $\alpha_k \in \mathbb{R}$, $\beta_k \geq 0$, $\sigma_k > 0$ and where W_k are independent standard Brownian motions. We assume, without loss of generality, that these signals take value zero at the initial time.

The laws of motion described above are the *actual* law of motions, determined by the actual effort level the agent exerts over time. The market, however, does not observe the agent's effort, and therefore believes in a law of motion that depends on the conjecture it forms about the agent's action—and which may be different from the actual law of motions off equilibrium path. Specifically, if the market conjectures the agent exerts effort a_t^* at time t, the market believes that the output and signals follow the laws of motion

$$\mathrm{d}X_t = (a_t^* + \theta_t) \,\mathrm{d}t + \sigma_1 \,\mathrm{d}W_{1,t},$$

and

$$\mathrm{d}S_{k,t} = (\alpha_k a_t^* + \beta_k \theta_t) \,\mathrm{d}t + \sigma_k \,\mathrm{d}W_{k,t}.$$

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entered output linearly. We would then re-define the state to be the relevant projection.

[13] In the case of a company, besides earnings, there is a large variety of indicators of performance (profitability, income gearing, liquidity, market capitalization, etc.). In the case of sovereign credit ratings, Moody's and Standard & Poor's list numerous economic, social, and political factors that underlie their rating (Moody's 1991; Moody's 1995; Standard & Poor's 1994); similarly, workers are evaluated according to a variety of measures of performance, both objective and subjective (see Baker, Gibbons and Murphy, 1994).

[14] $K = 1$ is the special case in which there is no such additional information.

Throughout we use the star notation to refer to the market's conjectures; in particular, $\mathbf{E}^*$ refers to the expectation under the market's conjectures, while $\mathbf{E}$ is the expectation under the actual law of motion.

In this section, we assume that the output and all the signals are publicly observed. On the other hand, neither the agent nor the market observe ability directly. Hence, as long as the market's expectations about the agent's effort level are correct, learning about his type is symmetric.

Output can be viewed as a special kind of signal, S_1, with $\alpha_1 = \beta_1 = 1$, and we use this notation whenever convenient. However, output is also payoff-relevant, as it enters the market's payoff. Given the cumulative payment process to the agent π, the market retains

$$\int_0^\infty e^{-rt}(\mathrm{d}X_t - \mathrm{d}\pi_t),$$

whereas the agent gets

$$\int_0^\infty e^{-rt}(\mathrm{d}\pi_t - c(a_t)\,\mathrm{d}t),$$

where $r > 0$ is the common discount rate. Plainly, efficiency calls for setting a_t at the constant value solving $c'(a_t) = 1$.[15] The market is modeled as competitive, so that it pays the agent its expected output at all times "upfront," as formalized in Definition 2.1 below.

We denote by $\mathcal{F}_t$ the market information available at time t. It includes the information generated by the signal processes S_k, $k = 1, \ldots, K$, up to time t, and some initial information $\mathcal{F}_0$.[16] A (public) strategy for the agent specifies an effort level as a function of time and the public information. It is represented by a stochastic process A, such that for all t, A_t is measurable with respect to $\mathcal{F}_t$. We denote by $\mathcal{A}$ the collection of all such processes.

Our focus is on deterministic equilibria. That is, we restrict attention to equilibria in which the agent's effort is a deterministic function of time, independently of the (public and private) history. This is not without loss of generality (other equilibria exist), but this is what is implicitly done throughout the literature, and arguably allows to interpret incentives as driven by career concerns exclusively, as opposed to "stick-and-carrot" effects.[17] One benefit of such equilibria is that they are robust

[15] Only the agent's impatience matters for equilibrium analysis, and this is how we interpret the parameter r. However, equal discounting is necessary for transfers to be irrelevant for efficiency.

[16] Formally $\{\mathcal{F}_t\}$ is the filtration generated by the signals S_k for $k = 1, \ldots, K$ and the initial information $\mathcal{F}_0$.

[17] Indeed, as is well known, versions of folk theorems arise quite generally in repeated games when one player takes myopic best-replies, as the market does here. Incomplete information complicates

to the specification of information available to the agent, because no information is needed in order to carry out the equilibrium strategy.

Furthermore, we focus on stationary equilibria, in which effort is constant, and the market belief process about the agent's type is also stationary. Stationarity is ensured by providing initial information to the market with the required precision about the type. In this section, we model such initial information as a signal $S^I := \theta_1 + \epsilon$ where ϵ is an independent noise with mean zero and variance $\Sigma/(1 - 2\Sigma/\gamma^2)$, where Σ is the (exogenous) variance of the public information defined below.[18]

Definition 2.1 *An equilibrium is a profile (a^*, π) where a^* is the effort level of the agent at all times and π is the cumulative payment process, such that*

1. *(Zero-profit) For all τ,*

$$\pi_\tau = \int_0^\tau \mathbf{E}^*[a^* + \theta_t \mid \mathcal{F}_t] \, \mathrm{d}t.$$

2. *(Optimal effort)*

$$a^* \in \underset{A \in \mathcal{A}}{\operatorname{argmax}} \, \mathbf{E} \left[\int_0^\infty e^{-rt} (\mathrm{d}\pi_t - c(A_t) \, \mathrm{d}t) \right].$$

Throughout the paper, we assume that all the optimization programs have suprema that are bounded. (*Sufficient conditions to be added.*)

The zero-profit condition is equivalent to saying that the market pays the agent the expected output at all times. That is, if the agent is expected to put in effort a_t^* at time t, then $\mathrm{d}\pi_t = \mathbf{E}^*[\mathrm{d}X_t \mid \mathcal{F}_t] = (\mu_t + a_t^*) \, \mathrm{d}t$, where $\mu_t := \mathbf{E}^*[\theta_t \mid \mathcal{F}_t]$ is the market's best estimate of the agent's type. Hence, what the agent is concerned with is his expected discounted reputation.

Lemma 2.2 *Given a cumulative payment process that satisfies the zero-profit condition, the stationary effort level a^* maximizes the agent's payoff if and only if it maximizes*

$$\mathbf{E} \left[\int_0^\infty e^{-rt} (\mu_t - c(A_t)) \, \mathrm{d}t \right],$$

the problem. It does not obviate it. Holmström refers to such constructions as "explicit long-term contracts;" nonetheless, they are self-enforcing.

[18] Therefore, $\mathcal{F}_0$ is the information generated by S^I.

over $\mathcal{A}$, where $\mu_t = \mathbf{E}^*[\theta_t \mid \mathcal{F}_t]$ is derived using effort level a^* as the market's expectation.

Note that this lemma makes no reference to the output process any longer. Hence, output provides a foundation for why the agent cares about his discounted reputation—reputation is valued because it affects the market's belief about the output's value (provided $\beta_1 > 0$)—but, taking this as granted, output is nothing more but a signal entering the market's filtering problem.

Define

$$m_\beta := \sum_{k=1}^K \frac{\beta_k^2}{\sigma_k^2},\ m_{\alpha\beta} := \sum_{k=1}^K \frac{\alpha_k \beta_k}{\sigma_k^2},$$

as well as

$$\kappa := \sqrt{1+\gamma^2 m_\beta}.$$

In this section, we assume that $m_{\alpha\beta} \geq 0$. (Otherwise, the unique equilibrium effort level is 0, cf.(4).) Because $\beta_1 = 1$, $m_\beta > 0$. The parameter κ determines the rate at which the mean belief "decays," in the sense that μ_t satisfies

$$\mathrm{d}\mu_t = \sum_k \frac{\beta_k}{\sigma_k^2}(\mathrm{d}S_{k,t} - \alpha_k a^* \mathrm{d}t) - \kappa \mu_t.$$

An important quantity is

$$\Sigma := \frac{\kappa - 1}{m_\beta}.$$

This gives the value of the variance of the agent's type given the market's information, at all times.

Theorem 2.3 *There exists a unique stationary equilibrium. It is characterized by the (unique) effort level that solves*

$$c'(a^*) = \Sigma \frac{m_{\alpha\beta}}{\kappa + r}. \tag{4}$$

Equation (4) is a standard optimality condition for investment in any productive capital, and makes clear that the market's mean belief is an asset that the agent manages. This asset depreciates at rate κ, to be added to the discount rate when evaluating the net present value of effort. Investment in effort has productivity $m_{\alpha\beta}$, which measures how additional effort substitutes for higher ability. In turn, ability is converted in belief at price Σ, which measures belief responsiveness and that the agent does not affect.

Effort is increasing in the agent's patience, but does not converge to first-best effort even as discounting vanishes.[19] This is in contrast to Holmström's model with changing types, and is due in particular to mean-reversion, which eventually erases the reputational benefits from an instantaneous increase in effort. Because, there is more than one signal here, equilibrium effort might higher or lower than the social optimum. (When output is the only signal, it is readily checked from (4) that effort is too low.) Note that a signal that would only be informative about the type, or the effort, but not both (that is, such that $\alpha_k \beta_k = 0$) plays no role in incentives if it weren't for Σ. Still, such signals are not irrelevant, as they do affect learning via Σ. If $\alpha_k \beta_k < 0$, then such a signal actually depresses effort. As expected, the fiance does not depend on α_k, as the market adjusts for the equilibrium effort—it only depends on the noise in the mean-reverting process, as well as on m_β, the signal-to-noise ratio in the learning process.

Before introducing ratings, we make a few observations that are useful in understanding and interpreting later results.

Lemma 2.4 *It holds that*

$$\mathbf{Var}[\theta_t \mid \mu_t] + \mathbf{Var}[\mu_t] = \frac{\gamma^2}{2} \; (= \mathbf{Var}[\theta_t]).$$

That is, the precision of the belief (as measured by the variance of the type conditional on the belief) and its stability over time (as measured by the variance of the belief) are perfect substitutes: it is not possible to provide information that is both precise and stable, two properties of rating policies that are often quoted as being desirable (see, *e.g.*, Cantor and Mann, 2006). If stability is of foremost importance, a lower precision is desirable. The proof of the following is immediate and omitted.

Lemma 2.5 *Equilibrium effort increases in α_k and decreases in σ_k, $k = 1, \ldots, K$. It admits an interior maximum with respect to β_k, and is single-peaked in γ.*

To understand these comparative statics, it is easiest to think of the agent's incentives to increase his effort permanently by some small amount. This increases his reputation at all later times, as measured by μ_t. A higher α_k makes this increase more pronounced, as the sensitivity of reputation to effort is proportional to this coefficient. Noise in the learning process (as measured by σ_k) dampens the benefit from the increase in effort, as it slows down learning, and the agent is impatient. Finally, the role of β_k is ambiguous: if it is zero, then the market dismisses signal k in terms of learning; if it

[19]See Cisternas (2012) for the same observation in a model with human capital accumulation.

is very high, then the small variation in the signal caused by the increase in effort will be (wrongly) attributed to the type being higher than it is, but by an amount proportional to β_k^{-1}, which is negligible, and so not worth the increased effort: a higher β_k makes signal k more relevant for the reputation, but less manipulable. The interpretation of the comparative statics with respect to γ is similar and omitted.

3 Rating Systems

We now introduce a long-lived intermediary (she) who designs a rating system. This intermediary can be thought of as a "reputational intermediary," an independent professional whose purpose is to emit a credible quality signal about the agent.[20]

We no longer take for granted that the market observes output or signals. In fact, until Section 5.1, we assume that the market's exclusive source of information is the intermediary.[21] The intermediary observes the output $X = S_1$, as well as the other signals $S_2, \ldots, S_K$. Given her information, the intermediary releases a rating $\mathbf{Y}_t$ to the market (a scalar or a vector of numbers), as specified below. This structure is summarized in Figure 1.

We assume that the agent observes the output, the signals and the ratings, in addition to his effort level, although our focus on equilibria in which effort is deterministic makes this assumption innocuous, as explained in Section 2.

We assume that the intermediary commits to the rating policy—the (possibly random) map from signals to ratings. Her objective is to maximize effort by the agent. Obviously, this might result in excessive effort, relative to the efficient amount, but this is easily remedied by adding white noise to the rating, as explained below. That is, if (constant) effort a can be induced via some rating policy, then so can all effort levels in $[0, a]$. Hence, our goal here is to characterize the implementable effort levels; that is, we seek to characterize the maximum action level, which is also the effort level that maximizes expected discounted output.[22]

One can think of many examples of rating systems. In the case of a one-dimensional

[20] Reputational intermediaries do not only include so-called rating agencies, but also, in some of their roles, underwriters, accountants, lawyers, attorneys, investment bankers, auditors, etc. (see Coffee, 1997).

[21] The relative importance of exclusive vs. non-exclusive information varies widely across industries, and even within an industry: in the credit rating industry, solicited ratings are based on both public and confidential information; unsolicited ratings, on the other hand, rely exclusively on public information.

[22] While effort is the only variable that is relevant for efficiency in our model, in many applications the precision of the ratings matters as well, and this trade-off will be discussed in Section 4.4.

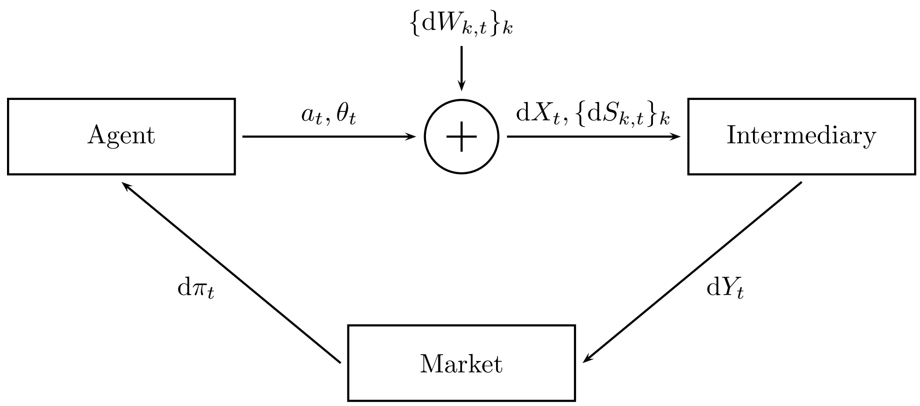

Figure 1: Flow of Information and Payments between Participants

signal, for instance, the system can involve *exponential smoothing* (as allegedly used by Business Week in its business school ranking), which involves setting

$$Y_t = \int_{s=0}^{t} e^{-\delta(t-s)} \mathrm{d}S_s,$$

for some choice of $\delta > 0$. The rating system can be a *moving window* (as commonly used in consumer credit ratings or BBB grades) whereby

$$Y_t = \{\mathrm{d}S_s : s \in [t-T, t)\},$$

for some $T > 0$. It can involve *periodic reviews*, in which the rating gets revised at predetermined dates. And so forth. A detailed comparison between some of these common policies is given in Section 6.

Our objective is to solve for the best system. However, there are some rating systems that make the problem trivial: suppose for instance that one of the signals perfectly reveals the agent's effort. Then it suffices for the rating system to raise a "red flag" (ostensibly stopping providing any rating whatsoever in the future) as soon as it finds out that the agent deviates from (say) the efficient effort level, to ensure that any deviation is unattractive in the first place.[23] We view such a system as unrealistic: in punishing the agent, the rating also "punishes" the market. The

[23]More sophisticated schemes can be devised that work even there is some small noise in the signal about effort, while inducing efficient effort at all times.

specific history should affect the content of the rating, but not the quality of the information that it conveys, nor the equilibrium effort level induced.

To rule out such equilibria, we focus attention on *stationary* rating systems, whereby calendar time is irrelevant: the rating is only a function of the past signal realizations, but not of t.[24,25] (To define stationarity formally, it is necessary to include a fictitious history of outputs and signals for times $t \leq 0$ as we do in the definitions below.) This, however, precludes a discussion of periodic systems.[26]

Second, we assume that the ratings and the other random variables of the model have a jointly Gaussian distribution. This restriction is in line with stationarity (if the "initial" belief were normally distributed, so should later beliefs be), and also makes the model tractable: this is the class of information systems for which standard filtering techniques apply (as well as, serendipitously, Holmström's analysis). Given Lemma 2.2, it is without loss that the rating is equal to the market belief, and nothing more.

We continue to work in a stationary environment. Because ratings refer to past signal realizations, it is natural to introduce a fictitious history of the agent's ability at time 0, so as to define stationary rating systems formally.

The type process θ continues to be a stationary Ornstein-Uhlenbeck process, but is now defined over the entire real line. For any t, θ_t has mean zero and variance $\gamma^2/2$, and the law of θ is defined for all $t \in \mathbb{R}$ by

$$\mathrm{d}\theta_t = -\theta_t \,\mathrm{d}t + \gamma \,\mathrm{d}Z_t,$$

where Z is an independent "bi-directional" standard Brownian motion defined over the real line.[27]

Similarly, we introduce an infinite history of signals and outputs, and define the processes X and S_k, $k = 2, \dots, K$ over the real line. When the agent exerts effort level a_t at time $t \geq 0$ (and by convention $a_t = 0$ for $t < 0$), these processes follow the laws of motion

$$\mathrm{d}X_t = (\theta_t + a_t) \,\mathrm{d}t + \sigma_1 \,\mathrm{d}W_{1,t},$$

[24]This is not to say that stationarity is the only way to rule them out, but it is a natural way to do so.

[25]As explained below, this is weaker than requiring the rating to be a Markov function of the intermediary's belief.

[26]We can nevertheless examine specific families of periodic ratings, such as the one mentioned in the example above, see the discussion in Section 6.

[27]More precisely, Z is defined by joining two independent standard Brownian motions, Z^+ and Z^-, by letting $Z_t = Z_t^+$ if $t \geq 0$ and $Z_t = Z_t^-$ if $t \geq 0$.

and

$$\mathrm{d}S_{k,t} = (\alpha_k a_t + \beta_k \theta_t)\,\mathrm{d}t + \sigma_k\,\mathrm{d}W_{k,t},$$

for independent standard bi-directional Brownian motions W_k, $k = 1, \ldots, K$, with $X_0 = 0$, $S_{k,0} = 0$.[28] As in the benchmark model with public signals, off-path, the market may not properly anticipate the agent's effort levels. It is then possible that it forms a different belief regarding the law of the signals and outputs, where the conjectured action a_t^* at time t replaces a_t in the equations above. We continue to use the star notation to refer to the conjectures of the market.

For any time $t \in \mathbb{R}$, we denote by $\mathcal{G}_t$ the information generated by the history of signals $S_{k,s}$ for $k = 1, \ldots, K$ and $s \leq t$. This is the information observed by the intermediary and the agent (who additionally observes his own actions).

Definition 3.1 *A rating process is defined by a vector process* $\mathbf{Y}$ *such that the following holds:*

1. *For all* t, $\mathbf{Y}_t$ *is measurable with respect to the intermediary's information* $\mathcal{G}_t$.
2. $(\mathbf{Y}_t, S_{1,t}, \ldots, S_{K,t})$ *is a jointly normal and jointly stationary process.*
3. *For all* k, $\Delta \mapsto \mathbf{Cov}[\mathbf{Y}_t, S_{k,t-\Delta}]$ *is continuously differentiable.*
4. *The mean rating is zero under the market's conjecture:* $\mathbf{E}^*[\mathbf{Y}] = 0$.

The first condition states that the rating at any given time cannot be contingent on information that is not yet realized. In addition there may be some additional noise, but as the noise may be represented as an extra dummy signal with no drift, we do not model the extra source of noise explicitly. As will be shown, the optimal rating system does not make use of the extra source of noise.

The second condition concerns the normality and the stationary of the ratings. Stationarity and normality do not only rule out periodic ratings, they also rule out

[28] Formally, we can introduce these virtual histories by introducing dummy variables $\widetilde{\theta}$, $\widetilde{X}$, $\widetilde{S}_k$. To do so, we introduce the processes that define the fictitious history, whose law of motion is defined by

$$\mathrm{d}\widetilde{\theta}_t = \widetilde{\theta}_t \mathrm{d}t + \gamma \mathrm{d}\widetilde{Z}_t,$$

$$\mathrm{d}\widetilde{X}_t = \widetilde{\theta}_t \mathrm{d}t + \sigma_1 \mathrm{d}\widetilde{W}_{0,t},$$

$$\mathrm{d}\widetilde{S}_{k,t} = \beta_k \widetilde{\theta}_t \mathrm{d}t + \sigma_k \mathrm{d}\widetilde{W}_{k,t},$$

where $\widetilde{Z}$ and $\widetilde{W}_k$, $k = 1, \ldots, K$ are independent standard Brownian motions, $\widetilde{\theta}_0 = \theta_0$, $\widetilde{X}_0 = 0$ and $\widetilde{S}_{k,0} = 0$. For $t < 0$, we let $\theta_t = \widetilde{\theta}_{-t}$ and $S_{k,t} = \widetilde{S}_{k,-t}$.

coarse rating systems such as, say, the practice of badges on eBay. At first sight, such a system is stationary, but not Gaussian. Yet, in terms of the corresponding belief process, stationarity also fails. The last condition is simply a normalization.

These assumptions ensure that a rating policy admits an alternative, analytic representation that is central to our analysis.

Lemma 3.2 *Fix a rating process* $\mathbf{Y}$*. Given a market conjectured effort level* a^**, there exist vector-valued functions* $\mathbf{u}_k$*,* $k = 1, \ldots, K$*, such that*

$$\mathbf{Y}_t = \sum_{k=1}^{K} \int_{-\infty}^{t} \mathbf{u}_k(t-s)(\mathrm{d}S_{k,s} - \alpha_k a^* \,\mathrm{d}s).$$

Moreover, it holds that, for all $\Delta \geq 0$*,*

$$\mathbf{u}_k(\Delta) = \frac{\beta_k \gamma^2}{\sigma_k^2} C \left(e^{\kappa\Delta} + \frac{\kappa - 1}{\kappa + 1} e^{-\kappa\Delta} \right) - \frac{f_k'(\Delta)}{\sigma_k^2} - \frac{\beta_k \gamma^2}{\sigma_k^2 \kappa} \int_0^{\Delta} \sinh(\kappa(\Delta - s)) F'(s) \,\mathrm{d}s,$$

with

$$f_k(\Delta) \coloneqq \mathbf{Cov}[\mathbf{Y}_t, S_{k,t-\Delta}], \quad F(\Delta) \coloneqq \sum_{k=1}^{K} \frac{\beta_k}{\sigma_k^2} f_k(\Delta),$$

and

$$C \coloneqq \frac{1}{2\kappa} \int_0^{\infty} F'(j) e^{-\kappa j} \,\mathrm{d}j.$$

In signal processing terms, this means that (stationary, Gaussian) rating systems can be represented by linear time-invariant filters, with the functions $\mathbf{u}_k$ being the impulse responses. Whenever $u_k(t) = e^{-\delta_k t}$ for some $\delta_k > 0$, the term impulse response will directly refer to the parameter δ_k. The converse of Lemma 3.2 is obvious: given a continuous linear time-invariant filter, the resulting process Y is a rating system.[29]

The intermediary's belief is one such filter, obtained by setting $u_k = (\beta_k/\sigma_k^2)e^{-\kappa t}$. But there is no reason that a given rating be "Markovian" with respect to (that is, can be deduced from) the intermediary's belief—indeed, we will see that the optimal rating policy fails to be. Nor need a rating admit a finite-dimensional representation, as the functions u_k are entirely arbitrary.

[29]This representation admits no obvious extension to the case of non-stationary Gaussian systems. There are well-known examples of Brownian motions (in their own filtration) that are constructed from another Brownian motion in very different and surprising ways. See Jeulin and Yor (1979).

An important issue that was glossed over so far pertains to the status of past ratings: do they remain in the public domain? If so, we refer to the rating as *public*. Otherwise, it is *confidential* (or private). We take this distinction as exogenous, as there might be technological, institutional or regulatory constraints that prevent ratings from being private (or conversely impose confidentiality). However, given the intermediary's commitment to the rating policy, a public rating is a special case of confidential ratings, as nothing prevents the intermediary from encoding all past confidential ratings into the current one.[30] Yet as we shall see, optimal public and confidential rating policies markedly differ.

For a given rating process $\mathbf{Y}$, let $\mathcal{F}_t$ denote the information available to the market at time t. We distinguish between confidential and public ratings. In the confidential case, $\mathcal{F}_t$ is the information contained in the rating $\mathbf{Y}_t$. In the public case, $\mathcal{F}_t$ is the information contained in Y_s for all $s \leq t$ (including negative values of s).[31] In other words, ratings are public whenever $\mathcal{F}_t \subseteq \mathcal{F}_{t'}$ whenever $t' \geq t$. The same rating process can thus be used for two different rating policies, and a rating policy is formally defined by a rating process together with the qualifier "confidential" or "public". Alternatively, it can be defined by the information made available to the market, captured by the family of information sets $\{F\}_t$. We will use the two definitions interchangeably throughout the paper.

A strategy for the agent continues to be captured by a stochastic process $\{A\}_t$ that generates the effort paths as a function of the agent's information; that is, for all t, A_t is measurable with respect to $\mathcal{F}_t$. We denote by $\mathcal{A}$ the set of the agent's strategies.

Definition 3.3 *Given a rating policy defined by $\{\mathcal{F}\}_t$, a stationary equilibrium is a profile (a^*, π), where a^* is the effort exerted at every time and π is the cumulative payment process, such that*

1. *(Zero-profit) For all $\tau \geq 0$,*

$$\pi_\tau = \int_0^\tau \mathbf{E}^*[a^* + \theta_t \mid \mathcal{F}_t] \, \mathrm{d}t.$$

[30] As an empirical matter, it is not always easy to tell whether ratings are public because the intermediary chooses to make them so, or because implementing confidentiality is difficult. Standard & Poor's prides itself that its "public ratings opinions are disseminated broadly and free of charge," but this has not always been the case, as investors (the "market") had to pay subscription fees to consult the ratings. Besides, credit rating agencies run ancillary (private) consulting businesses. Confidential credit ratings perdure, but the overwhelming amount of credit ratings is public nowadays.

[31] Formally, for confidential ratings, $\mathcal{F}_t$ is the σ-algebra generated by $\mathbf{Y}_t$, while for public ratings, $\{\mathcal{F}_t\}$ is the filtration generated by $\mathbf{Y}$.

2. (Optimal effort)

$$a^* \in \operatorname*{argmax}_{A \in \mathcal{A}} \mathbf{E}\left[\int_0^\infty e^{-rt}(\mathrm{d}\pi_t - c(A_t)\,\mathrm{d}t)\right].$$

The proof of the following lemma is relegated to the end of this section (see Proposition 3.10).

Lemma 3.4 *For any public or confidential rating policy, there exists a unique stationary equilibrium.*

If a is the stationary equilibrium effort level associated to a rating policy, we say that the rating policy *implements* effort level a.

Lemma 3.5 *If there exists an (exclusive) rating system that implements effort a, then for any $a' \in [0, a]$, there exists a rating system that implements effort a'.*

Indeed, adding white noise to a rating process depresses effort: if Y implements some a, then $Y + \sigma W$ (where W is some independent standard Brownian motion, and $\sigma \geq 0$) implements lower effort levels that can be continuously adjusted with σ. The proof is immediate and omitted.

This lemma relies on exclusivity: if the market had access to additional sources of information in addition to the intermediary, zero effort might not be implementable. Because equilibrium effort based on this alternative source of information might be too high (see the discussion after Thm.2.3), it can happen that the intermediary could usefully depress effort, and there are better ways to do so than to disclose no information whatsoever. Just as hiding some information can enhance effort, revealing some can depress it. We return to this issue in Section 5.1.

A process that takes an important role is the belief process of the market, μ, defined by $\mu_t = \mathbf{E}^*[\theta_t \mid \mathcal{F}_t]$. As in the benchmark model of Section 2, the agent is concerned with is his expected discounted reputation, as defined by the next lemma, the immediate counterpart of Lemma 2.2. The proof is the same as for Lemma 2.2, and so we omit it.

Lemma 3.6 *Given a payment process that satisfies the zero-profit condition, the stationary effort level a^* maximizes the agent's payoff if and only if it maximizes*

$$\mathbf{E}\left[\int_0^\infty e^{-rt}(\mu_t - c(A_t))\,\mathrm{d}t\right],$$

over $\mathcal{A}$, where $\mu_t = \mathbf{E}^*[\theta_t \mid \mathcal{F}_t]$ is derived using effort level a^* as the market's expectation.

Hence, to boost incentives, the rating system must maximize the right-hand side, namely, the agent's reputation, net of the cost.[32]

The rating process is itself a belief process, in the sense that the (alternative) rating process defined by $\{\widetilde{Y}_t\} := \{\mu_t\}$ is a well-defined scalar rating. By Lemma 3.6, this alternative rating policy implements the same effort as the original rating policy. In particular, the restriction to scalar rating systems is without loss of generality: the range of actions they implement is the same as multi-dimensional rating policies. (This lemma is reminiscent of the revelation principle, although one must pay attention to preserving the properties of Definition 3.1 when applying it.)

Lemma 3.7 *For any (public/confidential) multi-dimensional rating policy that implements effort level a, there exists a (public/confidential) scalar rating policy that implements a.*

The following is an intuitive characterization of mean beliefs that can serve as an alternative definition.

Proposition 3.8 *Let Y be a scalar rating process in the sense of Definition 3.1. Then, Y is:*

1. *A belief for confidential ratings if and only if, for all t, $\mathbf{E}^*[\theta_t \mid Y_t] = Y_t$.*
2. *A belief for public ratings if and only if, for all t, $\mathbf{E}^*[\theta_t \mid \{Y_s\}_{s \leq t}] = Y_t$.*

By Lemma 3.7 we can restrict attention to scalar rating policies without loss, and to compute the equilibrium under a rating policy, we must compute the associated market belief. With confidential scalar ratings, the market belief process is easily obtained, it is proportional to the rating process itself, as

$$\mathbf{E}^*[\theta_t \mid Y_t] = \frac{\mathbf{Cov}[\theta_t, Y_t]}{\mathbf{Var}[Y_t]} Y_t. \tag{5}$$

However getting the market belief associated with public scalar ratings is much more involved, and quickly becomes intractable. When dealing with public ratings, and in the spirit of Proposition 3.8, we restrict attention further to rating processes that are proportional to the market beliefs of public ratings, so as to continue to use (5). The conditions required for a scalar rating to be proportional to the market belief process of a public rating policy can be stated in simple terms: a scalar rating process is proportional to the market belief under a public rating policy if and only if the intertemporal correlations of these scalar ratings equal the intertemporal correlations of the agent types. It is the object of the following lemma.

[32] Again, all this assumes is $\beta_1 > 0$; if $\beta_1 < 0$, the objective would be to minimize the reputation.

Lemma 3.9 *A scalar rating process Y is proportional to the market belief induced by a public rating policy if and only if for every $\Delta \geq 0$,*

$$\frac{\mathbf{Cov}[Y_{t+\Delta}, Y_t]}{\sqrt{\mathbf{Var}[Y_t]}\sqrt{\mathbf{Var}[Y_{t+\Delta}]}} = \frac{\mathbf{Cov}[\theta_{t+\Delta}, \theta_t]}{\sqrt{\mathbf{Var}[\theta_t]}\sqrt{\mathbf{Var}[\theta_{t+\Delta}]}} \quad \left(= e^{-\Delta}\right).$$

Using the reduced-form belief process as rating process, or, for convenience, a multiple of it to normalize the variance of ratings to one, we may prove equilibrium existence and uniqueness, and characterize the equilibrium effort level. The next result might be technical, but it lies the foundations for the optimization programs that the intermediary face and that we solve in Section 4.

Proposition 3.10 *Let $\{Y_t\}$ be a scalar rating process with $\mathbf{Var}[Y_t] = 1$. Then* (a) *if $\{Y_t\}$ is the rating of a confidential policy, or* (b) *if $\{Y_t\}$ is the rating of a public policy and is such that Y_t is proportional to the market belief $\mathbf{E}^*[\theta_t \mid \{Y_s\}_{s \leq t}]$, then there exists a unique stationary equilibrium. It is characterized by the (unique) effort level a^* that solves:*

$$c'(a^*) = \frac{\gamma^2}{2}\left[\sum_{k=1}^{K} \beta_k \int_0^{\infty} u_k(t) e^{-t}\, \mathrm{d}t\right]\left[\sum_{k=1}^{K} \alpha_k \int_0^{\infty} u_k(t) e^{-rt}\, \mathrm{d}t\right], \tag{6}$$

where we use the linear representation of Y given in Lemma 3.2.

Therefore in the confidential setting we may think of the optimal rating policy as the solution to the problem of maximizing

$$\frac{\mathbf{Cov}[Y_t, \theta_t]}{\mathbf{Var}[Y_t]}\left[\sum_{k=1}^{K} \alpha_k \int_0^{\infty} u_k(t) e^{-rt}\, \mathrm{d}t\right], \tag{7}$$

over $\{u_k\}_{k=1}^{K}$, noting that $\mathbf{Cov}[Y_t, \theta_t]$ is equal to the first two factors in (6). We have divided through by the variance (which by definition is quadratic in u_k, as is the right-hand side of (6)) to ensure that a solution to (7) can always be rescaled to yield a maximizer to the right-hand side of (6) subject to $\mathbf{Var}[Y_t] = 1$.

The first term (namely, the ratio) can be thought as a measure of the sensitivity of the system to effort, while the second conveys incentives in a more direct and transparent fashion, by amplifying the usual career concerns as measure by α_k.

The equilibrium referred to in Proposition 3.10 is unique in the broader class of deterministic equilibria, as the proof makes clear. That is, it is the unique equilibrium as long as we focus on equilibria in which the agent's strategy depends on calendar time only, and not on the public signals or his past actions. On the other hand, there might be additional equilibria in public strategies, see ft.17.

4 Optimal Ratings

4.1 Persistence vs. Sensitivity

To build intuition regarding the desirability of rating systems, let us start with a simple example: exponential smoothing as a confidential rating. Suppose that the intermediary wishes to use the rating

$$Y_t = \sum_k \frac{\beta_k}{\sigma_k^2} \int_{-\infty}^{t} e^{-\delta(t-j)} \mathrm{d}S_{k,j},$$

where $\delta > 0$ need not equal κ. If it does, then *ipso facto* the intermediary discloses her private belief, and we are back to the analysis of Section 2. If it does not, then in general the market belief would be affected by the observation of past ratings, if this information were available; but it is not, by definition of confidentiality.

It is not hard to derive effort as a function of such a system, namely:

$$c'(a) = \frac{1}{r+\delta} \frac{\gamma^2 m_{\alpha\beta}\delta}{1+\gamma^2 m_\beta + \delta}. \tag{8}$$

The first term (namely, $\frac{1}{r+\delta}$) captures the fact that, not only are future returns from effort discounted because of impatience, but also because future ratings discount past signals at rate δ. The fact that ratings discount older signals is unfortunate for effort, as current effort only enters the current signal. Clearly, the lower δ, the larger this term. Rating persistence increases the impact of current effort on future ratings.

But it also decreases the impact of ratings on beliefs. This is captured by the second term. The impulse response δ affects how *sensitive* the market belief is to effort. If δ is close to zero, this term is zero as well. This is because ability is imperfectly persistent: recent signals are more useful for inference than old signals, so that excessive persistence makes the rating less useful. Instead, if δ is large, the rating is very informative about the latest signals, and so about current type. This is especially important when mean-reversion or noise γ are large (if the noise were nearly zero, one could use old signals to extrapolate the current type nearly as efficiently as one could with the current signal).

The intermediary must trade off persistence with sensitivity. The resulting first-order condition (with respect to δ) gives as optimal solution

$$\delta = \sqrt{1+\gamma^2 m_\beta}\sqrt{r} = \kappa\sqrt{r}. \tag{9}$$

Treating δ independently of γ (as we have while considering (8)) wasn't an option in the baseline of Section 2: there, interpreting Y as the belief, it holds that $\delta = \kappa$, which is a function of γ by definition, and so (8) reduces to

$$c'(a) = \frac{1}{\kappa + r}\frac{(\kappa - 1)m_{\alpha\beta}}{m_\beta},$$

which is precisely (4) from Theorem 2.3. The trade-off has disappeared: if only we could choose κ in the baseline model, we would prefer a higher value.[33]

Let us return to (9). Not surprisingly, the more patient the agent, the more the intermediary's choice is guided by the first term of (8), leading her to settle for high persistence (low δ) despite the poor sensitivity that it implies. The intermediary finds Bayesian updating (*e.g.*, $\delta = \kappa$) too persistent or not depending upon

$$r \lesseqgtr 1, \tag{10}$$

namely, whether or not patience outweighs the rate of mean-reversion (normalized to 1). If $r = 1$, then she is content with Bayesian updating. This highlights the compromise between persistence and sensitivity that the intermediary faces: Including fresh signals is desirable for sensitivity; including old ones is desirable for persistence. It does not imply that the intermediary finds this rating system (based on one exponential function) advisable. If anything, more sophisticated rating systems might help manage this trade-off.

To shed some light on this, let us turn to a richer example. Departing from our convention regarding the output process, let us assume that output is solely a function of ability, not of effort: $\beta \coloneqq \beta_1 > 0, \alpha_1 = 0$, while the unique other signal is purely about effort: $\alpha \coloneqq \alpha_2 > 0, \beta_2 = 0$, and set $\sigma \coloneqq \sigma_1 = \sigma_2$. Consider the best rating system among the two-parameter family

$$u_1(t) = \frac{\beta}{\sigma^2}e^{-\kappa t}, \quad u_2(t) = c\frac{\beta}{\sigma^2}\sqrt{\delta}e^{-\delta t}.$$

with parameters $c \in \mathbb{R}, \delta > 0$. This family is restrictive: in particular, the first signal enters ratings the way it enters the intermediary's belief, and so distortions only

[33]The trade-off vanishes because (8) is increasing in γ.

appear via the second signal, which affects her rating, but not her belief.

Computing the marginal cost, given a pair (c, δ), we obtain

$$c'(a) = \frac{\sqrt{\delta}}{r+\delta} \frac{c}{1+c^2} \frac{2\alpha\beta}{(1+\kappa)\sigma^2}.$$

Comparing with (8), the first denominator is familiar: it is the impact on persistence of the impulse rate δ that is chosen, to be added to the discount rate. An increase in effort at time t will be reflected in the rating at time $t+\Delta$, for Δ, but gets discounted twice: by a factor $e^{-r\Delta}$ by the agent, and by a factor $e^{-\delta\Delta}$ by the market. Integrating over all $\Delta \geq 0$, and given the normalization factor $\sqrt{\delta}$ added in front of u_2, this gives a boost to incentives at time t proportional to $\sqrt{\delta}/(r+\delta)$.

The constant c in front of u_2 further boosts incentives (it amplifies the return of effort), but it also affects sensitivity: increasing c leads to an increase in the market's variance, which depresses the sensitivity of the market's belief to effort, and hence depresses effort. This is reflected by the denominator $1+c^2$. Because of the normalization factor $\sqrt{\delta}$, the choice of δ does not affect the variance in the market's belief (plainly, once u_2 is squared and integrated over all $\Delta \geq 0$, δ vanishes).

The higher the rate δ, the higher sensitivity, for the same reason as before. The weight c is the additional parameter here: make c too small, and sensitivity disappears, as the impulse response δ becomes useless if it is assigned no weight (effort does not enter output, and so incentives disappear if the rating does not confound beliefs); make it too big, and sensitivity vanishes as well, because the uninformative term $e^{-\delta t}$ then overwhelms the informative branch $e^{-\kappa t}$ in the rating system. The objective is maximized for an intermediate value of c.

Hence, the maximization problem is entirely separable: parameters α, β, κ are irrelevant for the choice of maximizers; furthermore $c/(1+c^2)$ is maximum at $c=1$, and $\sqrt{\delta}/(r+\delta)$ at $\delta = r$. That is, *independently of* δ, the best weight to assign to the second term is 1; and *independently of* c, the best pick for an impulse response is r. The intermediary's favorite level of persistence might be lower than the "natural" rate at which beliefs decay, if the agent is sufficiently impatient.

4.2 Confidential Ratings

In this section, we solve for the optimal rating system when ratings are confidential. We introduce

$$m_\alpha := \sum_{k=1}^{K} \frac{\alpha_k^2}{\sigma_k^2},$$

and

$$\lambda := (\kappa-1)\sqrt{r}(1+r)m_{\alpha\beta}+(\kappa-r)\sqrt{\Delta}, \quad \Delta := (r+\kappa)^2(m_\alpha m_\beta - m_{\alpha\beta}^2)+(1+r)^2 m_{\alpha\beta}^2.$$

We assume throughout the remainder of the paper that $\kappa \neq r$, $r \neq 1$, and $\lambda \neq 0$.

Theorem 4.1 *The optimal rating system is unique and given by*[34]

$$u_k(t) = c_k \frac{\sqrt{r}}{\lambda} e^{-rt} + \frac{\beta_k}{\sigma_k^2} e^{-\kappa t},$$

with coefficients

$$c_k := (\kappa^2 - r^2) m_\beta \frac{\alpha_k}{\sigma_k^2} - (\kappa^2 - 1) m_{\alpha\beta} \frac{\beta_k}{\sigma_k^2}.$$

This result has at least three remarkable features: two exponentials suffice; the impulse responses are the discount rate and the belief response; and the coefficients on the term involving the belief impulse are precisely equal to what they should be to compute the intermediary's belief: these weights are not distorted. Put another way, it holds that

$$Y_t = \delta U_t + (1-\delta)\mu_t^*,$$

for some $\delta \in \mathbb{R}$, where μ_t^* is the intermediary's belief at time t and $\{U_t\}$ is the process solving

$$\mathrm{d}U_t = \frac{\sqrt{r}}{\lambda} \sum_k c_k(\mathrm{d}S_{k,t} - \alpha_k a^* \mathrm{d}t) - rU_t.$$

Several important conclusions follow.

- *White noise is harmful*: If a signal is such $\alpha_k = \beta_k = 0$, its weight in the rating process is zero: Irrelevant noise has no use, as it depresses effort.

- *All signals matter, but none should be disclosed*: Except for non-generic parameter configurations (such as a signal being white noise, precisely), the optimal rating involves *all* signals. Some might be weighted negatively, as explained below, when performance along that dimension adversely impacts the rating. Nonetheless, because these are independent sources of information about the

[34]Recall that we focus on "direct mechanisms" whereby the rating is equal to the market mean belief. Obviously, one-to-one transformations (positive affine transformations in particular) of this rating are equivalent. Throughout, uniqueness is to be understood as up to such a transformation. We use whichever scalar multiple is convenient.

agent's type and effort, it is always beneficial to aggregate them somehow. *Even if* two signals are independently distributed, using both allows the intermediary to reduce the noise in the recommendation, and hence to boost effort.

- *Two states are necessary and sufficient*: The intermediary only needs to keep track of the pair $\{\mu^*, U\}$: her private belief (that she does not disclose) and some *incentive* state U decaying at rate r. These two states are combined with some fixed weights, in a way that prevents the market from backing out either one.

To make sense of the optimal policy, it is useful to consider special cases.

First, consider the case in which all signals have the same parameters, namely, $\alpha_k = \beta_k = 1$, $\sigma_k = \sigma$ for all k. Then, for all k,

$$u_k(t) = u(t) := \frac{1}{\sigma^2}\left[\frac{1-\sqrt{r}}{\kappa-\sqrt{r}}\sqrt{r}e^{-rt} + e^{-\kappa t}\right].$$

Hence, whether the incentive state is added or subtracted from the belief state depends on two familiar comparisons, given (9) and (10). If $\delta = \kappa\sqrt{r}$, as defined in (9), lies in between r and κ, the "favorite" impulse response is in between the two rates appearing in u_k. Hence, balancing both by assigning them weights of the same sign is a good idea. This occurs when $\sqrt{r} < 1$ or $\sqrt{r} > \kappa$. For values $\sqrt{r} \in (1, \kappa)$, both rates are either too high or too low, leading to weights of different signs.

Of course, the rating is meant to boost effort, not depress it; hence, whether or not the incentive term has a negative coefficient, the entire expression $u(t)$ is always positive *for some* t. If $\sqrt{r} < 1$ or $\sqrt{r} > \kappa$, then u is positive for all t, and decreasing with t: older signals contribute less than newer ones. On the other hand, if $1 < \sqrt{r} < \sqrt{\kappa}$, then $u(0) > 0$, but it is single-troughed, and negative for all t above some threshold. Hence, signals from a given (old) vintage unambiguously decrease the current rating: very recent signals boost the rating, but older signals are detrimental. In this case, an agent with a better record is "held to higher standards:" having performed well in the past makes the current rating more severe. Finally, if if $\sqrt{\kappa} < \sqrt{r} < \kappa$, then $u(0) < 0$ and u is single-peaked, and positive for all t above some threshold. The agent is then hold to low standards if he performed well in the past: in fact, his rating is highest if he overperformed in the past and underperformed recently. But the agent cares about future ratings and knows that his rating will suffer eventually if he decides to underperform now. See Figure 2 for an illustration of the different cases as discounting varies.

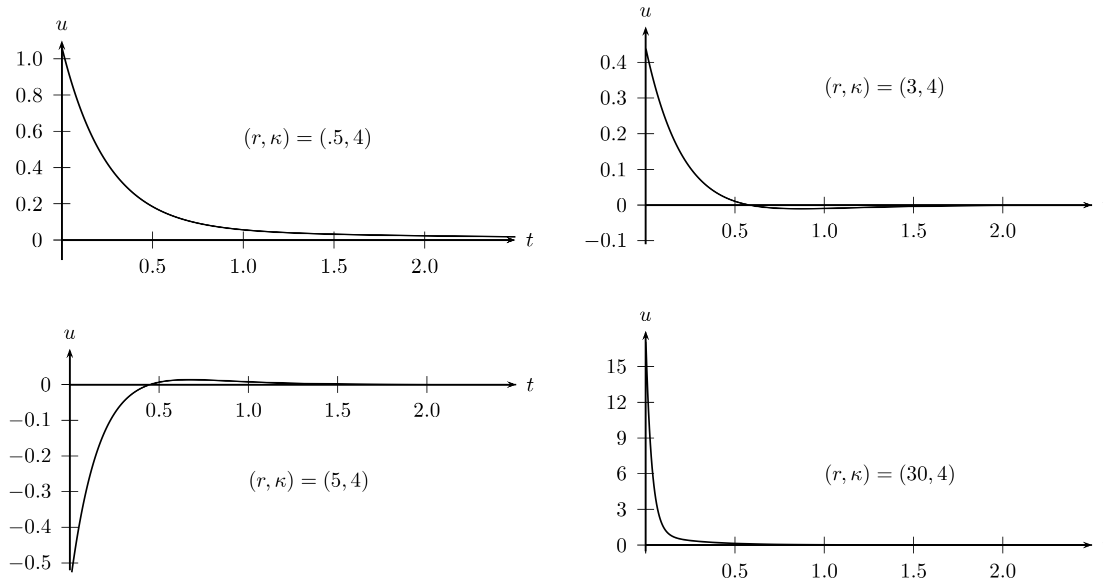

Figure 2: Rating in the case of homogenous signals ($\alpha = \beta = \sigma = 1$).

To understand this rather surprising feature of the solution, recall that (taking for granted the decomposition into two branches, with exponents κ and r, as suggested by our earlier example), the choice of the coefficient to place on the branch with exponent r results from a trade-off between two terms, a sensitivity factor $\mathbf{Cov}[Y_t, \theta_t]/\mathbf{Var}[Y_t]$ (recall (7)) and a second factor that clearly benefits from a higher coefficient (which roughly corresponds to persistence in the previous examples). The sensitivity is maximized by a *negative* value for this coefficient, as correlation is maximized when the coefficient is zero, yet this ratio isn't the correlation between Y and θ: variance increases faster than standard deviation. If sensitivity is paramount, the coefficient is negative. In that case, depending upon whether $r \gtrless \kappa$, this implies that recent or late positive innovations in the signals are hurting the current rating, and the direction goes the wrong way in terms of preferences: penalizing older (newer) good signal realizations is especially costly when the agent is very patient (impatient), so that the second factor, favoring to a positive coefficient, takes over once $r < 1$ or $r > \kappa^2$.

Note that, as $r \to 0$, the weight on the incentive term vanishes, while as $r \to \infty$, it is the exponential itself that makes the first term disappear (for $t > 0$). Hence, the intermediary reveals her belief in these two extreme cases. This suggests that patience plays an ambiguous role in the informativeness of the rating (cf. Lemma 4.2 below).

Second, consider the case in which (despite our normalization regarding output being the first signal) the first K_0 signals are informative about effort only ($\beta_k = 0$) while the remaining ones are exclusively about ability ($\alpha_k = 0$). Then the formula above gives that, for $k \leq K_0$,

$$u_k(t) = (\kappa^2 - r^2) m_\beta \frac{\alpha_k}{\sigma_k^2} e^{-rt},$$

while for $k \geq K_0 + 1$,

$$u_k(t) = \frac{\beta_k}{\sigma_k^2} e^{-\kappa t}.$$

That is, signals that are uninformative from a learning point of view are assigned to the incentive term exclusively, and the informative signals to the learning term. This generalizes our second example from Section 4.1, and establishes that the two-parameter family considered there contains the optimal rating system.

More generally, we may rewrite the rating process as

$$u_k(t) = \frac{\beta_k}{\sigma_k^2} \left[\left((\kappa^2 - r^2) \frac{\alpha_k}{\beta_k} - (\kappa^2 - 1) \frac{m_{\alpha\beta}}{m_\beta} \right) \frac{\sqrt{r} m_\beta}{\lambda} e^{-rt} + e^{-\kappa t} \right].$$

Noting that the factor β_k / σ_k^2 is the adjustment that corrects for the possibly differing means and variances of the innovation processes $\mathrm{d}S_k$, we see that signals are ordered according to the ratios α_k / β_k in terms of their importance in the incentive term, as is intuitive. See Figure 3 for an illustration (in which the agent is held to high standards).

The proofs of Theorem 4.1 and Theorem 4.4 below can be found in Appendix C. Roughly, they rely on calculus of variations. Proposition 3.10 delivers the objective, Proposition 3.8 the constraints. Lemma 3.2 is used to turn these probabilistic constraints (on Y) into analytic ones (on $\{u_k\}_{k=1}^K$). Neither the objective nor the constraints are standard, as they involve multiple integrals (as ratings stretch backward in time, while the payoff is an integral over future dates), yet the control is one-dimensional, with delayed arguments appearing. Further, (in the public case) a continuum of constraints must be satisfied. The trick is to guess a "weighted average" of these constraints to replace the program with a relaxed isoperimetric program whose solution happens to satisfy the original set of constraints. In Appendix B, we derive the necessary machinery to attack such problems (Proposition 26 is the Euler-Lagrange equation for our context, that we apply to solve for the unique admissible solution).

Given the optimal rating policy, it is immediate to derive the market belief variance.

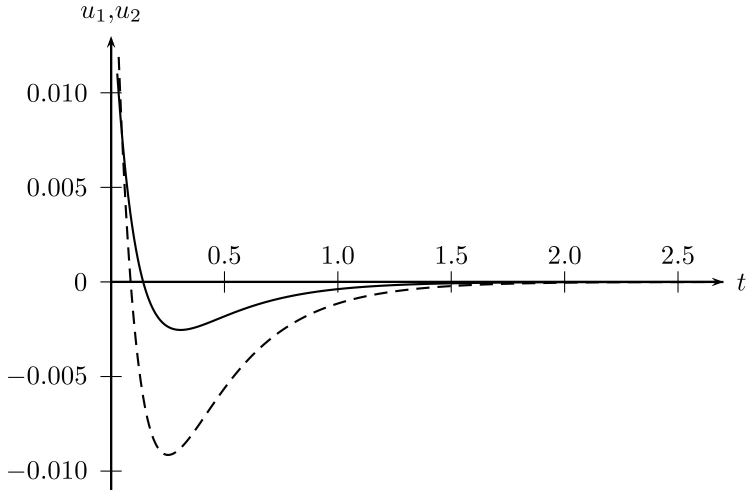

Figure 3: Confidential u_1 (solid), u_2 (dashed), $(\alpha_2, \beta_2, \sigma_2, \gamma, \sigma_1, r) = (2, 3, 1, 1, 1, 10)$, $K = 2$.

The proofs of the next two lemmas are straightforward and omitted.

Lemma 4.2 *The variance of the belief in the confidential case is given by*

$$\mathbf{Var}\, \mu^{\mathrm{c}} = \frac{(\kappa-1)^2}{4m_\beta}(1 + 2m_{\alpha\beta}\sqrt{r/\Delta}).$$

As Figure 4 illustrates, variance is maximum for intermediate level of patience. When the agent is very patient (very impatient), the rating puts emphasis on older (recent) signals, in a way that isn't particularly useful for learning. As a result, the belief is imprecise, and its variance low.

Finally, we are interested in the performance of the rating.

Lemma 4.3 *The maximum marginal cost induced by the optimal confidential rating policy is given by*

$$c'(a^{\mathrm{c}}) = \frac{\kappa - 1}{4(\kappa + r)m_\beta}\left(2m_{\alpha\beta} + \sqrt{\Delta/r}\right).$$

Surprisingly, this expression need not be globally monotone in r. This is because, as is clear from Figure 3, some past innovations can affect the rating adversely. Hence, it isn't the case that a more patient agent necessarily works harder for a given rating policy. For a given policy, he might become more sensitive to the future adverse

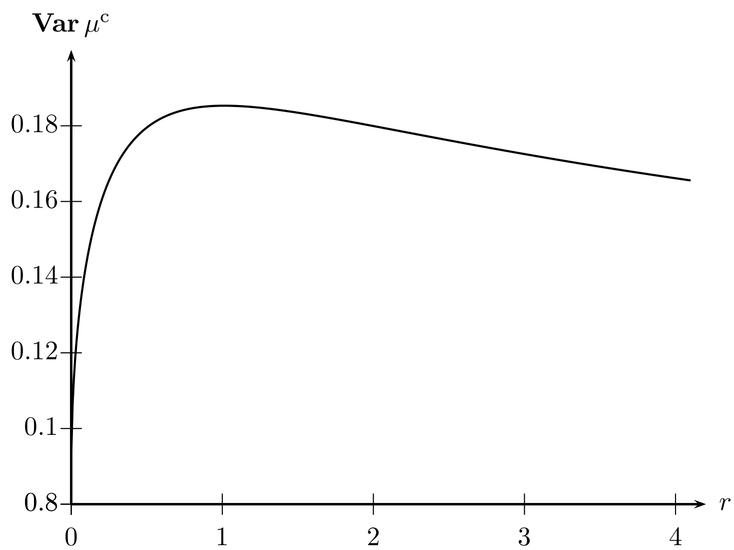

Figure 4: Market variance as a function of the discount rate, confidential ratings $((\alpha_2, \beta_2, \sigma_2, \gamma, \sigma_1) = (3, 2, 1, 1, 2))$, $K = 2$.

impact of his effort today on some of his future ratings. (Of course, this is only a small part of the story, as the rating policy isn't arbitrary.)

Despite this surprising feature, effort tends to infinity as $r \downarrow 0$, and to 0 as $r \to \infty$: any given level of effort can (cannot) be achieved if the agent is sufficiently patient (impatient). Effort is also unambiguously increasing in κ (hence, in γ), as well as in m_α.

4.3 Public Ratings

In this section, we solve for the optimal rating policy when ratings are public, *i.e.*, past ratings are freely available to the market. The definitions of $m_\alpha, m_\beta, m_{\alpha\beta}$ as well as κ, λ are unchanged. Recall that $\kappa \neq r$, and $r \neq 1$.

Theorem 4.4 *The optimal public rating policy is unique and given by*

$$u_k(t) = d_k \frac{\sqrt{r}}{\lambda} e^{-\sqrt{r}t} + \frac{\beta_k}{\sigma_k^2} e^{-\kappa t},$$

with coefficients

$$d_k := \frac{\kappa - \sqrt{r}}{\kappa - r} c_k + \lambda \frac{\sqrt{r} - 1}{\kappa - r} \frac{\beta_k}{\sigma_k^2}.$$

Relative to the confidential case, the coefficient on the belief term is unchanged. As in the confidential case, it follows that this rating policy can be implemented by a two-state automaton, with states (μ_t^*, V_t), and a rating

$$Y_t = \delta V_t + (1 - \delta)\mu_t^*,$$

for some $\delta \in \mathbb{R}$, where μ_t^* is the *intermediary*'s belief at time t and $\{V_t\}$ is a process satisfying

$$\mathrm{d}V_t = \frac{\sqrt{r}}{\lambda} \sum_k d_k (\mathrm{d}S_{k,t} - \alpha_k a^* \mathrm{d}t) - \sqrt{r} V_t.$$

Nonetheless, both the impulse response and the coefficient on the incentive term change.

First, the impulse response on the incentive term is now the geometric mean of the discount rate and the mean-reversion rate.[35] To understand this change, let us go back to one of our earlier examples, in which output is solely a function of ability, not of effort: $\beta := \beta_1 > 0, \alpha_1 = 0$; and the second signal is purely about effort: $\alpha := \alpha_2 > 0, \beta_2 = 0$, with $\sigma := \sigma_1 = \sigma_2$. Recall that the optimal confidential rating policy was characterized by

$$u_1(t) = \frac{\beta}{\sigma^2} e^{-\kappa t}, \quad u_2(t) = \frac{\beta}{\sigma^2} c \sqrt{\delta} e^{-\delta t}.$$

with $c = 1$, $\delta = r$. However, such a system typically fails to satisfy the publicness constraint: such a rating covaries over time in a way that is inconsistent with the way beliefs are supposed to mean-revert. One way to address this is to add a second term to u_1 to manipulate the intertemporal correlation of ratings, so as to make their evolution consistent with the way beliefs evolve. Namely, we can re-define u_1 as

$$u_1(t) = \frac{\beta}{\sigma^2} e^{-\kappa t} - d e^{-\delta t},$$

where d is carefully chosen so as to yield the appropriate covariance in the rating that would be indistinguishable from telling the truth. (Typically, for r small, this

[35]While the mean-reversion rate has been normalized to one, this is simply a time change, and it is easy to check that without this normalization, this impulse response is indeed the harmonic mean.

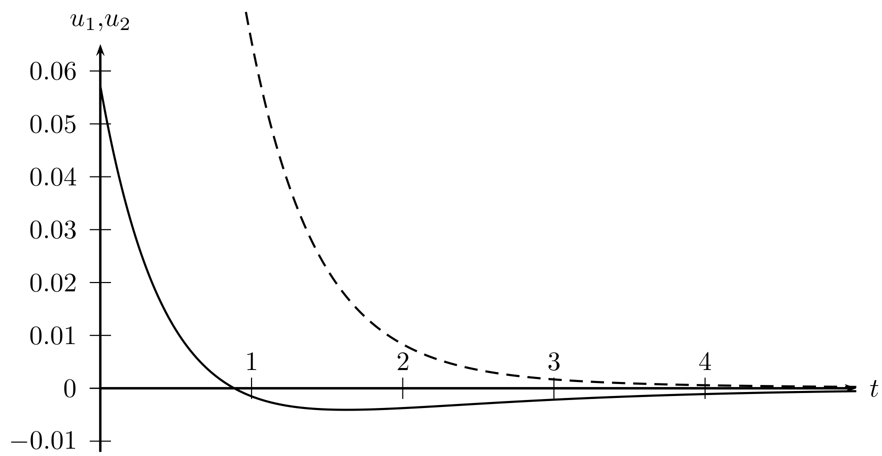

Figure 5: Public u_1 (solid), u_2 (dashed), $(\alpha_1, \beta_1, \sigma_1, \gamma, \sigma_2, r) = (3, 2, 1, 1, 2, 1/2)$, $K = 2$.

requires $d < 0$ to "counteract" $c = 1$ and simulate mean-reversion.) Of course, doing so comes at a cost in terms of sensitivity, since ratings become less informative. The more extreme the distortion due to the incentive term, the costlier the correction to generate the "right" mean-reversion. Shading the impulse response δ towards the natural rate at which ratings should co-vary –the rate of mean-reversion, precisely– reduces the cost of this correction. The optimal rating system settles this trade-off in a particular simple way, half-way between the rate that is optimal without the constraint (namely, r) and the rate at which ratings must covary (namely, 1). As a result, we obtain in this example the following rating policy

$$u_1(t) = \frac{\beta}{\sigma^2}\left[e^{-\kappa t} - \frac{1-\sqrt{r}}{\kappa - r}\sqrt{r}e^{-\sqrt{r}t}\right], \quad u_2(t) = \frac{\beta}{\sigma^2}\frac{\kappa - \sqrt{r}}{\kappa - r}\sqrt{r}e^{-\sqrt{r}t}.$$

Moving on to the coefficients on the incentive vs. belief term, the formulas of the public and confidential case might look very close, they differ in ways that have a significant impact. For instance, when signals are collinear (α_k/β_k independent of k, σ_k unrestricted), as in the i.i.d. case, then full disclosure is optimal: $d_k = 0$ for all k. This is easy to understand: collinear signals are equivalent to a single signal with higher precision. Yet with a single signal, and a "continuum" of constraints imposed by publicness, the only distortion that can be introduced is white noise, which is detrimental to effort. This is in contrast with the confidential case, in which the intermediary still has some leeway. In particular, with only one signal ($K = 1$), full

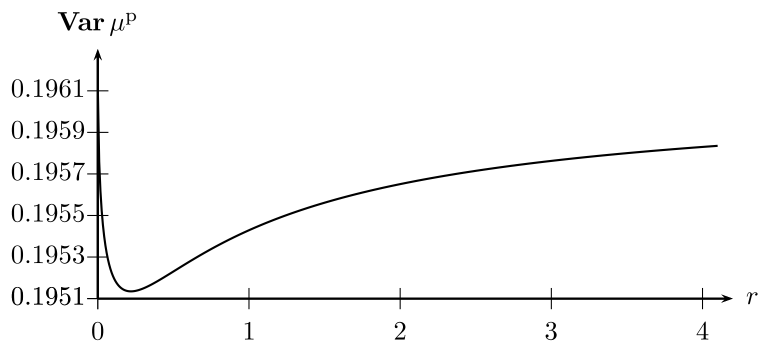

Figure 6: Market variance as a function of the discount rate, public ratings $((\alpha_2, \beta_2, \sigma_2, \gamma, \sigma_1) = (3, 2, 1, 1, 2))$, $K = 2$.

disclosure obtains in the public case. Not so in the confidential case. This is not to say that the agent exerts no effort: career concerns play a role (see Lemma 4.6), and are amplified by the intermediary's disclosure policy. But this policy is rather dull.

Overall, the time-profile of ratings look qualitatively similar to the confidential case; in particular, handicapping old or recent ratings arise here as well for intermediate discount rates. See Figure 5 for an illustration of the optimal policy.

How does market variance vary with the discount rate under the optimal policy? As Figure 6 makes clear, it is single-bottomed, with transparency resulting under extreme patience or impatience. Plainly, this wasn't the case under confidential ratings. With a public rating system, the weights on each exponential term are no longer independent: high persistence means significant learning over time, in the sense that the weight on the incentive term *must* become negligible, relative to the weight on the learning term. Similarly, very recent but very informative signals give rise to perfect learning.

Lemma 4.5 *The variance of the belief in the public case is given by*

$$\mathbf{Var}\,\mu^{\mathrm{p}} = \frac{2(1+r)}{(1+\sqrt{r})^2}\,\mathbf{Var}\,\mu^{\mathrm{c}},$$

where $\mathbf{Var}\,\mu^{\mathrm{c}}$ *is given by Lemma 4.2.*

The proof of this lemma and the next are straightforward and omitted. Because $2(1+r)/(1+\sqrt{r})^2 \in [1, 2]$, Lemma 4.5 implies that the variance of the market belief is always higher in the public case—equivalently, the precision of its information is larger: the market is better informed under the optimal public system (but at best, "twice" as well). It is readily verified that this variance tends to the benchmark value under transparency (equal to $(\kappa-1)^2/(2m_\beta)$) when either $r \to 0$ or $r \to \infty$. That is, when the agent is either very patient or impatient, the intermediary reveals almost all of her information.

Finally, we are interested in the performance of the rating.

Lemma 4.6 *The maximum marginal cost induced by the optimal public rating policy is given by*

$$c'(a^{\mathrm{p}}) = \frac{4\sqrt{r}}{(1+\sqrt{r})^2} c'(a^{\mathrm{c}}),$$

where $c'(a^{\mathrm{c}})$ is given by Lemma 4.3.

Because $4\sqrt{r} \leq (1+\sqrt{r})^2$, effort is lower under the public than under the private scheme, as expected. Effort decreases with r, going from $\sqrt{m_\alpha m_\beta - m_{\alpha\beta}^2 + m_{\alpha\beta}^2/\kappa^2}$ to 0 as r goes from 0 to $+\infty$. Hence, in stark contrast with the confidential case, there is an absolute upper bound on effort that can be induced, independently of the agent's patience.

4.4 The Effort-Precision Trade-off

While we have chosen effort as the yardstick of efficiency, the quality of information available to the market is another important measure of the performance of the rating system. To some extent, quality and effort are substitutes: if the intermediary is entirely transparent, effort is generally not maximum. Yet these substitutes are imperfect, as the effort-maximizing policy does not leave the market entirely uninformed. Hence, there is a range of information precisions over which there is a trade-off between effort and this precision. Fixing precision, there is a maximum effort level that can be induced by public or confidential ratings.

As is easy to show (we omit the details), this maximum effort corresponds to a policy qualitatively similar to the ones derived in the previous subsections—only the weights on the exponentials vary. See Figure 7. The two curves map the variance in the market belief into the maximum effort that can be induced over the range of variances over which there is a trade-off. Our analysis so far has focused on maximizing effort only, corresponding to the highest point on these curves. Effort is higher in the confidential case for any given level of variance: not too surprisingly, private ratings manage to boost effort for a given level of informativeness. This means that publicness isn't simply about forcing the intermediary to provide better

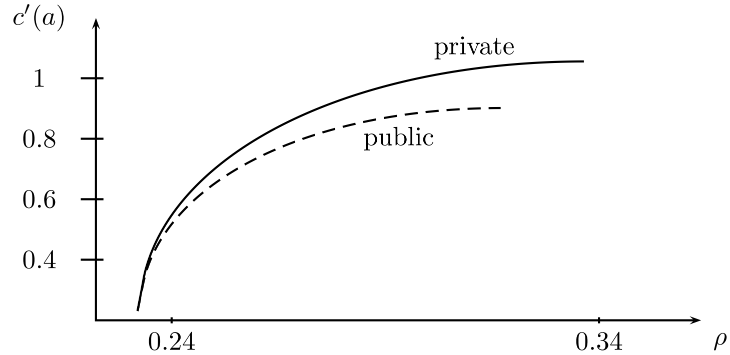

Figure 7: (Marginal cost of) effort as a function of maximum variance ρ, public vs. confidential ratings $((\beta_1, \beta_2, \alpha_1, \alpha_2, \gamma, r, \sigma_1, \sigma_2) = (3, 2, 1/3, 5, 1, 1/5, 1, 2))$.

information. A private rating system is able to simultaneously incentivize more effort and provide better information than a public one.

5 Extensions

5.1 Exclusivity

The information that rating systems provide might not be entirely proprietary. As discussed in ft.21, in credit ratings, solicited ratings are based on a mix of information that is widely available to market participants, as well as information that is exclusively accessible to the intermediary. We refer to this distinction as *exclusive* vs. *non-exclusive* information. In particular, given that the signal $S_1 = X$ is the agent's output (or equivalently profit), it makes sense to assume that the entire history of this signal is at the market's disposal whether the intermediary likes it or not. Yet, as one might expect, the intermediary does not ignore the fact that the market has direct access to this source of information: what she reveals about the exclusive signals reflects the characteristics of those signals she cannot hide from the market.

Specifically, fix $K_0 = 0, \ldots, K$. At time t, all participants (the agent, the intermediary and the market) observe $\{S_{k,s}\}_{s \leq t, k=1,\ldots,K_0}$ in addition to the rating of the intermediary—as before, we will consider both the case of public and confidential rating systems, according to whether or not past ratings are publicly available or not.

Signals $S_{k,t}$, $k \geq K_0 + 1$, are only observed by the intermediary.[36]

If $K_0 = 0$, no signal is accessible to the market, and we are back to exclusive ratings analyzed in Section 4, a special case of the results that follow. If $K_0 = K$, then the intermediary serves no purpose, and we are back to the model without an intermediary, examined in Section 2, a case that will be ignored in the sequel as the optimal rating policy is then irrelevant and undetermined. By our ordering convention, output is observed whenever any signal is observed, but this is a mere normalization: as explained, the agent is concerned about his discounted expected reputation, and the role of output in this reputation is that of a signal like any other.

As mentioned, Lemma 3.5 is no longer valid: because the market observes signals $k = 1, \ldots, K_0$, reputational concerns arise whether the intermediary is present or not: simply because effort a is implementable does not mean that the entire range $[0, a]$ is. In fact, it might be that these signals already lead to excessive equilibrium effort. An intermediary can also help depress excessive effort through her rating system. For instance, if one of her exclusive signals provides very precise information about the agent's effort, disclosing it dampens reputational concerns. Hence, it might be of as much interest to characterize the lowest implementable effort level as it is to find the highest (intermediate levels are then implementable as well). There is no fundamental asymmetry between these two problems, and the resulting rating systems are remarkably similar. We comment on this below, but keeping in line with our analysis so far, focus on identifying the highest equilibrium effort level.

As we have done in the public exclusive setting, when dealing with non-exclusive signals, it is useful to restrict attention to rating processes that are proportional to the market beliefs.[37] The conditions required for a scalar rating to be proportional to the market belief process with non-exclusive signals $S_1, \ldots, S_{K_n}$ can be stated as follows: for every non-exclusive signal S_k, the product of the correlation between type and rating and the correlation between present signal and future rating must equal the correlation between current signal and future type. The conditions are captured in the following lemma.

[36] One might wonder whether the number of non-exclusive signals is irrelevant, given the Gaussian assumption. Indeed, the theorems below suggest that it nearly is, as only a few parameters, m^n_β, m^n_α, and $m^n_{\alpha\beta}$ (as defined below) summarize the optimal rating system. However, these parameters are not independent of each other, and one needs at least two non-exclusive as well as at least two exclusive signals to span all the possible values of these parameters.

[37] In the case of confidential non-exclusive policies, it is possible, though tedious, to compute the market beliefs associated with any rating process; and doing so it becomes possible to derive the optimal policy. However, in the case of public non-exclusive policies, such computations are not feasible and the restriction to ratings proportional to market beliefs is essential.

Lemma 5.1 *A scalar rating process Y is proportional to the market belief induced by a public or confidential rating policy with non-exclusive signals $S_1, \ldots, S_{K_n}$ if and only if for every $k = 1, \ldots, K_n$ and every $\Delta \geq 0$,*

$$\frac{\mathbf{Cov}[\theta_t, Y_t]}{\sqrt{\mathbf{Var}[\theta_t]}\sqrt{\mathbf{Var}[Y_t]}} \times \frac{\mathbf{Cov}[S_{k,t}, Y_{t+\Delta}]}{\sqrt{\mathbf{Var}[S_{k,t}]}\sqrt{\mathbf{Var}[Y_{t+\Delta}]}} = \frac{\mathbf{Cov}[S_{k,t}, \theta_{t+\Delta}]}{\sqrt{\mathbf{Var}[S_{k,t}]}\sqrt{\mathbf{Var}[\theta_{t+\Delta}]}}.$$

All proofs for this section are in the online appendix (Hörner and Lambert, 2015).

5.1.1 Private Non-Exclusive Ratings

Even if the intermediary does not disclose any information, the market nonetheless learns about the agent's type via the available signals (while the theorem below applies also to the case $K_0 = 0$, for the sake of discussion we focus on $K_0 \geq 1$). This motivates the definition of

$$\hat{\kappa} := \sqrt{1 + \gamma^2 \sum_{k=1}^{K_0} \frac{\beta_k^2}{\sigma_k^2}},$$

which is precisely the impulse response of the market's belief in the absence of any additional information.

We must introduce partial sums akin to $m_\alpha, m_\beta, m_{\alpha\beta}$, but applying to exclusive or non-exclusive signals only. Write m_k^n, m_k^e for these sums, *e.g.*,

$$m_\alpha^n := \sum_{k=1}^{K_0} \frac{\alpha_k^2}{\sigma_k^2}, \; m_\beta^n := \sum_{k=1}^{K_0} \frac{\beta_k^2}{\sigma_k^2}, \; m_{\alpha\beta}^n := \sum_{k=1}^{K_0} \frac{\alpha_k \beta_k}{\sigma_k^2},$$

and similarly

$$m_\alpha^e := \sum_{k=K_0+1}^{K} \frac{\alpha_k^2}{\sigma_k^2}, \; m_\beta^e := \sum_{k=K_0+1}^{K} \frac{\beta_k^2}{\sigma_k^2}, \; m_{\alpha\beta}^e := \sum_{k=K_0+1}^{K} \frac{\alpha_k \beta_k}{\sigma_k^2}.$$

We assume throughout that either $m_{\alpha\beta}^n \geq 0$ or $m_{\alpha\beta} \geq 0$, ensuring that positive effort can be achieved in equilibrium (by either disclosing no or all exclusive information).[38]

More generally, we add superscripts n, e (for non-exclusive and exclusive) whenever convenient, with the meaning being clear from the context.

[38]These assumptions are stronger than necessary. The theorem below delivers a rating system and therefore a value to the objective function. If this value is positive, the rating system is optimal. If it is negative, then the unique equilibrium has zero effort.

It turns out that Theorem 4.1 holds word for word, provided we redefine Δ. So, let

$$\lambda := (\kappa - 1)\left(\sqrt{r}(1+r)m_{\alpha\beta} + (\kappa^2 - r^2)\sqrt{\Delta}\right),$$

where

$$\Delta := \frac{(\kappa+1)(\hat{\kappa}+1)}{2(\kappa-\hat{\kappa})}\left[\frac{m_\alpha^e m_\beta^e}{\kappa^2-\hat{\kappa}^2} + \frac{(1+2r+\hat{\kappa})(m_{\alpha\beta}^n)^2}{(r+\hat{\kappa})^2(\hat{\kappa}+1)} - \frac{(1+2r+\kappa)m_{\alpha\beta}^2}{(r+\kappa)^2(\kappa+1)}\right].$$

With these slightly generalized formulas, we restate Theorem 4.1.

Theorem 5.2 *The optimal rating system is unique. It is given by (for $k \geq K_0$)*

$$u_k(t) = c_k \frac{\sqrt{r}}{\lambda} e^{-rt} + \frac{\beta_k}{\sigma_k^2} e^{-\kappa t},$$

with coefficients

$$c_k := (\kappa^2 - r^2) m_\beta \frac{\alpha_k}{\sigma_k^2} + (1-\kappa^2) m_{\alpha\beta} \frac{\beta_k}{\sigma_k^2}.$$

With non-exclusive signals, uniqueness is not simply up to a factor of proportionality: because there are signals that are observed by the market, there is some leeway in the intermediary's task: she can give a rather terse rating based on all the signals she is the only one to observe (signals $k \geq K_0$), and let the market derive the belief from the combination of this rating and the commonly observed ones. Or the intermediary spoon-feeds this belief to this market, committing to do so in a way that accounts for the availability of the first K_0 signals, so that the market may disregard them. In Theorem 5.2, we have opted for the first option. While the second option would be a more logical choice if we take beliefs as the "canonical message space," it also leads to a specification that is more ungainly, of the form

$$u_k(t) = c_k e^{-rt} + \frac{\beta_k}{\sigma_k^2} e^{-\kappa t} + d_k e^{-\hat{\kappa} t},$$

for some choice of c_k, d_k. (See our supplementary appendix.)

As is clear from the theorem, the rating system depends on the existence of non-exclusive signals, but it does not rely on the signal realizations of the non-exclusive signals, only on their characteristics. In this sense, it is not necessary for the intermediary to observe these non-exclusive signals, only to be aware of them.

To appreciate the role of non-exclusivity, let us revisit the example with signals that have identical parameters. That is, there are two signals, and $\alpha \coloneqq \alpha_1 = \alpha_2$, $\beta \coloneqq \beta_1 = \beta_2$, as well as $\sigma_1 = \sigma_2 =: \sigma$. Applying the formula, we get

$$u_2(t) = \frac{1}{\sigma^2}\left[\frac{1-r}{\frac{(\kappa-r)\sqrt{2(\kappa^2+1)r+\sqrt{2}\sqrt{\kappa^2+1}(r+1)^2+\kappa(r-1)^2+(r+1)^2}}{\sqrt{2}\sqrt{\kappa^2+1}+2r}+(\kappa-1)\sqrt{r}}\sqrt{r}e^{-rt}+e^{-\kappa t}\right],$$

to be compared with

$$\frac{1}{\sigma^2}\left[\frac{1-\sqrt{r}}{\kappa-\sqrt{r}}\sqrt{r}e^{-rt}+e^{-\kappa t}\right]$$

in the exclusive (confidential) case. Obviously, the expression is much more complicated in the non-exclusive case, but the two are not hard to compare nonetheless. In particular, the range of values (r, κ) over which the coefficient on the incentive term is positive is strictly larger with non-exclusive ratings (being negative if and only if r is in an interval of values that is a proper subset of $[\kappa, \kappa^2]$). This shouldn't come too much as a surprise: the rating system puts more emphasis on the incentive term to compensate the unbiased non-exclusive signal. Taken in isolation, the rating is less reliable than it would be absent the non-exclusive signal. In this sense, the quality of the information that is freely available, and the quality of the information provided by the rating are substitutes.

5.1.2 Public Non-Exclusive Ratings

Unlike in the confidential case, the generalization to non-exclusive ratings involve some significant changes in two ways. First, the case $K_0 = K - 1$ is very special: only one signal is exclusively observed by the intermediary, yet public ratings impose a "continuum" of constraints. While the intermediary retains some flexibility (she could disclose nothing, for instance), it turns out that the constraint is strong enough that transparency is optimal. Loosely speaking, with only one signal privately observed, and ratings that must be public, rating systems only differ in the amount of noise they add to the signal, and introducing noise is undesirable *per se*.

Second, when $K_0 = 1, \dots, K - 2$ (which presupposes at least three available signals, including the output), the impulse response on the incentive state is no longer $\sqrt{r}$, but rather some value $\delta > 0$ (which can be smaller or larger than $\sqrt{r}$) that solves an uninspiring polynomial equation of degree 6.

Theorem 5.3 *The public non-exclusive rating policy is unique and as follows:*

1. If $K_0 = 1, \ldots, K-2$, then, for signals $k \le K_0$,

$$u_k^n(t) = c^n \frac{\beta_k}{\sigma_k^2} e^{-\delta t} + \frac{\beta_k}{\sigma_k^2} e^{-\kappa t},$$

and for signals $k > K_0$,

$$u_k^e(t) = \left(c^e \frac{\beta_k}{\sigma_k^2} + d^e \frac{\alpha_k}{\sigma_k^2} \right) e^{-\delta t} + \frac{\beta_k}{\sigma_k^2} e^{-\kappa t},$$

for some constants c^n, c^e, d^e and $\delta > 0$ given in Appendix A.

2. If $K_0 = K - 1$, then, for all k,

$$u_k(t) = \frac{\beta_k}{\sigma_k^2} e^{-\kappa t}.$$

All the parameters of the rating system are elementary functions of δ, the root of the polynomial of degree 6. This polynomial is irreducible, and in fact admits no solution in terms of radicals. Nonetheless, it always admits exactly two positive roots, and we indicate how to pick the correct one in Appendix A.

Interestingly, and intuitively, the non-exclusive signals enter in the rating, but the coefficient α_k for $k \le K_0$ only matters via $m_\alpha, m_{\alpha\beta}$ and so does not enter the weight assigned to the incentive term corresponding to the k-th signal any differently than it does affect any other signal: because these signals are observed, the component due to equilibrium effort can be backed out by the market as well as it can by the intermediary, and so does not directly matter. Nonetheless, the intermediary does account for the fact that the agent's career concerns independently of the rating system, and it gets reflected in the optimal system.

5.2 Multiple Actions

Rating systems are often criticized not for providing insufficient incentives, but biasing incentives. That is to say, when actions have multiple dimensions, a poorly designed system might gear the agent's focus away from those actions that are most productive towards those that are effective at improving the rating.[39]

[39] Moral hazard takes many forms. In credit rating, for instance, both shirking and risk-shifting by the issuer are costly moral hazard activities that rating systems might encourage (see Langohr and Langohr, 2009, Ch.3).

Our model can be extended to tackle this issue. Suppose now that there is not one, but L effort levels a_ℓ, $\ell = 1, \ldots, L$, and assume that the cost of effort is additively separable, namely, with some abuse of notation,

$$c(a_1, \ldots, a_L) = \textstyle\sum_\ell c(a_\ell).$$

For a discussion of how restrictive this is, see Holmstrom and Milgrom (1991). In fact, let us further assume quadratic cost: $c(a_\ell) = ca_\ell^2$, $c > 0$. Signals are now defined by their law

$$\mathrm{d}S_{k,t} = (\textstyle\sum_\ell \alpha_{k,\ell} a_{\ell,t} + \beta_k \theta_t)\,\mathrm{d}t + \sigma_k \mathrm{d}W_{k,t},$$

for all $k = 1, \ldots, K$. We focus on confidential rating systems. The model is unchanged from Section 4.2 in all other respects. Assume that $\sum_\ell \alpha_{1,\ell} \neq 0$.

It turns out that the optimal rating system for multi-dimensional actions is equivalent to the one derived in Section 4.2 for a fictitious model with one-dimensional effort a, and coefficients

$$\alpha_k := \frac{\sum_\ell \alpha_{1,\ell}\alpha_{k,\ell}}{\sum_\ell \alpha_{1,\ell}},$$

where signals $\tilde{S}_k$ follow

$$\mathrm{d}\tilde{S}_{k,t} = (\alpha_k a_t + \beta_k \theta_t)\,\mathrm{d}t + \sigma_k \mathrm{d}W_{k,t},$$

for all $k = 1, \ldots, K$.

To illustrate how an optimal rating system might induce the agent to engage in unproductive activities, consider the following example. Output is only a function of effort a_1 ($\beta_1 = 0$); however, the signal S_2 reflects both effort a_2 and the agent's type; namely,

$$\mathrm{d}S_{1,t} = a_{1,t}\mathrm{d}t + \sigma_1 \mathrm{d}W_{1,t},$$

and

$$\mathrm{d}S_{2,t} = (a_{2,t} + \theta_t)\,\mathrm{d}t + \sigma_2 \mathrm{d}W_{2,t}.$$

Absent any rating system, if either only the first, or if both signals are observed, the unique equilibrium involves $a_\ell = 0$, $\ell = 1, 2$. Indeed, the action a_1 does not affect learning about the type, and while the action a_2 does, the type does not affect the expectation about output.

It is easy to derive the optimal (exclusive, confidential) rating system, given by

$$u_1(t) = \frac{\sqrt{r}}{\sigma_1} e^{-rt}, \quad u_2(t) = \frac{e^{-\kappa t}}{\sigma_2^2}.$$

The signal that is irrelevant for learning does not get discarded, but rather exclusively assigned to the incentive term; conversely, the signal that matters for learning only matters for the learning term. This leads to positive effort on both dimensions, namely,

$$c'(a_1) = \frac{\kappa - 1}{4\sqrt{r}\sigma_1}, \quad c'(a_2) = \frac{\kappa - 1}{2(r+\kappa)\sigma_2^2},$$

and conditional market (belief) variance

$$\frac{1}{4}(\kappa - 1)^2\sigma_2^2.$$

Unproductive effort in the unobservable dimension that affects learning is the price to pay for effort in the productive activity.

6 Performance of Standard Policies

So far, we have focussed on the best rating policy, establishing that it is a simple mixture rating system. Nonetheless, many systems do not use mixtures. Here, we illustrate how our methods also allow us to compare some standard policies that are used in practice. As mentioned in Section 3, exponential smoothing and moving windows are two systems that are commonly implemented. We argue that a properly calibrated exponential smoothing rating system outperforms any moving window rating system. For simplicity, we focus on private exclusive ratings with just one additional signal (simply denoted S, or S_t at time t).

Formally, in the case of exponential smoothing, the intermediary releases signal

$$Y_t = \int_{-\infty}^{t} e^{-\lambda(t-j)}(c\,\mathrm{d}X_j + (1-c)\,\mathrm{d}S_j)$$

at time t, where $\lambda > 0$ is the coefficient of smoothing, and c is the relative weight put on the output. With a moving window, the intermediary releases a signal

$$Y_t = \int_{t-\Delta}^{t} c\,\mathrm{d}X_j + (1-c)\,\mathrm{d}S_j,$$

where $\Delta > 0$ is the size of the moving window, and c is the relative weight put on the output. The *optimal* exponential smoothing (resp., moving window) system is defined by the choice of (c, λ) (resp., (c, Δ)) such that equilibrium effort is maximum. It is not hard to show the following.

Lemma 6.1 *The optimal exponential smoothing system yields higher effort than any moving window system.*

In fact, the proof shows something stronger: for any given weight c, the best rating system using exponential smoothing with that weight on the output outperforms the best moving window under the same constraint.

Add discussion of periodic reviews.

7 Concluding Comments

Our analysis makes several restrictive assumptions.

First, we have assumed that effort and ability are substitutes. While this follows Holmström (1999) and most of the literature on career concerns, it is restrictive, as the analysis of Dewatripont, Jewitt and Tirole (1999) makes clear. Building on the recent work of Cisternas (2015), it might be possible to extend the analysis to technologies for which effort and ability would be complements.

Second, we have restricted attention to stationary systems. This is largely motivated by tractability. As pointed out, we do not know how to extend such an elementary building block as the representation lemma (Lemma 3.2) to non-stationary Gaussian processes, an obvious prerequisite to an analysis of the non-stationary case.

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A Missing Formulas for Theorem 5.3

The missing formulas for Theorem 5.3 are (writing n_k instead of m_k^n, for concision),

$$c^n := -\frac{\kappa-1}{\delta-1} - \frac{\Lambda_1 R_\beta(\delta+\kappa-r-1)}{z(\delta-1)\left(\delta+\kappa-R_\beta\right)},$$

$$c^e := \frac{(\delta-r)\left(m_{\alpha\beta}R_\beta+z\right)}{(r-\kappa)z}, \quad d^e := \frac{(\delta-r)(\kappa+r)m_\beta R_\beta}{\left(\kappa^2-1\right)z},$$

where

$$\begin{aligned}
\Lambda_1 &:= \frac{\lambda_1(\kappa+r)\left(\left(1-\delta^2\right)m_\beta+\left(\kappa^2-1\right)n_\beta\right)}{(\delta-1)m_\beta(r-\delta)},\\
R_\beta &:= \frac{(\kappa-1)\left((\delta-1)(r+1)m_\beta+(\kappa+1)n_\beta(r+1-\delta-\kappa)\right)}{(\delta-1)m_\beta(r-\delta)},\\
z &:= \frac{m_{\alpha\beta}\left(\left(r^2-1\right)m_\beta-\left(\kappa^2-1\right)n_\beta\right)}{(\delta-\kappa)m_\beta}\\
&\quad+\frac{\left(r^2-\kappa^2\right)\left(\left(\kappa^2-1\right)\lambda_1 n_\beta-(\delta-1)m_\beta\left((\delta+1)\lambda_1+n_{\alpha\beta}(r-\delta)\right)\right)}{(\delta-1)(\delta-\kappa)m_\beta(r-\delta)},
\end{aligned}$$

in terms of λ_1 and δ.

The parameter λ_1 is a function of δ, and we accordingly write $\lambda_1(\delta)$ when convenient. It holds that

$$\lambda_1 = \frac{(r-\delta)\left((\kappa-1)\sigma_\beta\left(r(\delta+\kappa+1)-\delta^2\right)+(\delta+\kappa)\left(\delta^2-\kappa r\right)\right)(A_1+A_2)}{(1-\kappa)\sigma_\beta D_1+\sigma_{\alpha\beta}(\kappa+r)D_2},$$

where

$$\begin{aligned}
A_1 &= \left(\kappa^2-1\right)m_{\alpha\beta}^2\left((\delta+\kappa)^2-(\kappa+1)\sigma_\beta(2\delta+\kappa-1)\right)\left(\left(\kappa^2-1\right)\sigma_\beta+2\sigma_{\alpha\beta}\left(r^2-\kappa^2\right)\right),\\
A_2 &= (\kappa+r)^2\left(x^2\sigma_\alpha m_\alpha m_\beta-(\kappa+1)\sigma_{\alpha\beta}^2 m_{\alpha\beta}^2((\delta-1)(\delta+r)(r-\kappa)+x(\delta+\kappa-r-1))\right),
\end{aligned}$$

with

$$x := (\kappa+1)\sigma_\beta(\delta+\kappa-r-1)+(\delta+\kappa)(r-\kappa);$$

The expressions for D_1 and D_2 are somewhat unwieldy, unfortunately. It holds that

$$
\begin{aligned}
D_1 = {} & (\kappa-1)(\kappa+1)^2\sigma_\beta^2\left(\delta^4-r^4-2r^3+2r^2\left(2\delta^2+2\delta\kappa+\kappa^2-1\right)-2\delta^2 r(2\delta+2\kappa-1)\right)\\
& -(\kappa+1)\sigma_\beta(\delta+\kappa)\left(\delta^3\left(\delta^2+3\delta\kappa+\kappa-1\right)+r^4(\delta-2\kappa+1)+r^3(-\delta(\kappa-3)-3\kappa+1)\right.\\
& \qquad +r^2\left(-2\delta^3+\delta^2(3\kappa-1)+\delta\left(4\kappa^2-\kappa+1\right)+4\kappa\left(\kappa^2-1\right)\right)\\
& \qquad \left.+\delta^2 r\left(-3\delta(\kappa+1)-8\kappa^2+3\kappa+3\right)\right)\\
& +(\delta+\kappa)^2\left(\delta^3(2\delta\kappa+\delta+\kappa)+r^4(\delta-\kappa)+(\delta+1)r^3(\delta-\kappa)\right.\\
& \qquad \left.+r^2\left(-\delta^3+\delta^2(\kappa+1)-\delta\kappa+2\kappa^2(\kappa+1)\right)-\delta^2 r\left(\delta^2-\delta\kappa+\delta+\kappa(4\kappa+3)\right)\right),
\end{aligned}
$$

and

$$
\begin{aligned}
D_2 = {} & \left(\kappa^2-1\right)\sigma_\beta^2\left((\delta-1)\delta^3(\kappa-1)+r^3\left(-\left(2\delta^2+3\delta\kappa+\delta+2\kappa^2+\kappa-1\right)\right)\right.\\
& \qquad +r^2\left(4\delta^3+\delta^2(7\kappa+1)+\delta\left(4\kappa^2+\kappa-1\right)+2\kappa\left(\kappa^2-1\right)\right)\\
& \qquad \left.+\delta^2 r\left(-2\delta^2-5\delta\kappa+\delta-4\kappa^2+\kappa+1\right)\right)\\
& +\sigma_\beta(\delta+\kappa)\left(-\delta^3\left(\delta^2(\kappa+1)+\delta\left(3\kappa^2-1\right)+(1-\kappa)\kappa\right)\right.\\
& \qquad +r^3\left(\delta^2(\kappa-1)+\delta\left(3\kappa^2-1\right)+\kappa\left(4\kappa^2+\kappa-3\right)\right)\\
& \qquad +r^2\left(\delta^3(1-3\kappa)+\delta^2\left(3-9\kappa^2\right)-\delta\kappa\left(4\kappa^2+\kappa-1\right)-4\kappa^2\left(\kappa^2-1\right)\right)\\
& \left.+\delta^2 r\left(\delta^2(3\kappa+1)+5\delta\kappa^2+\delta+\kappa\left(8\kappa^2-\kappa-5\right)\right)\right)-2(\delta+\kappa)^2(r-\kappa)\left(\delta^2-\kappa r\right)^2.
\end{aligned}
$$

Finally, regarding δ, consider the polynomial

$$
\tilde{P}(z)=b_0+b_1 z+b_2 z^2+b_3 z^3+b_4 z^4+b_5 z^5+z^6,
$$

with

$$
\begin{aligned}
b_0 &\coloneqq \zeta\left(\zeta+\psi g_{\alpha\beta}\right),\\
b_1 &\coloneqq \zeta\left(2\eta_\beta+g_{\alpha\beta}\right),\\
b_2 &\coloneqq \frac{1}{2}\left(-2\eta_\beta\left(2\zeta-\eta_\beta\right)-g_{\alpha\beta}\left((4\psi-1)\eta_\beta+\psi\right)-|g_{\alpha\beta}|\sqrt{\zeta+\psi^2-2\psi\eta_\beta}\right),\\
b_3 &\coloneqq -2\left(\eta_\beta^2+\zeta\right)-g_{\alpha\beta}\left(\eta_\beta+\psi\right)-|g_{\alpha\beta}|\sqrt{\zeta+\psi^2-2\psi\eta_\beta},\\
b_4 &\coloneqq \frac{1}{2}\left(2\left(\eta_\beta-2\right)\eta_\beta+g_{\alpha\beta}\left(\eta_\beta+\psi\right)-|g_{\alpha\beta}|\sqrt{\zeta+\psi^2-2\psi\eta_\beta}\right),\\
b_5 &\coloneqq 2\eta_\beta+g_{\alpha\beta},
\end{aligned}
$$

where $\sigma_\beta = 1 - n_\beta/m_\beta$, $\sigma_\alpha = 1 - n_\alpha/m_\alpha$, $\sigma_{\alpha\beta} = 1 - n_{\alpha\beta}/m_{\alpha\beta}$ and

$$\begin{aligned}
\eta_\beta &:= \frac{\kappa(1-\sigma_\beta)+\sigma_\beta}{r}, \quad \zeta := \frac{\kappa^2(1-\sigma_\beta)+\sigma_\beta}{r^2}, \\
g_{\alpha\beta} &:= \frac{2(\kappa-1)(r+1)^2\chi(\chi+1)m_{\alpha\beta}^2}{r\left(\sigma_\alpha m_\alpha m_\beta(\kappa+r)^2+(\kappa-1)m_{\alpha\beta}^2\left(2(r+1)\chi-(\kappa-1)\sigma_\beta\right)\right)}, \\
\psi &:= \frac{(\kappa-1)\sigma_\beta+\chi(\kappa(\chi+2)+\chi)}{2r\chi(\chi+1)}, \quad \chi := \frac{(\kappa-1)\sigma_\beta-\sigma_{\alpha\beta}(\kappa+r)}{r+1}.
\end{aligned}$$

In the supplementary appendix, we prove

Lemma A.1 *The polynomial $\tilde{P}$ is irreducible and admits no solutions in terms of radicals. It has exactly two positive distinct roots $\tilde{\delta}_-, \tilde{\delta}_+$. Let $\delta_- = r\tilde{\delta}_-, \delta_+ = r\tilde{\delta}_+$. It holds that either $(\delta_-^2 - r)\lambda_1(\delta_-) < 0$ or $(\delta_+^2 - r)\lambda_1(\delta_+) < 0$, but not both. The parameter δ is equal to δ_- if $(\delta_-^2 - r)\lambda_1(\delta_-) < 0$, and to δ_+ otherwise.*

B Some Useful Mathematical Results

Throughout this section, we fix the integers $N \geq 1$, $K \geq 0$, $M \geq 1$, $\ell \geq 1$. We let $F : \mathbb{R}_+^N \times \mathbb{R}^{K\times M} \to \mathbb{R}$, and $G_j : \Omega \times \mathbb{R}^{K\times M} \to \mathbb{R}$, $j = 1, \ldots, \ell$ be piecewise differentiable functions. For all $k \leq K$, we define $\phi_k : \Omega \to \mathbb{R}_+$ as a shifted projection in the following sense: for every k, we require that $\phi_k = x_i + \Delta$ for some i and $\Delta \geq 0$.

First, we consider the problem of maximizing

$$\int_{\mathbb{R}_+^N} F(x, \mathbf{u}(\phi_1(x)), \mathbf{u}(\phi_2(x)), \ldots, \mathbf{u}(\phi_K(x)))\, \mathrm{d}x \tag{11}$$

with respect to $\mathbf{u} : \mathbb{R}_+ \to \mathbb{R}^M$.

We define $F_{i,k}(x, (y_{1,1}; \ldots; y_{M,1}), \ldots, (y_{1,K}; \ldots; y_{M,K}))$ as

$$\frac{\partial F(x, (y_{1,1}; \ldots; y_{M,1}), \ldots, (y_{1,K}; \ldots; y_{M,K}))}{\partial y_{i,k}}.$$

The following result is an extension of the Euler-Lagrange first-order conditions adapted to this problem.

Proposition B.1 *Suppose $\mathbf{u}^*$ is a solution to the optimization problem* (11) *with no* *constraints. Then*

$$\sum_k \int_{\mathbb{R}_+^N \cap \{\phi_k = t\}} F_{i,k}(x, \mathbf{u}^*(\phi_1(x)), \mathbf{u}^*(\phi_2(x)), \ldots, \mathbf{u}(\phi_K(x)))\,\mathrm{d}x = 0,$$

where we observe that $\mathbb{R}_+^N \cap \{\phi_k = t\}$ *is a rectangle of* $\mathbb{R}_+^N$*, which may be of smaller dimension, and the integral is taken with respect to the associated Lebesgue measure.*

Second, we consider the problem of maximizing (11) with respect to $\mathbf{u} : \mathbb{R}_+ \to \mathbb{R}^M$ and subject to the constraints

$$\int_{\mathbb{R}_+^N} G_j(x, \mathbf{u}(\phi_1(x)), \ldots, \mathbf{u}(\phi_K(x)))\,\mathrm{d}x = 0, \tag{12}$$

for $j = 1, \ldots, \ell$.

Proposition B.2 *Suppose* $\mathbf{u}^*$ *is a solution to the optimization problem* (11) *subject to the constraints* (12)*. Then there exist* $\lambda_0, \lambda_1, \ldots, \lambda_\ell$*, such that* $\mathbf{u}^*$ *maximizes*

$$\int_{\mathbb{R}_+^N} L(x, \mathbf{u}(\phi_1(x)), \ldots, \mathbf{u}(\phi_K(x)))\,\mathrm{d}x,$$

where $L \coloneqq \lambda_0 F + \sum_{j=1}^{\ell} \lambda_j G_j$*. In addition, if* $\mathbf{u}^*$ *is not an extremal of* $\sum_{j=1}^{\ell} \lambda_j G_j$*, then we can choose* $\lambda_0 = 1$ *without loss of generality.*

The second proposition is an extension of the multiplier theorem for isoperimetric problems in the calculus of variations, and it can be proved in a similar fashion (see, for example, Burns, 2014).

Proof of Proposition B.1.

For $\mathbf{u} : \mathbb{R}_+ \to \mathbb{R}^M$, let

$$J(\mathbf{u}) \coloneqq \int_\Omega F(x, \mathbf{u}(\phi_1(x)), \mathbf{u}(\phi_2(x)), \ldots, \mathbf{u}(\phi_K(x)))\,\mathrm{d}x.$$

Suppose that $J(\mathbf{u})$ is maximized for $\mathbf{u} = \mathbf{u}^*$. Then, it is necessary that for all i and all t,

$$\sum_k \int_{\mathbb{R}_+^N \cap \{\phi_k = t\}} F_{i,k}(x, \mathbf{u}^*(\phi_1(x)), \mathbf{u}^*(\phi_2(x)), \ldots, \mathbf{u}(\phi_K(x)))\,\mathrm{d}x = 0,$$

where each integral over a set of the form $\mathbb{R}_+^N \cap \{\phi_k = t\}$ is taken with respect to the Lebesgue measure on that set.

The proof relies on a classic variational argument. Fix i and let $\mathbf{v} : \mathbb{R}_+ \to \mathbb{R}^M$, where $\mathbf{v} = (v_1, \ldots, v_M)$ and $v_j = 0$ for $j \neq i$. Let $j(\epsilon) \coloneqq J(\mathbf{u}^* + \epsilon \mathbf{v})$. We observe that j is maximized at $\epsilon = 0$, thus $j'(0) = 0$.

We differentiate under the integral sign as allowed by the regularity conditions imposed on F, and we get

$$j'(0) = \int_{\mathbb{R}_+^N} \sum_k F_{i,k} F(x, \mathbf{u}^*(\phi_1(x)), \ldots, \mathbf{u}^*(\phi_K(x)))\eta_i(\phi_k(x))\,\mathrm{d}x.$$

Now suppose by contradiction that for some t,

$$\sum_k \int_{\mathbb{R}_+^N \cap \{\phi_k = t\}} F_{i,k}(x, \mathbf{u}^*(\phi_1(x)), \mathbf{u}^*(\phi_2(x)), \ldots, \mathbf{u}(\phi_K(x)))\,\mathrm{d}x$$

is non-zero. For example, suppose that value is positive. Observe that each integral of the sum is continuous with respect to t, and so by continuity,

$$\sum_k \int_{\mathbb{R}_+^N \cap \{\phi_k = t_0\}} F_{i,k}(x, \mathbf{u}^*(\phi_1(x)), \mathbf{u}^*(\phi_2(x)), \ldots, \mathbf{u}(\phi_K(x)))\,\mathrm{d}x$$

is positive on any small enough interval of t_0's that includes t. Let I_t be such an interval, and let η_i be a function that is zero outside of I_t and one inside of I_t (the value at the boundary is irrelevant).

We have that

$$\begin{aligned} 0 \neq \sum_k \int_{\Omega \cap \{\phi_k \in I_t\}} & F_{i,k}(x, \mathbf{u}^*(\phi_1(x)), \mathbf{u}^*(\phi_2(x)), \ldots, \mathbf{u}(\phi_K(x)))\,\mathrm{d}x \\ &= \sum_k \int_\Omega F_{i,k}(x, \mathbf{u}^*(\phi_1(x)), \mathbf{u}^*(\phi_2(x)), \ldots, \mathbf{u}(\phi_K(x)))\eta_i(\phi_k(x))\,\mathrm{d}x, \end{aligned}$$

which contradicts $j'(0) = 0$. ■

C Proofs

C.1 Proof of Lemma 2.2 and Lemma 3.6

If the cumulative payment process satisfies the zero-profit condition, then the agent who chooses effort strategy $A = \{A_t\}$ makes (*ex ante*) payoff

$$\mathbf{E}\left[\int_0^\infty (a^* + \mu_t - c(A_t))\, e^{-rt}\,\mathrm{d}t\right],$$

with μ_t the market's belief about the agent's type

$$\mu_t := \mathbf{E}^*[\theta_t|\mathcal{F}_t],$$

where a^* denotes the market's conjectured (stationary) effort level.

Observe that the agent has no impact on the market's conjectured effort level. Thus, the agent's strategy is optimal if and only if it maximizes

$$\mathbf{E}\left[\int_0^\infty (\mu_t - c(A_t))\, e^{-rt}\,\mathrm{d}t\right].$$

C.2 Proof of Theorem 2.3

Let $\mu_t := \mathbf{E}^*[\theta_t \mid \mathcal{F}_t]$ and recall that $\Sigma = \mathbf{Var}[\theta_t \mid \mathcal{F}_t]$ which by assumption is constant.

We prove that given a cumulative payment process that satisfies the zero-profit condition, there exists an optimal effort strategy for the agent; that it is unique and pinned down by the first-order condition given in Theorem 2.3. This, in turn, yields existence of a unique equilibrium.

Let us fix a cumulative payment process that satisfies the zero-profit condition, and suppose that the agent follows effort strategy $A = \{A_t\}$. The agent's time-0 (*ex post*) payoff is then

$$\int_0^\infty [a^* + \mu_t - c(A_t)]\, e^{-rt}\,\mathrm{d}t, \tag{13}$$

where a^* is the effort level conjectured by the market. Maximizing the agent's *ex ante* payoff is equivalent to maximizing the agent's *ex post* payoff, up to probability zero events of information realization. Hence, we seek conditions on A_t so as to maximize

(13), which is equivalent to maximizing

$$\int_0^\infty \left[\mu_t - c(A_t)\right] e^{-rt}\,\mathrm{d}t. \tag{14}$$

By standard linear filtering arguments,

$$\begin{aligned}
\mathrm{d}\mu_t &= -\mu_t\,\mathrm{d}t + \Sigma \sum_{k=1}^{K} \frac{\beta_k}{\sigma_k^2}\left[\mathrm{d}S_{k,t} - \alpha_k a^*\,\mathrm{d}t - \beta_k \mu_t\,\mathrm{d}t\right] \\
&= -\left(1 + \Sigma m_\beta\right)\mu_t\,\mathrm{d}t + \Sigma \sum_{k=1}^{K} \frac{\beta_k}{\sigma_k^2}\left[\mathrm{d}S_{k,t} - \alpha_k a^*\,\mathrm{d}t\right].
\end{aligned}$$

After integration:

$$\mu_t = \Sigma \sum_{k=1}^{K} \frac{\beta_k}{\sigma_k^2} \int_0^t e^{-(t-s)\left(1+\Sigma m_\beta\right)}\left[\mathrm{d}S_{k,s} - \alpha_k a^*\,\mathrm{d}s\right].$$

As

$$\mathrm{d}S_{k,s} = \left(\alpha_k A_s + \beta_k \theta_s\right)\mathrm{d}s + \sigma_k\,\mathrm{d}W_{k,s},$$

maximizing (14) is the same as maximizing

$$\int_0^\infty \int_0^t \left[\Sigma \sum_{k=1}^{K} \frac{\beta_k}{\sigma_k^2}\alpha_k A_s\right] e^{-(t-s)\left(1+\Sigma m_\beta\right)} e^{-rt}\,\mathrm{d}s\,\mathrm{d}t - \int_0^\infty c(A_t)e^{-rt}\,\mathrm{d}t.$$

Let us rewrite

$$\int_0^\infty \int_0^t \left[\Sigma \sum_{k=1}^{K} \frac{\beta_k}{\sigma_k^2}\alpha_k A_s\right] e^{-(t-s)\left(1+\Sigma m_\beta\right)} e^{-rt}\,\mathrm{d}s\,\mathrm{d}t,$$

as

$$\int_0^\infty \int_0^\infty 1_{s\leq t} \left[\Sigma \sum_{k=1}^K \frac{\beta_k}{\sigma_k^2}\alpha_k A_s\right] e^{-(t-s)\left(1+\Sigma m_\beta\right)} e^{-rt}\,\mathrm{d}s\,\mathrm{d}t$$

$$= \int_0^\infty \int_s^\infty \left[\Sigma \sum_{k=1}^K \frac{\beta_k}{\sigma_k^2}\alpha_k A_s\right] e^{-(t-s)\left(1+\Sigma m_\beta\right)} e^{-rt}\,\mathrm{d}t\,\mathrm{d}s$$

$$= \int_0^\infty A_s e^{-rs} \left[\int_s^\infty \left[\Sigma \sum_{k=1}^K \frac{\beta_k}{\sigma_k^2}\alpha_k\right] e^{-(t-s)\left(1+\Sigma m_\beta\right)} e^{-r(t-s)}\right] \mathrm{d}t\,\mathrm{d}s$$

$$= \int_0^\infty A_s e^{-rs} \left[\int_0^\infty \Sigma m_{\alpha\beta} e^{-\left(1+r+\Sigma m_\beta\right)\Delta}\right] \mathrm{d}\Delta\,\mathrm{d}s.$$

Thus maximizing (13) is equivalent (up to measure zero sets) to maximizing

$$\int_0^\infty A_s e^{-rs} \left[\int_0^\infty \Sigma m_{\alpha\beta} e^{-\left(1+r+\Sigma m_\beta\right)\Delta}\right] \mathrm{d}\Delta\,\mathrm{d}s - \int_0^\infty c(A_t)e^{-rt}\,\mathrm{d}t,$$

which is equivalent to maximizing

$$A_s \left[\int_0^\infty \Sigma m_{\alpha\beta} e^{-\left(1+r+\Sigma m_\beta\right)\Delta}\right] \mathrm{d}\Delta - c(A_s) = A_s \frac{\Sigma m_{\alpha\beta}}{1+\Sigma m_\beta + r} - c(A_s), \tag{15}$$

for (almost) every s. By strict concavity, (15) is maximized if and only if

$$c'(A_t) = \frac{\Sigma m_{\alpha\beta}}{1+\Sigma m_\beta + r} = \frac{\Sigma m_{\alpha\beta}}{\kappa^2 + r}. \tag{16}$$

Therefore the agent wants to choose an effort level that is constant over time and determined by (16).

C.3 Proof of Lemma 2.4

We note that θ_t and μ_t are jointly normal, and as μ_t is the market belief, $\mathbf{Cov}[\theta_t, \mu_t] = \mathbf{Var}[\mu_t]$, so applying the projection formulas:

$$\begin{aligned}\mathbf{Var}[\theta_t \mid \mu_t] &= \mathbf{Var}[\theta_t] - \frac{\mathbf{Cov}[\theta_t, \mu_t]^2}{\mathbf{Var}[\mu_t]} \\ &= \frac{\gamma^2}{2} - \mathbf{Var}[\mu_t].\end{aligned}$$

C.4 Proof of Lemma 3.2

We prove the lemma for the case of Y scalar, without loss of generality.

Necessity. We start by showing that if Y has the linear representation stated in the lemma, then the associated $\{u_k\}_{k=1}^K$ are uniquely determined, conditional on some regularity conditions. Recall that, as defined in Lemma 3.2, $f_k(\Delta) = \mathbf{Cov}[Y_t, S_{k,t-\Delta}]$. Computing,

$$\begin{aligned} f_k(\Delta) &= \sum_{i=1}^{K} \int_0^\infty u_i(s)\,\mathbf{Cov}[\mathrm{d}S_{i,t-s}, S_{k,t-\Delta}] \\ &= \sigma_k^2 \int_\Delta^t u_k(s)\,\mathrm{d}s + \frac{\beta_k\gamma^2}{2} \int_0^\infty \int_0^{t-\Delta} U(s) e^{-|t-s-j|}\,\mathrm{d}j\,\mathrm{d}s, \end{aligned}$$

where $U(t) \coloneqq \sum_{k=1}^K \beta_k u_k(t)$. Then:

$$\begin{aligned} f_k'(\Delta) &= -\sigma_k^2 u_k(\Delta) - \frac{\beta_k\gamma^2}{2} \int_0^\infty U(s) e^{-|\Delta-s|}\,\mathrm{d}s, \\ f_k''(\Delta) &= -\sigma_k^2 u_k'(\Delta) + \frac{\beta_k\gamma^2}{2} \int_0^\Delta U(s) e^{-(\Delta-s)}\,\mathrm{d}s - \frac{\beta_k\gamma^2}{2} \int_\Delta^\infty U(s) e^{+(\Delta-s)}\,\mathrm{d}s, \\ f_k'''(\Delta) &= -\sigma_k^2 u_k''(\Delta) + \beta_k\gamma^2 U(\Delta) - \frac{\beta_k\gamma^2}{2} \int_0^\Delta U(s) e^{-(\Delta-s)}\,\mathrm{d}s - \frac{\beta_k\gamma^2}{2} \int_\Delta^\infty U(s) e^{+(\Delta-s)}\,\mathrm{d}s, \end{aligned}$$

and so

$$f_k'(\Delta) - f_k'''(\Delta) = \sigma_k^2 u''(\Delta) - \sigma_k^2 u(\Delta) - \beta_k\gamma^2 U(\Delta). \tag{17}$$

Let us multiply (17) by β_k/σ_k^2 and sum over k. We get an ODE in U:

$$F'(\Delta) - F'''(\Delta) = U''(\Delta) - U(\Delta) - \gamma^2 m_\beta U(\Delta) = U''(\Delta) - \kappa U(\Delta).$$

After integrating by parts, the general solution to this ordinary differential equation can be written as

$$U(\Delta) = C_1 e^{\kappa\Delta} + C_2 e^{-\kappa\Delta} - F'(\Delta) - \frac{\kappa^2 - 1}{\kappa} \int_0^\Delta \sinh(\kappa(\Delta - s)) F'(s)\,\mathrm{d}s,$$

for some constants C_1 and C_2. Additionally, C_1 and C_2 should be chosen such that the following equation

$$F'(\Delta) = -U(\Delta) - \frac{\kappa^2 - 1}{2} \int_0^\infty U(s) e^{-|\Delta - s|} \, \mathrm{d}s$$

is satisfied for every $\Delta \geq 0$.

For a start, we will work as if $U(s) = 0$ if $s \geq A$, for A "large" and then we will send A to infinity. We have:

$$-U(\Delta) = F'(\Delta) - C_1 e^{\kappa\Delta} - C_2 e^{-\kappa\Delta} + \frac{\kappa^2 - 1}{\kappa} \int_0^\Delta \sinh(\kappa(\Delta - s)) F'(s) \, \mathrm{d}s,$$

and

$$\begin{aligned}\int_0^A U(s) e^{-|\Delta - s|} \, \mathrm{d}s = C_1 \int_0^A e^{\kappa s} e^{-|\Delta - s|} \, \mathrm{d}s + C_2 \int_0^A e^{-\kappa s} e^{-|\Delta - s|} \, \mathrm{d}s - \int_0^A F'(s) e^{-|\Delta - s|} \, \mathrm{d}s \\ - \frac{\kappa^2 - 1}{\kappa} \int_0^A \int_0^s \sinh(\kappa(s - j)) F'(j) \, \mathrm{d}j \, \mathrm{d}s.\end{aligned}$$

Then we compute

$$\begin{aligned}C_1 \int_0^A e^{+\kappa s} e^{-|\Delta - s|} \, \mathrm{d}s &= C_1 \left[\frac{e^{A(\kappa - 1) + \Delta}}{\kappa - 1} - \frac{e^{-\Delta}}{\kappa + 1} + \frac{e^{\kappa\Delta}}{\kappa + 1} - \frac{e^{\kappa\Delta}}{\kappa - 1} \right], \\ C_2 \int_0^A e^{-\kappa s} e^{-|\Delta - s|} \, \mathrm{d}s &= C_2 \left[-\frac{e^{-A(\kappa + 1) + \Delta}}{\kappa + 1} + \frac{e^{-\Delta}}{\kappa - 1} + \frac{e^{-\kappa\Delta}}{\kappa + 1} - \frac{e^{-\kappa\Delta}}{\kappa - 1} \right].\end{aligned}$$

Thus:

$$
\begin{aligned}
&\int_0^A \int_0^s \sinh(\kappa(s-j))F'(j)e^{-|\Delta-s|}\,\mathrm{d}j\,\mathrm{d}s \\
&\qquad = \int_0^A \int_0^A 1_{j\leq s} \sinh(\kappa(s-j))F'(j)e^{-|\Delta-s|}\,\mathrm{d}j\,\mathrm{d}s \\
&\qquad = \int_0^A F'(j) \int_j^A \sinh(\kappa(s-j))e^{-|\Delta-s|}\,\mathrm{d}s\,\mathrm{d}j \\
&\qquad = \int_0^\Delta F'(j) \int_j^\Delta \sinh(\kappa(s-j))e^{-|\Delta-s|}\,\mathrm{d}s\,\mathrm{d}j \\
&\qquad\qquad + \int_0^\Delta F'(j) \int_\Delta^A \sinh(\kappa(s-j))e^{-|\Delta-s|}\,\mathrm{d}s\,\mathrm{d}j \\
&\qquad\qquad + \int_\Delta^A F'(j) \int_j^A \sinh(\kappa(s-j))e^{-|\Delta-s|}\,\mathrm{d}s\,\mathrm{d}j \\
&\qquad = -\frac{\kappa}{\kappa^2-1} \int_0^A F'(j)e^{-|\Delta-j|}\,\mathrm{d}j \\
&\qquad\qquad + \frac{e^{-A+\Delta}}{\kappa^2-1} \int_0^A F'(j)\left[\kappa\cosh(\kappa(A-j)) + \sinh(\kappa(A-j))\right]\mathrm{d}j \\
&\qquad\qquad - \frac{2}{\kappa^2-1} \int_0^\Delta \sinh(\kappa(\Delta-j))F'(j)\,\mathrm{d}j.
\end{aligned}
$$

We want to determine C_1 and C_2 such that the following equality holds:

$$
\begin{aligned}
F'(\Delta) = {} & F'(\Delta) - \sigma^2 C_1 e^{\kappa\Delta} - \sigma^2 C_2 e^{-\kappa\Delta} \\
& + \frac{\kappa^2-1}{\kappa}\int_0^\Delta \sinh(\kappa(\Delta-s))F'(s)\,\mathrm{d}s \\
& - \frac{\kappa^2-1}{2}C_1\left[\frac{e^{A(\kappa-1)+\Delta}}{\kappa-1} - \frac{e^{-\Delta}}{\kappa+1} + \frac{e^{\kappa\Delta}}{\kappa+1} - \frac{e^{\kappa\Delta}}{\kappa-1}\right] \\
& - \frac{\kappa^2-1}{2}C_2\left[-\frac{e^{-A(\kappa+1)+\Delta}}{\kappa+1} + \frac{e^{-\Delta}}{\kappa-1} + \frac{e^{-\kappa\Delta}}{\kappa+1} - \frac{e^{-\kappa\Delta}}{\kappa-1}\right] \\
& + \frac{\kappa^2-1}{2}\int_0^A F'(s)e^{-|\Delta-s|}\,\mathrm{d}s \\
& - \frac{\kappa^2-1}{2}\frac{\kappa^2-1}{\kappa}\frac{\kappa}{\kappa^2-1}\int_0^A F'(j)e^{-|\Delta-j|}\,\mathrm{d}j \\
& + \frac{\kappa^2-1}{2}\frac{\kappa^2-1}{\kappa}\frac{e^{-A+\Delta}}{\kappa^2-1}\int_0^A F'(j)\left[\kappa\cosh(\kappa(A-j)) + \sinh(\kappa(A-j))\right]\mathrm{d}j \\
& - \frac{\kappa^2-1}{2}\frac{\kappa^2-1}{\kappa}\frac{2}{\kappa^2-1}\int_0^\Delta \sinh(\kappa(\Delta-j))F'(j)\,\mathrm{d}j.
\end{aligned}
\tag{18}
$$

After simplification the last equation becomes:

$$
\begin{aligned}
F'(\Delta) = {} & F'(\Delta) - C_1 e^{\kappa\Delta} - C_2 e^{-\kappa\Delta} \\
& - \frac{\kappa^2-1}{2}C_1\left[\frac{e^{A(\kappa-1)+\Delta}}{\kappa-1} - \frac{e^{-\Delta}}{\kappa+1} + \frac{e^{\kappa\Delta}}{\kappa+1} - \frac{e^{\kappa\Delta}}{\kappa-1}\right] \\
& - \frac{\kappa^2-1}{2}C_2\left[-\frac{e^{-A(\kappa+1)+\Delta}}{\kappa+1} + \frac{e^{-\Delta}}{\kappa-1} + \frac{e^{-\kappa\Delta}}{\kappa+1} - \frac{e^{-\kappa\Delta}}{\kappa-1}\right] \\
& + \frac{\kappa^2-1}{2}\frac{\kappa^2-1}{\kappa}\frac{e^{-A+\Delta}}{\kappa^2-1}\int_0^A F'(j)\left[\kappa\cosh(\kappa(A-j)) + \sinh(\kappa(A-j))\right]\mathrm{d}j,
\end{aligned}
$$

which becomes

$$
\begin{aligned}
0 = &-\frac{\kappa^2-1}{2} C_1 \left[\frac{e^{A(\kappa-1)+\Delta}}{\kappa-1} - \frac{e^{-\Delta}}{\kappa+1}\right] \\
&- \frac{\kappa^2-1}{2} C_2 \left[-\frac{e^{-A(\kappa+1)+\Delta}}{\kappa+1} + \frac{e^{-\Delta}}{\kappa-1}\right] \\
&+ \frac{\kappa^2-1}{2} \frac{e^{-A+\Delta}}{\kappa} \int_0^A F'(j) \left[\kappa \cosh(\kappa(A-j)) + \sinh(\kappa(A-j))\right] \mathrm{d}j,
\end{aligned}
$$

so

$$
\begin{aligned}
0 = &-C_1 \left[\frac{e^{A(\kappa-1)+\Delta}}{\kappa-1} - \frac{e^{-\Delta}}{\kappa+1}\right] \\
&- C_2 \left[-\frac{e^{-A(\kappa+1)+\Delta}}{\kappa+1} + \frac{e^{-\Delta}}{\kappa-1}\right] \\
&+ \frac{e^{-A+\Delta}}{\kappa} \int_0^A F'(j) \left[\kappa \cosh(\kappa(A-j)) + \sinh(\kappa(A-j))\right] \mathrm{d}j.
\end{aligned}
$$

Let us pull out the term in A from the inside of the integral:

$$
\begin{aligned}
\int_0^A F'(j) \left[\kappa \cosh(\kappa(A-j)) + \sinh(\kappa(A-j))\right] \mathrm{d}j = &\frac{e^{\kappa A}}{2}(\kappa+1) \int_0^A f'(j) e^{-\kappa j} \,\mathrm{d}j \\
&+ \frac{e^{-\kappa A}}{2}(\kappa-1) \int_0^A F'(j) e^{\kappa j} \,\mathrm{d}j.
\end{aligned}
$$

We are then getting a system of two equations with two unknowns:

$$
\begin{aligned}
&- C_1 \frac{1}{\kappa-1} e^{A(\kappa-1)} + C_2 \frac{1}{\kappa+1} e^{-A(\kappa+1)} \\
&\qquad + \frac{e^{A(\kappa-1)}}{2\kappa}(\kappa+1) \int_0^A f'(j) e^{-\kappa j} \,\mathrm{d}j + \frac{e^{-A(\kappa+1)}}{2\kappa}(\kappa-1) \int_0^A f'(j) e^{\kappa j} \,\mathrm{d}j = 0,
\end{aligned}
$$

and

$$
\frac{C_1}{\kappa+1} = \frac{C_2}{\kappa-1}.
$$

So here are the constants:

$$
C_1 = \frac{1}{2\kappa} \frac{e^{A(\kappa-1)}(\kappa+1)^2(\kappa^2-1) \int_0^A f'(j) e^{-\kappa j} \,\mathrm{d}j + e^{-A(\kappa+1)}(\kappa+1)^2(\kappa-1)^2 \int_0^A f'(j) e^{\kappa j} \,\mathrm{d}j}{(\kappa+1)^2 e^{A(\kappa-1)} - (\kappa-1)^2 e^{-A(\kappa+1)}},
$$

and

$$C_2 = \frac{\kappa - 1}{\kappa + 1} C_1,$$

so

$$C_2 = \frac{1}{2\kappa} \frac{e^{A(\kappa-1)}(\kappa+1)^2(\kappa-1)^2 \int_0^A f'(j)e^{-\kappa j}\,\mathrm{d}j + e^{-A(\kappa+1)}(\kappa^2-1)(\kappa-1)^2 \int_0^A f'(j)e^{\kappa j}\,\mathrm{d}j}{(\kappa+1)^2 e^{A(\kappa-1)} - (\kappa-1)^2 e^{-A(\kappa+1)}}.$$

Now we send A to infinity:

$$C_1 \to \frac{\kappa^2 - 1}{2\kappa} \int_0^\infty f'(j)e^{-\kappa j}\,\mathrm{d}j,$$

and

$$C_2 \to \frac{(\kappa - 1)^2}{2\kappa} \int_0^\infty f'(j)e^{-\kappa j}\,\mathrm{d}j.$$

Therefore, subject to regularity conditions, we must have:

$$U(\Delta) = C_1 e^{\kappa\Delta} + C_2 e^{-\kappa\Delta} - F'(\Delta) - \frac{\kappa^2 - 1}{\kappa} \int_0^\Delta \sinh(\kappa(\Delta - s))F'(s)\,\mathrm{d}s.$$

with

$$C_1 = \frac{\kappa^2 - 1}{2\kappa} \int_0^\infty F'(j)e^{-\kappa j}\,\mathrm{d}j,$$
$$C_2 = \frac{(\kappa - 1)^2}{2\kappa} \int_0^\infty F'(j)e^{-\kappa j}\,\mathrm{d}j.$$

From this we get

$$u_k(\Delta) = C_1 \frac{\beta_k\gamma^2}{\sigma_k^2(\kappa^2-1)} e^{\kappa\Delta} + C_2 \frac{\beta_k\gamma^2}{\sigma_k^2(\kappa^2-1)} e^{-\kappa\Delta} - \frac{f_k'(\Delta)}{\sigma_k^2} - \frac{\beta_k\gamma^2}{\sigma_k^2\kappa} \int_0^\Delta \sinh(\kappa(\Delta - s))F'(s)\,\mathrm{d}s,$$

which upon rearranging is the desired result.

Sufficiency. We now prove that if $\{u_k\}_{k=1}^K$ is as in the statement of the lemma, then Y has the linear representation as in the lemma.

Assume u_k (all k) is square integrable (which is implied by the finite variance of Y_t) and integrable (which is implied by the fact that covariances exist and are finite) and let

$$Z_t = \sum_{k=1}^{K} \int_{-\infty}^{t} u_k(t-s)(\mathrm{d}S_{k,s} - \alpha a^* \,\mathrm{d}s).$$

If we have $\mathbf{Cov}[Y_t - Z_t, S_{k,t-\Delta}] = 0$ for every Δ and k, then Y_t and $S_{k,t-\Delta}$ are independent for every Δ and k. As by assumption $Y_t - Z_t$ is measurable with respect to the information generated by the past signals $S_{k,t-\Delta}$, $\Delta \geq 0$, $k = 1, \ldots, K$, it implies that $\mathbf{Var}[Y_t - Z_t] = 0$ and thus $Y_t = Z_t$. So the proof reduces to showing that $\mathbf{Cov}[Y_t - Z_t, S_{k,t-\Delta}] = 0$ for every $\Delta \geq 0$, $k = 1, \ldots, K$.

Let $g_k(\Delta) := \mathbf{Cov}[Z_t, S_{k,t-\Delta}]$. Then for all $\Delta \geq 0$,

$$\begin{aligned} g_k(\Delta) &= \sum_{i=1}^{K} \int_0^{\infty} u_i(s)\, \mathbf{Cov}[\mathrm{d}S_{i,t-s}, S_{k,t-\Delta}] \\ &= \sigma_k^2 \int_{\Delta}^{t} u_k(s)\,\mathrm{d}s + \frac{\beta_k \gamma^2}{2} \int_0^{\infty} \int_0^{t-\Delta} U(s) e^{-|t-s-j|}\,\mathrm{d}j\,\mathrm{d}s, \end{aligned}$$

and

$$g_k'(\Delta) = -\sigma_k^2 u_k(\Delta) - \frac{\beta_k \gamma^2}{2} \int_0^{\infty} U(s) e^{-|\Delta - s|}\,\mathrm{d}s.$$

We have, by assumption,

$$u_k(\Delta) = C_1 \frac{\beta_k \gamma^2}{\sigma_k^2(\kappa^2-1)} e^{\kappa\Delta} + C_2 \frac{\beta_k \gamma^2}{\sigma_k^2(\kappa^2-1)} e^{-\kappa\Delta} - \frac{f_k'(\Delta)}{\sigma_k^2} - \frac{\beta_k \gamma^2}{\sigma_k^2 \kappa} \int_0^{\Delta} \sinh(\kappa(\Delta - s)) F'(s)\,\mathrm{d}s. \tag{19}$$

So:

$$-\sigma_k^2 u_k(\Delta) = f'(\Delta) - C_1 \frac{\beta_k \gamma^2}{\kappa^2 - 1} e^{\kappa\Delta} - C_2 \frac{\beta_k \gamma^2}{\kappa^2 - 1} e^{-\kappa\Delta} + \frac{\beta_k \gamma^2}{\kappa} \int_0^{\Delta} \sinh(\kappa(\Delta - s)) F'(s)\,\mathrm{d}s.$$

Further, multiplying (19) by β_k and summing over k, we have

$$U(\Delta) = C_1 e^{\kappa\Delta} + C_2 e^{-\kappa\Delta} - F'(\Delta) - \frac{\kappa^2 - 1}{\kappa} \int_0^{\Delta} \sinh(\kappa(\Delta - s)) F'(s)\,\mathrm{d}s.$$

We have

$$\int_0^{\infty} U(s) e^{-|\Delta - s|}\,\mathrm{d}s = \lim_{A \to \infty} \int_0^{A} U(s) e^{-|\Delta - s|}\,\mathrm{d}s.$$

Then

$$\int_0^A U(s)e^{-|\Delta-s|}\,\mathrm{d}s = C_1\int_0^A e^{\kappa s}e^{-|\Delta-s|}\,\mathrm{d}s + C_2\int_0^A e^{-\kappa s}e^{-|\Delta-s|}\,\mathrm{d}s - \int_0^A F'(s)e^{-|\Delta-s|}\,\mathrm{d}s$$
$$- \frac{\kappa^2-1}{\kappa}\int_0^A\int_0^s \sinh(\kappa(s-j))F'(j)\,\mathrm{d}j\,\mathrm{d}s.$$

We compute:

$$\begin{aligned}
C_1\int_0^A e^{+\kappa s}e^{-|\Delta-s|}\,\mathrm{d}s &= C_1\left[\frac{e^{A(\kappa-1)+\Delta}}{\kappa-1} - \frac{e^{-\Delta}}{\kappa+1} + \frac{e^{\kappa\Delta}}{\kappa+1} - \frac{e^{\kappa\Delta}}{\kappa-1}\right],\\
C_2\int_0^A e^{-\kappa s}e^{-|\Delta-s|}\,\mathrm{d}s &= C_2\left[-\frac{e^{-A(\kappa+1)+\Delta}}{\kappa+1} + \frac{e^{-\Delta}}{\kappa-1} + \frac{e^{-\kappa\Delta}}{\kappa+1} - \frac{e^{-\kappa\Delta}}{\kappa-1}\right].
\end{aligned}$$

Then, for any $A > \Delta$:

$$\begin{aligned}
&\int_0^A\int_0^s \sinh(\kappa(s-j))F'(j)e^{-|\Delta-s|}\,\mathrm{d}j\,\mathrm{d}s\\
&\qquad= \int_0^A\int_0^A 1_{j\le s}\sinh(\kappa(s-j))F'(j)e^{-|\Delta-s|}\,\mathrm{d}j\,\mathrm{d}s\\
&\qquad= \int_0^A F'(j)\int_j^A \sinh(\kappa(s-j))e^{-|\Delta-s|}\,\mathrm{d}s\,\mathrm{d}j\\
&\qquad= \int_0^\Delta F'(j)\int_j^\Delta \sinh(\kappa(s-j))e^{-|\Delta-s|}\,\mathrm{d}s\,\mathrm{d}j\\
&\qquad\quad+ \int_0^\Delta F'(j)\int_\Delta^A \sinh(\kappa(s-j))e^{-|\Delta-s|}\,\mathrm{d}s\,\mathrm{d}j\\
&\qquad\quad+ \int_\Delta^A F'(j)\int_j^A \sinh(\kappa(s-j))e^{-|\Delta-s|}\,\mathrm{d}s\,\mathrm{d}j\\
&\qquad= -\frac{\kappa}{\kappa^2-1}\int_0^A F'(j)e^{-|\Delta-j|}\,\mathrm{d}j\\
&\qquad\quad+ \frac{e^{-A+\Delta}}{\kappa^2-1}\int_0^A F'(j)\left[\kappa\cosh(\kappa(A-j)) + \sinh(\kappa(A-j))\right]\mathrm{d}j\\
&\qquad\quad- \frac{2}{\kappa^2-1}\int_0^\Delta \sinh(\kappa(\Delta-j))F'(j)\,\mathrm{d}j.
\end{aligned}$$

$$
\begin{aligned}
\int_0^A U(s)e^{-|\Delta-s|}\,\mathrm{d}s = C_1 &\left[\frac{e^{A(\kappa-1)+\Delta}}{\kappa-1} - \frac{e^{-\Delta}}{\kappa+1} + \frac{e^{\kappa\Delta}}{\kappa+1} - \frac{e^{\kappa\Delta}}{\kappa-1}\right] \\
&+ C_2\left[-\frac{e^{-A(\kappa+1)+\Delta}}{\kappa+1} + \frac{e^{-\Delta}}{\kappa-1} + \frac{e^{-\kappa\Delta}}{\kappa+1} - \frac{e^{-\kappa\Delta}}{\kappa-1}\right] \\
&- \int_0^A F'(s)e^{-|\Delta-s|}\,\mathrm{d}s \\
&+ \frac{\kappa^2-1}{\kappa}\frac{\kappa}{\kappa^2-1}\int_0^A F'(j)e^{-|\Delta-j|}\,\mathrm{d}j \\
&- \frac{\kappa^2-1}{\kappa}\frac{e^{-A+\Delta}}{\kappa^2-1}\int_0^A F'(j)\left[\kappa\cosh(\kappa(A-j)) + \sinh(\kappa(A-j))\right]\mathrm{d}j \\
&+ \frac{\kappa^2-1}{\kappa}\frac{2}{\kappa^2-1}\int_0^\Delta \sinh(\kappa(\Delta-j))F'(j)\,\mathrm{d}j.
\end{aligned}
\tag{20}
$$

After some re-ordering,

$$
\begin{aligned}
\int_0^A U(s)e^{-|\Delta-s|}\,\mathrm{d}s = \frac{2}{\kappa}&\int_0^\Delta \sinh(\kappa(\Delta-j))F'(j)\,\mathrm{d}j \\
&+ C_1\left[\frac{e^{\kappa\Delta}}{\kappa+1} - \frac{e^{\kappa\Delta}}{\kappa-1}\right] \\
&+ C_2\left[\frac{e^{-\kappa\Delta}}{\kappa+1} - \frac{e^{-\kappa\Delta}}{\kappa-1}\right] \\
&+ C_1\left[\frac{e^{A(\kappa-1)+\Delta}}{\kappa-1} - \frac{e^{-\Delta}}{\kappa+1}\right] \\
&+ C_2\left[-\frac{e^{-A(\kappa+1)+\Delta}}{\kappa+1} + \frac{e^{-\Delta}}{\kappa-1}\right] \\
&+ \frac{\kappa^2-1}{\kappa}\frac{\kappa}{\kappa^2-1}\int_0^A F'(j)e^{-|\Delta-j|}\,\mathrm{d}j \\
&- \frac{\kappa^2-1}{\kappa}\frac{e^{-A+\Delta}}{\kappa^2-1}\int_0^A F'(j)\left[\kappa\cosh(\kappa(A-j)) + \sinh(\kappa(A-j))\right]\mathrm{d}j.
\end{aligned}
$$

Using

$$C_1 = \frac{\kappa^2 - 1}{2\kappa} \int_0^\infty F'(j) e^{-\kappa j} \, \mathrm{d}j,$$
$$C_2 = \frac{(\kappa - 1)^2}{2\kappa} \int_0^\infty F'(j) e^{-\kappa j} \, \mathrm{d}j,$$

we get that

$$\begin{aligned}
&C_1 \left[\frac{e^{A(\kappa-1)+\Delta}}{\kappa - 1} - \frac{e^{-\Delta}}{\kappa + 1} \right] \\
&\qquad + C_2 \left[-\frac{e^{-A(\kappa+1)+\Delta}}{\kappa + 1} + \frac{e^{-\Delta}}{\kappa - 1} \right] \\
&\qquad + \frac{\kappa^2 - 1}{\kappa} \frac{\kappa}{\kappa^2 - 1} \int_0^A F'(j) e^{-|\Delta - j|} \, \mathrm{d}j \\
&\qquad - \frac{\kappa^2 - 1}{\kappa} \frac{e^{-A+\Delta}}{\kappa^2 - 1} \int_0^A F'(j) \left[\kappa \cosh(\kappa(A - j)) + \sinh(\kappa(A - j)) \right] \mathrm{d}j
\end{aligned}$$

converges to 0 as $A \to \infty$.

Thus,

$$\begin{aligned}
\int_0^\infty U(s) e^{-|\Delta - s|} \, \mathrm{d}s = \frac{2}{\kappa} &\int_0^\Delta \sinh(\kappa(\Delta - j)) F'(j) \, \mathrm{d}j \\
&+ C_1 \left[\frac{e^{\kappa\Delta}}{\kappa + 1} - \frac{e^{\kappa\Delta}}{\kappa - 1} \right] \\
&+ C_2 \left[\frac{e^{-\kappa\Delta}}{\kappa + 1} - \frac{e^{-\kappa\Delta}}{\kappa - 1} \right].
\end{aligned}$$

So

$$\begin{aligned}
g'(\Delta) = f'(\Delta) &- C_1 \frac{\beta_k \gamma^2}{\kappa^2 - 1} e^{\kappa\Delta} - C_2 \frac{\beta_k \gamma^2}{\kappa^2 - 1} e^{-\kappa\Delta} \\
&+ \frac{\beta_k \gamma^2}{\kappa} \int_0^\Delta \sinh(\kappa(\Delta - s)) F'(s) \, \mathrm{d}s \\
&- \frac{\beta_k \gamma^2}{2} \int_0^\infty U(s) e^{-|\Delta - s|} \, \mathrm{d}s.
\end{aligned}$$

After plugging in the expression for the last term,

$$
\begin{aligned}
g_k'(\Delta) = f'(\Delta) &- C_1 \frac{\beta_k \gamma^2}{\kappa^2 - 1} e^{\kappa \Delta} - C_2 \frac{\beta_k \gamma^2}{\kappa^2 - 1} e^{-\kappa \Delta} \\
&+ \frac{\beta_k \gamma^2}{\kappa} \int_0^{\Delta} \sinh(\kappa(\Delta - s)) F'(s) \, \mathrm{d}s \\
&- \frac{\beta_k \gamma^2}{\kappa} \int_0^{\Delta} \sinh(\kappa(\Delta - j)) F'(j) \, \mathrm{d}j \\
&- \frac{\beta_k \gamma^2}{2} C_1 \left[\frac{e^{\kappa \Delta}}{\kappa + 1} - \frac{e^{\kappa \Delta}}{\kappa - 1} \right] \\
&- \frac{\beta_k \gamma^2}{2} C_2 \left[\frac{e^{-\kappa \Delta}}{\kappa + 1} - \frac{e^{-\kappa \Delta}}{\kappa - 1} \right] \\
= f'(\Delta)&.
\end{aligned}
$$

So we get that $g_k' = f_k'$ and as $f_k(0) = g_k(0) = 0$, it implies that $f = g$, which concludes the proof.

C.5 Proof of Lemma 3.7

First, let us consider a confidential rating policy with rating process $\mathbf{Y}$. Let Z be the process defined as

$$
Z_t \coloneqq \mathbf{E}^*[\theta_t \mid \mathbf{Y}_t] = \mathbf{Cov}[\theta_t, \mathbf{Y}_t] \, \mathbf{Var}[\mathbf{Y}_t]^{-1} \mathbf{Y}_t, \tag{21}
$$

which is a scalar rating process that satisfies the conditions of Definition 3.3. Additionally, the confidential rating policy based on Z generates (by definition) the same market belief as the original rating policy, and thus implements the same effort level.

Now, let us consider a public rating policy with rating process $\mathbf{Y}$. Let Z be the process defined as

$$
Z_t \coloneqq \mathbf{E}^*[\theta_t \mid \{\mathbf{Y}_s\}_{s \le t}].
$$

A linear filtering argument yields (with n the dimension of $\mathbf{Y}$):

$$
Z_t - Z_{t-\Delta} = \sum_{j=1}^{n} \int_{t-\Delta}^{t} w_j(s) \, \mathrm{d}Y_{j,s},
$$

for some functions w_j. We note that, due to the mean-reverting nature of θ, $Z_{t-\Delta} \to 0$ as $\Delta \to \infty$, both in the mean-square sense and in the almost sure sense. Hence, sending Δ to infinity, we get

$$Z_t = \sum_{j=1}^{n} \int_{-\infty}^{t} w_j(s)\, \mathrm{d}Y_{j,s},$$

and it follows that Z is a scalar rating process that satisfies Definition 3.3. Additionally, as above, the confidential rating policy based on Z generates (by definition) the same market belief as the original public rating policy, and thus implements the same effort level.

C.6 Proof of Proposition 3.8

The proof is immediate.

Part (1). If Y is the belief for a confidential rating policy, then $Y_t = \mu_t$, where, by definition, $\mu_t = \mathbf{E}^*[\theta_t|Y_t]$. Conversely, if $Y_t = \mathbf{E}^*[\theta_t|Y_t]$, then by definition $\mu_t = \mathbf{E}^*[\theta_t|Y_t]$, so $Y_t = \mu_t$, and Y is also the belief for a confidential rating policy.

Part (2). If Y is the belief for a public rating policy, then $Y_t = \mu_t$, where, by definition, $\mu_t = \mathbf{E}^*[\theta_t|\{Y_s\}_{s\le t}]$. Conversely, if $Y_t = \mathbf{E}^*[\theta_t|\{Y_s\}_{s\le t}]$, then by definition $\mu_t = \mathbf{E}^*[\theta_t|\{Y_s\}_{s\le t}]$, so $Y_t = \mu_t$, and Y is also a belief for a public rating policy.

C.7 Proof of Lemma 3.9

Note that the correlation between types $\theta_t, \theta_{t+\Delta}$ must satisfy

$$\frac{\mathbf{Cov}[\theta_t, \theta_{t+\Delta}]}{\sqrt{\mathbf{Var}[\theta_t]}\sqrt{\mathbf{Var}[\theta_{t+\Delta}]}} = e^{-\Delta}.$$

since, as $\{\theta_t\}$ is a stationary Ornstein-Uhlenbeck process with mean-reverting rate 1 and volatility γ,

$$\mathbf{Cov}[\theta_t, \theta_{t+\Delta}] = \frac{\gamma^2}{2} e^{-\Delta},$$

and

$$\mathbf{Var}[\theta_t] = \mathbf{Var}[\theta_{t+\Delta}] = \frac{\gamma^2}{2}.$$

Let μ be the market belief process induced by a public rating policy. The random variable θ_t is then independent from every μ_s, $s \le t$, conditionally on μ_t, as μ_t carries all relevant information about θ_t. Thus, $\mathbf{Cov}[\theta_t, \mu_s \mid \mu_t] = 0$.

The projection formulas yield

$$\mathbf{Cov}[\theta_t, \mu_s \mid \mu_t] = \mathbf{Cov}[\theta_t, \mu_s] - \frac{\mathbf{Cov}[\theta_t, \mu_t]\,\mathbf{Cov}[\mu_s, \mu_t]}{\mathbf{Var}[\mu_t]},$$

so

$$\mathbf{Cov}[\mu_s, \mu_t] = \mathbf{Var}[\mu_t]\frac{\mathbf{Cov}[\theta_t, \mu_s]}{\mathbf{Cov}[\theta_t, \mu_t]} = \mathbf{Var}[\mu_s]\frac{\mathbf{Cov}[\theta_t, \mu_s]}{\mathbf{Cov}[\theta_s, \mu_s]},$$

where we used the stationarity of the pair (μ, θ). Besides, there exist $u_1, \ldots, u_K$, such that μ_t can be written as

$$\mu_t = \sum_{k=1}^{K} \int_0^\infty u_k(j)[\mathrm{d}S_{k,j} - \alpha_k \,\mathrm{d}j].$$

Hence, recalling that $\mathbf{Cov}[\theta_t, \theta_s] = \gamma^2 e^{-|t-s|}/2$,

$$\mathbf{Cov}[\mu_s, \theta_s] = \frac{\gamma^2}{2}\sum_{k=1}^{K} \int_0^\infty u_k(j)\beta_k e^{-j} \,\mathrm{d}j,$$

and, for $t \geq s$,

$$\mathbf{Cov}[\mu_s, \theta_t] = \frac{\gamma^2}{2}\sum_{k=1}^{K} \int_0^\infty u_k(j)\beta_k e^{-(t-s+j)} \,\mathrm{d}j = e^{-(t-s)}\,\mathbf{Cov}[\mu_s, \theta_s].$$

Taking $t = s + \Delta$ with any $\Delta \geq 0$,

$$\mathbf{Cov}[\mu_s, \mu_{s+\Delta}] = \mathbf{Var}[\mu_s]e^{-\Delta}.$$

Now, let Y be a scalar rating process that satisfies

$$\mathbf{Cov}[Y_{t+\Delta}, Y_t] = \mathbf{Var}[Y_t]e^{-\Delta},$$

for every $\Delta \geq 0$. There exist $u_1, \ldots, u_K$, such that Y_t can be written as

$$Y_t = \sum_{k=1}^{K} \int_0^\infty u_k(j)[\mathrm{d}S_{k,j} - \alpha_k \,\mathrm{d}j],$$

so that, as above, for $s \leq t$,

$$\mathbf{Cov}[Y_s, \theta_t] = e^{-(t-s)}\,\mathbf{Cov}[Y_s, \theta_s] = e^{-(t-s)}\,\mathbf{Cov}[Y_t, \theta_t],$$

using the stationarity of (Y, θ), and we have by assumption on Y that

$$e^{-(t-s)} = \frac{\mathbf{Cov}[Y_t, Y_s]}{\mathbf{Var}[Y_s]} = \frac{\mathbf{Cov}[Y_t, Y_s]}{\mathbf{Var}[Y_t]}.$$

Therefore,

$$\mathbf{Cov}[\theta_t, Y_s \mid Y_t] = \mathbf{Cov}[\theta_t, Y_s] - \frac{\mathbf{Cov}[\theta_t, Y_t]\,\mathbf{Cov}[Y_s, Y_t]}{\mathbf{Var}[Y_t]} = 0.$$

As θ and Y are jointly normal, it implies that θ_t and Y_s are independent conditionally on Y_t for every $s \leq t$, and the market belief associated to the public rating policy with rating process Y satisfies

$$\mu_t = \mathbf{E}^*[\theta_t \mid \{Y_S\}_{s \leq t}] = \mathbf{E}^*[\theta_t \mid Y_t] = \frac{\mathbf{Cov}[\theta_t, Y_t]}{\mathbf{Var}[Y_t]} Y_t,$$

where we observe that $\mathbf{E}^*[\theta_t] = \mathbf{E}^*[Y_t] = 0$. By stationarity, both $\mathbf{Cov}[\theta_t, Y_t]$ and $\mathbf{Var}[Y_t]$ are constant, and the rating process Y is proportional to the belief process μ.

C.8 Proof of Proposition 3.10

Let

$$\begin{aligned}
\mu_t &:= \mathbf{E}^*[\theta_t \mid Y_t] \\
&= \mathbf{Cov}[Y_t, \theta_t] Y_t \\
&= \mathbf{Cov}[Y_t, \theta_t] \sum_{k=1}^{K} \int_{-\infty}^{t} u_k(t-s)\left[\mathrm{d}S_{k,s} - \alpha_k a^*\,\mathrm{d}s\right],
\end{aligned}$$

where a^* is the effort level conjectured by the market. Observe that by stationarity, $\mathbf{Cov}[Y_t, \theta_t]$ is constant.

We prove that, given a cumulative payment process that satisfies the zero-profit condition, there exists an optimal effort strategy for the agent, which it is unique and pinned down by the first-order condition given in Proposition 3.10. This, in turn, yields existence of a unique equilibrium.

Let us fix a cumulative payment process that satisfies the zero-profit condition, and suppose that the agent follows effort strategy $A = \{A_t\}$. The agent's time-0 (*ex* *post*) payoff is then

$$\int_0^\infty \left[a^* + \mu_t - c(A_t)\right] e^{-rt}\,\mathrm{d}t. \tag{22}$$

Maximizing the agent's *ex ante* payoff is equivalent to maximizing the agent's *ex post* payoff, up to probability zero events. Hence, we seek conditions on A_t that maximize (22).

Note that maximizing (22) is equivalent to maximizing

$$\mathbf{Cov}[Y_t, \theta_t] \int_0^\infty \int_0^t \sum_{k=1}^K \alpha_k A_s e^{-rt}\,\mathrm{d}s\,\mathrm{d}t - \int_0^\infty c(A_t) e^{-rt}\,\mathrm{d}t. \tag{23}$$

Let us re-write

$$\begin{aligned}
&\mathbf{Cov}[Y_t, \theta_t] \int_0^\infty \int_0^t \sum_{k=1}^K \alpha_k A_s e^{-rt}\,\mathrm{d}s\,\mathrm{d}t \\
&\quad = \mathbf{Cov}[Y_t, \theta_t] \int_0^\infty \int_s^{+\infty} \sum_{k=1}^K u_k(t-s) \alpha_k A_s e^{-rt}\,\mathrm{d}t\,\mathrm{d}s \\
&\quad = \mathbf{Cov}[Y_t, \theta_t] \int_0^\infty A_s e^{-rs} \int_s^{+\infty} \sum_{k=1}^K u_k(t-s) \alpha_k e^{-r(t-s)}\,\mathrm{d}t\,\mathrm{d}s \\
&\quad = \mathbf{Cov}[Y_t, \theta_t] \int_0^\infty A_s e^{-rs} \int_0^\infty \sum_{k=1}^K \alpha_k u_k(\Delta) e^{-r\Delta}\,\mathrm{d}\Delta\,\mathrm{d}s.
\end{aligned}$$

Therefore, maximizing (23) is equivalent to maximizing

$$\mathbf{Cov}[Y_t, \theta_t] \int_0^\infty A_s e^{-rs} \int_0^\infty \sum_{k=1}^K \alpha_k u_k(\Delta) e^{-r\Delta}\,\mathrm{d}\Delta\,\mathrm{d}s - \int_0^\infty c(A_t) e^{-rt}\,\mathrm{d}t, \tag{24}$$

which is the same as maximizing

$$\mathbf{Cov}[Y_t, \theta_t] A_s \int_0^\infty \sum_{k=1}^K \alpha_k u_k(\Delta) e^{-r\Delta}\,\mathrm{d}\Delta - c(A_s), \tag{25}$$

for every s. By strict concavity, we get that (23), and thus (22), is maximized if and only if

$$c'(A_t) = \mathbf{Cov}[Y_t, \theta_t] \int_0^\infty \sum_{k=1}^{K} \alpha_k u_k(\Delta) e^{-r\Delta} \, \mathrm{d}\Delta, \tag{26}$$

for every t.

We note that $\mathbf{Cov}[Y_t, \theta_t]$ is constant and equal to

$$\mathbf{Cov}[Y_t, \theta_t] = \frac{\gamma^2}{2} \sum_{k=1}^{K} \beta_k \int_0^\infty u_k(s) e^{-s} \, \mathrm{d}s.$$

Hence, (22) is maximized if and only if

$$c'(A_t) = \frac{\gamma^2}{2} \left[\sum_{k=1}^{K} \beta_k \int_0^\infty u_k(t) e^{-t} \, \mathrm{d}t \right] \left[\sum_{k=1}^{K} \alpha_k \int_0^\infty u_k(t) e^{-rt} \, \mathrm{d}t \right],$$

for every t. Thus, the optimal effort strategy exists for the agent, it is unique (up to measure zero events and times), it is constant and pinned down by the last equation.

C.9 Proof of Theorem 4.1

Throughout this proof, we use the following shorthand notation:

$$\begin{aligned} U(t) &= \sum_{k=1}^{K} \beta_k u_k(t), \\ V(t) &= \sum_{k=1}^{K} \alpha_k u_k(t), \\ U_0 &= \int_0^\infty U(t) e^{-t} \, \mathrm{d}t, \\ V_0 &= \int_0^\infty V(t) e^{-rt} \, \mathrm{d}t. \end{aligned}$$

We seek to maximize $c'(a)$ (where a is the stationary action of the agent) among policies that satisfy the normalization condition that the rating has variance one.

Given a scalar rating process Y of the form

$$Y_t = \sum_{k=1}^{K} \int_0^\infty u_k(t-s)\left[\mathrm{d}S_{k,s} - \alpha_k a^* \,\mathrm{d}s\right],$$

we note that by Itô's isometry,

$$\mathbf{Var}[Y_t] = \sum_{k=1}^{K} \sigma_k^2 \int_0^\infty u_k(s)^2 \,\mathrm{d}s + \sum_{k=1}^{K}\sum_{k'=1}^{K} \int_{-\infty}^{t}\int_{-\infty}^{t} \beta_k \beta_{k'} u_k(t-i) u_{k'}(t-j)\, \mathbf{Cov}[\theta_i, \theta_j]\,\mathrm{d}i\,\mathrm{d}j,$$

and since θ is a stationary Ornstein-Uhlenbeck process with mean-reversion rate 1 and volatility σ, we have $\mathbf{Cov}[\theta_t, \theta_s] = \gamma^2 e^{-|t-s|}/2$, so that

$$\mathbf{Var}[Y_t] = \sum_{k=1}^{K} \sigma_k^2 \int_0^\infty u_k(s)^2 \,\mathrm{d}s + \frac{\gamma^2}{2} \int_0^\infty \int_0^\infty U(i)U(j) e^{-|j-i|}\,\mathrm{d}i\,\mathrm{d}j.$$

Together with Proposition 3.10, we get that the problem of maximizing $c'(a)$ among policies that satisfy the normalization condition reduces to choosing a vector of functions $\mathbf{u} = (u_1, \ldots, u_k)$ that maximizes

$$\left[\int_0^\infty V(t) e^{-rt}\,\mathrm{d}t\right]\left[\int_0^\infty U(t) e^{-t}\,\mathrm{d}t\right],$$

subject to

$$\frac{\gamma^2}{2} \int_0^\infty \int_0^\infty U(i)U(j) e^{-|j-i|}\,\mathrm{d}i\,\mathrm{d}j + \sum_{k=1}^{K} \sigma_k^2 \int_0^\infty u_k(t)^2 \,\mathrm{d}t = 1.$$

Observe that we can write both the objective and the constraint as a double integral. The objective is equal to

$$\int_0^\infty \int_0^\infty V(i)U(j) e^{-ri} e^{-j}\,\mathrm{d}i\,\mathrm{d}j,$$

while the constraint can be written as

$$\int_0^\infty \int_0^\infty \left(\frac{\gamma^2}{2} U(i)U(j) e^{-|j-i|}\,\mathrm{d}i\,\mathrm{d}j + \sum_{k=1}^{K} \sigma_k^2 u_k(j)^2 e^{-i}\right)\mathrm{d}i\,\mathrm{d}j = 1.$$

This allows us to apply the results of Proposition B.1. Let

$$L(\mathbf{u}, \lambda) = F(\mathbf{u}) + \lambda G(\mathbf{u}),$$

where F and G are defined as

$$F(\mathbf{u}) = \left[\int_0^\infty V(t)e^{-rt}\,\mathrm{d}t\right]\left[\int_0^\infty U(t)e^{-t}\,\mathrm{d}t\right],$$

and

$$G(\mathbf{u}) = \frac{\gamma^2}{2}\int_0^\infty\int_0^\infty U(i)U(j)e^{-|j-i|}\,\mathrm{d}i\,\mathrm{d}j + \sum_{k=1}^K \sigma_k^2 \int_0^\infty u_k(t)^2\,\mathrm{d}t.$$

Assume there exists a solution $\mathbf{u}^*$ to the relaxed optimization problem. Note that at the optimum, the objective is strictly positive, *i.e.*, $F(\mathbf{u}^*) > 0$, since the optimal solution does at least as well as full transparency (giving all information of all signals to the market) and full transparency induces a positive equilibrium effort and thus yields a positive value for F by our assumption that $m_{\alpha\beta} \geq 0$.

Proposition B.2 gives us existence of λ^* such that $\mathbf{u}^*$ maximizes $\mathbf{u} \mapsto L(\mathbf{u}, \lambda^*)$. It will be useful to observe that $\lambda^* < 0$. Indeed, at optimum, $F(\mathbf{u}^*) > 0$ and $G(\mathbf{u}^*) = 1$.

Proposition B.1 gives the first-order condition derived from Euler-Lagrange equations: if $\lambda = \lambda^*$ and $\mathbf{u} = \mathbf{u}^*$, then for all k and all t we have $L_k(t) = 0$, where we define

$$L_k(t) := \alpha_k U_0 e^{-rt} + \beta_k V_0 e^{-t} + \lambda\gamma^2\beta_k \int_0^\infty U(j)e^{-|t-j|}\,\mathrm{d}j + 2\lambda\sigma_k^2 u_k(t) = 0, \tag{27}$$

where U_0, V_0, U and V are defined as above as a function of $\mathbf{u}$.

We differentiate the above equation in the variable t twice, and get, for all k and all t:

$$\alpha_k U_0 r^2 e^{-rt} + \beta_k V_0 e^{-t} - 2\lambda\gamma^2\beta_k U(t) + \lambda\gamma^2\beta_k \int_0^\infty U(j)e^{-|t-j|}\,\mathrm{d}j + 2\lambda\sigma_k^2 u_k''(t) = 0. \tag{28}$$

The difference between (27) and (28) is

$$(1 - r^2)\alpha_k U_0 e^{-rt} + 2\lambda\gamma^2\beta_k U(t) + 2\lambda\sigma_k^2(u_k(t) - u_k''(t)) = 0. \tag{29}$$

In particular, multiplying (29) by β_k/σ_k^2 and summing over k, we get a linear differential equation that $U(t)$ should satisfy, namely,

$$(1-r^2)m_{\alpha\beta}U_0e^{-rt}+2\lambda\gamma^2m_\beta U(t)+2\lambda(U(t)-U''(t))=0,$$

where we recall that $m_\beta=\sum_k\beta_k^2/\sigma_k^2$, $m_{\alpha\beta}=\sum_k\alpha_k\beta_k/\sigma_k^2$, and $m_\alpha=\sum_k\alpha_k^2/\sigma_k^2$.

The characteristic polynomial has roots $\pm\sqrt{1+\gamma^2m_\beta}=\pm\kappa$. A particular solution is Ce^{-rt} for some constant C. If the solution is admissible, it is bounded, hence we get

$$U(t)=C_1e^{-rt}+C_2e^{-\kappa t},$$

for some constants C_1 and C_2.

For such U, u_k satisfies the linear differential equation (29), whose characteristic polynomial has roots ±1. A particular solution is a sum of scaled time exponentials e^{-rt} and $e^{-\kappa t}$. As every u_k is bounded, we must consider the negative root of the characteristic equation and we get that

$$u_k(t)=D_{1,k}e^{-rt}+D_{2,k}e^{-\kappa t}+D_{3,k}e^{-t}, \tag{30}$$

for some constants $D_{1,k},D_{2,k},D_{3,k}$.

Determination of the constants. We have established that the solution belongs to the family of functions that are sums of scaled time exponentials. We now solve for the constant factors.

We plug in the general form of u_k from (30) in the expression for L_k, and get:

$$L_k=L_{1,k}e^{-rt}+L_{2,k}e^{-\kappa t}+L_{3,k}e^{-t},$$

where the coefficients $L_{1,k}$, $L_{2,k}$, $L_{3,k}$ depend on the primitives of the model and the constants $D_{1,k},D_{2,k},D_{3,k}$. The condition that $L_k=0$ implies that $L_{1,k}=L_{2,k}=L_{3,k}=0$.

First, note that $U(t)$ does not include a term of the form e^{-t}, which implies that

$$\sum_{k=1}^K\beta_kD_{3,k}=0. \tag{31}$$

We also observe that

$$L_{2,k}=2\lambda\sigma_k^2D_{2,k}-\frac{2\gamma^2\lambda\beta_k\sum_{i=1}^K\beta_iD_{2,i}}{\kappa^2-1},$$

so that $L_{2,k} = 0$ for all k implies

$$D_{2,k} = a\frac{\beta_k}{\sigma_k^2}, \tag{32}$$

for some multiplier a. Next, we use (32) together with (31) to show that

$$\begin{aligned} L_{3,k} = {} & \frac{\beta_k}{2r}\sum_{i=1}^{K}\alpha_i D_{1,i} + \frac{\beta_k}{r+1}\sum_{i=1}^{K}\alpha_i D_{3,i} + \frac{\gamma^2\lambda\beta_k}{r-1}\sum_{i=1}^{K}\beta_i D_{1,i} \\ & + 2\lambda\sigma_k^2 D_{3,k} + \frac{a\gamma^2\lambda\beta_k m_\beta}{\kappa-1} + \frac{a\beta_k m_{\alpha\beta}}{\kappa+r}, \end{aligned}$$

and $L_{3,k} = 0$ for every k implies that $D_{3,k} = 0$ for all k. The equation $L_{3,k}/\beta_k = 0$ is linear in λ and then simplifies to:

$$\lambda\left(\frac{\gamma^2}{r-1}\sum_{i=1}^{K}\beta_i D_{1,i} + \frac{a\gamma^2 m_\beta}{\kappa-1}\right) + \frac{1}{2r}\sum_{i=1}^{K}\alpha_i D_{1,i} + \frac{a m_{\alpha\beta}}{\kappa+r} = 0. \tag{33}$$

Next, we use (32) together with (31) to show that

$$L_{1,k} = 2\lambda\sigma_k^2 D_{1,k} + \frac{a\alpha_k m_\beta}{\kappa+1} + \frac{((r-1)\alpha_k - 2\gamma^2\lambda\beta_k)}{r^2-1}\sum_{i=1}^{K}\beta_i D_{1,i},$$

and, since $L_{1,k} = 0$ must hold for every k, we get, since $\lambda \neq 0$,

$$\sigma_k^2 D_{1,k} = \left(\frac{\gamma^2\beta_k}{r^2-1} - \frac{\alpha_k}{2\lambda+2\lambda r}\right)\sum_{i=1}^{K}\beta_i D_{1,i} - \frac{a\alpha_k m_\beta}{2\kappa\lambda+2\lambda}. \tag{34}$$

We multiply (34) by β_k/σ_k^2 and sum over k to get

$$\left[(\kappa+1)\left((r-1)\left(m_{\alpha\beta} + 2\lambda(r+1)\right) - 2\gamma^2\lambda m_\beta\right)\right]\sum_{i=1}^{K}\beta_i D_{1,i} = -a\left(r^2-1\right)m_{\alpha\beta}m_\beta.$$

As by assumption $r \neq 1$, the right-hand side is non-zero, which implies

$$\left((r-1)\left(m_{\alpha\beta} + 2\lambda(r+1)\right) - 2\gamma^2\lambda m_\beta\right) \neq 0, \tag{35}$$

and thus

$$\sum_{i=1}^{K} \beta_i D_{1,i} = \frac{-a\left(r^2-1\right) m_{\alpha\beta} m_\beta}{(\kappa+1)\left((r-1)\left(m_{\alpha\beta}+2\lambda(r+1)\right)-2\gamma^2\lambda m_\beta\right)}. \tag{36}$$

Similarly, if we multiply (34) by α_k/σ_k^2 and sum over k, we get

$$\begin{aligned}\sum_{i=1}^{K} \alpha_i D_{1,i} &= \left(\frac{\gamma^2 m_{\alpha\beta}}{r^2-1} - \frac{m_\alpha}{2\lambda+2\lambda r}\right) \sum_{i=1}^{K} \beta_i D_{1,i} - \frac{a m_\alpha m_\beta}{2\kappa\lambda+2\lambda} \\ &= \frac{a m_\beta \left(m_\alpha\left(\gamma^2 m_\beta - r^2+1\right)-\gamma^2 m_{\alpha\beta}^2\right)}{(\kappa+1)\left((r-1)\left(m_{\alpha\beta}+2\lambda(r+1)\right)-2\gamma^2\lambda m_\beta\right)}.\end{aligned} \tag{37}$$

Putting together (33), (36) and (37) yields a quadratic equation in λ of the form

$$A\lambda^2 + B\lambda + C = 0, \tag{38}$$

which, after simplification and using that $\kappa^2 = 1+\gamma^2 m_\beta$,

$$\begin{aligned}A &= m_\beta \frac{\kappa+r}{1-\kappa}, \\ B &= \frac{m_{\alpha\beta}\left(\gamma^2 m_\beta\left(-2\kappa^2+r^2+1\right)+\left(\kappa^2-1\right)\left(r^2-1\right)\right)}{\gamma^2\left(\kappa^2-1\right)\left(\gamma^2 m_\beta - r^2+1\right)} \\ &= -\frac{2}{\gamma^2} m_{\alpha\beta}, \\ C &= \frac{m_\alpha m_\beta(\kappa+r)\left(r^2-\kappa^2\right)+m_{\alpha\beta}^2\left(\gamma^2 m_\beta(\kappa+r)-2(\kappa+1)(r-1)r\right)}{4\gamma^2(\kappa+1)r\left(r^2-\kappa^2\right)} \\ &= \frac{(\kappa-1)m_\alpha(\kappa+r)^2-\gamma^2 m_{\alpha\beta}^2(\kappa+2r-1)}{4\gamma^4 r(\kappa+r)}.\end{aligned}$$

As $\kappa > 1$, we immediately have $A < 0$. Also, C has the sign of

$$\begin{aligned}(\kappa-1)m_\alpha(\kappa+r)^2 - m_{\alpha\beta}^2(\kappa-1+2r)\gamma^2 = (\kappa-1)&m_\alpha(\kappa+r)^2 \\ &- m_{\alpha\beta}^2(\kappa-1+2r)m_\beta^{-1}(\kappa^2-1).\end{aligned}$$

By the Cauchy-Schwarz inequality, $m_\alpha m_\beta \geq m_{\alpha\beta}^2$, so:

$$\begin{aligned}&(\kappa-1)m_\alpha(\kappa+r)^2 - m_{\alpha\beta}^2(\kappa-1+2r)m_\beta^{-1}(\kappa^2-1) \\ &\quad\geq m_\alpha\left\{(\kappa-1)(\kappa+r)^2-(\kappa-1+2r)(\kappa^2-1)\right\} \\ &\quad= m_\alpha(\kappa-1)(1-r)^2 \\ &\quad> 0.\end{aligned}$$

Hence C is positive, $A \cdot C$ is negative, and Equation (38) has two roots, one positive and one negative. Besides, as $m_{\alpha\beta} > 0$ by assumption, $B < 0$. As we have already established that λ must be negative, we conclude that

$$\lambda = \frac{-B + \sqrt{B^2 - 4AC}}{2A}.$$

Pulling out the term $\sum_i \beta_i D_{1,i}$ in (34) using (36), we express $D_{1,k}$ as a solution of the linear equation. It follows that

$$D_{1,k} = a\frac{m_\beta \left[\gamma^2 m_{\alpha\beta}\frac{\beta_k}{\sigma_k^2} - (\kappa^2 - r^2)\frac{\alpha_k}{\sigma_k^2}\right]}{(1+\kappa)\left[2\lambda(\kappa^2 - r^2) + (1-r)m_{\alpha\beta}\right]},$$

where the denominator is non-zero by (35). We can simplify those expressions further. We define

$$\widetilde{\lambda} := (\kappa - 1)\sqrt{r}(1+r)m_{\alpha\beta} + (\kappa - r)\sqrt{\Delta},$$

with

$$\Delta = (r+\kappa)^2(m_\alpha m_\beta - m_{\alpha\beta}^2) + (1+r)^2 m_{\alpha\beta}^2.$$

Then, $D_{1,k} = a\sqrt{r}c_k/\widetilde{\lambda}$ with

$$c_k = (\kappa^2 - r^2)m_\beta\frac{\alpha_k}{\sigma_k^2} + (1-\kappa^2)m_{\alpha\beta}\frac{\beta_k}{\sigma_k^2}.$$

Note that, as a rating policy induces the same effort level up to a scaling of the rating policy, the constant multiplier a is irrelevant in the original optimization problem, and an optimal policy is given by

$$u_k(t) = c_k\frac{\sqrt{r}}{\widetilde{\lambda}}e^{-rt} + \frac{\beta_k}{\sigma_k^2}e^{-\kappa t},$$

for all k.

C.10 Proof of Theorem 4.4

Recall the shorthand notation that will be used throughout this proof as well:

$$U(t) \coloneqq \sum_{k=1}^{K} \beta_k u_k(t),$$
$$V(t) \coloneqq \sum_{k=1}^{K} \alpha_k u_k(t),$$
$$U_0 \coloneqq \int_0^\infty U(t)e^{-t}\,\mathrm{d}t,$$
$$V_0 \coloneqq \int_0^\infty V(t)e^{-rt}\,\mathrm{d}t.$$

Given a scalar rating process Y of the form

$$Y_t = \sum_{k=1}^{K} \int_0^\infty u_k(t-s)\left[\mathrm{d}S_{k,s} - \alpha_k a^* \,\mathrm{d}s\right],$$

we note as we do in the proof of Theorem 4.1 that by Itô's isometry, for $\Delta \geq 0$,

$$\begin{aligned}\mathbf{Cov}[Y_t, Y_{t+\Delta}] &= \sum_{k=1}^{K} \sigma_k^2 \int_0^\infty u_k(s)u_k(s+\Delta)\,\mathrm{d}s \\ &+ \sum_{k=1}^{K}\sum_{k'=1}^{K} \int_{-\infty}^{t}\int_{-\infty}^{t+\Delta} \beta_k\beta_{k'} u_k(t-i)u_{k'}(t+\Delta-j)\,\mathbf{Cov}[\theta_i,\theta_j]\,\mathrm{d}i\,\mathrm{d}j,\end{aligned}$$

so that as $\mathbf{Cov}[\theta_i,\theta_j] = \gamma^2 e^{-|i-j|}/2$, after a change of variables in the last term,

$$\mathbf{Cov}[Y_t, Y_{t+\Delta}] = \sum_{k=1}^{K} \sigma_k^2 \int_0^\infty u_k(s)u_k(s+\Delta)\,\mathrm{d}s + \frac{\gamma^2}{2}\int_0^\infty\int_0^\infty U(i)U(j)e^{-|j+\Delta-i|}\,\mathrm{d}i\,\mathrm{d}j.$$

We seek to maximize $c'(a)$ (where a is the stationary action of the agent) among policies that satisfy the condition that the rating is proportional to the belief of the market. By Lemma 3.9, this means choosing Y, or equivalently the vector function $\mathbf{u} = (u_1, \ldots, u_K)$ to maximize the objective given by Proposition 3.10, subject to $\mathbf{Cov}[Y_t, Y_{t+\Delta}] = e^{-\Delta}$ for every $\Delta \geq 0$. Using the expression for $\mathbf{Cov}[Y_t, Y_{t+\Delta}]$ just obtained, this is equivalent to maximizing

$$\frac{\gamma^2}{2}\left[\int_0^\infty U(t)e^{-t}\,\mathrm{d}t\right]\left[\int_0^\infty V(t)e^{-rt}\,\mathrm{d}t\right],$$

subject to the continuum of constraints

$$\sum_{k=1}^{K}\sigma_k^2\int_0^\infty u_k(j)u_k(j+\Delta)\,\mathrm{d}j+\frac{\gamma^2}{2}\int_0^\infty\int_0^\infty U(i)U(j)e^{-|j+\Delta-i|}\,\mathrm{d}i\,\mathrm{d}j=e^{-\Delta},$$

for every $\Delta \geq 0$.

The continuum of constraints makes it difficult to solve this optimization problem directly. Instead, we solve a relaxed optimization program with a single constraint: we maximize $F(\mathbf{u})$ defined as

$$F(\mathbf{u}) \coloneqq \left[\int_0^\infty U(t)e^{-t}\,\mathrm{d}t\right]\left[\int_0^\infty V(t)e^{-rt}\,\mathrm{d}t\right],$$

which is the original objectif without the constant factor $\gamma^2/2$, and subject to $G(\mathbf{u}) = \frac{2}{1+r}$, where we define

$$G(\mathbf{u}) \coloneqq g(0) + (1-r)\int_0^\infty e^{-r\Delta}g(\Delta)\,\mathrm{d}\Delta,$$

and with

$$g(\Delta) \coloneqq \sum_{k=1}^{K}\sigma_k^2\int_0^\infty u_k(j)u_k(j+\Delta)\,\mathrm{d}j+\frac{\gamma^2}{2}\int_0^\infty\int_0^\infty U(i)U(j)e^{-|j+\Delta-i|}\,\mathrm{d}i\,\mathrm{d}j.$$

As it turns out, the solution of this relaxed optimization problem satisfies the original continuum of constraints.

Focusing on the relaxed optimization problem, we begin with the necessary first-order conditions that pin down uniquely a smooth solution. We then show sufficiency.

Necessary conditions. Let

$$L(\mathbf{u},\lambda) \coloneqq F(\mathbf{u}) + \lambda G(\mathbf{u}).$$

Assume there exists an admissible solution $\mathbf{u}^*$ to the relaxed optimization problem. At optimum, $F(\mathbf{u}^*) > 0$, because the optimal solution does at least as well as full transparency, which satisfies the public constraint and induces a positive equilibrium effort by our assumption that $m_{\alpha\beta} \geq 0$ (and thus yields a positive value for F, which is positively proportional to the marginal equilibrium cost of effort).

Proposition B.2 gives us existence of λ^* such that $\mathbf{u}^*$ maximizes $\mathbf{u} \mapsto L(\mathbf{u}, \lambda^*)$. As in the confidential setting, it will be useful to observe that $\lambda^* < 0$. Indeed, at optimum, $F(\mathbf{u}^*) > 0$ and $G(\mathbf{u}^*) = 2/(1+r) > 0$.

Proposition B.1 gives the first-order condition derived from Euler-Lagrange equations: if $\lambda = \lambda^*$ and $\mathbf{u} = \mathbf{u}^*$ then for all k and all t, $L_k(t) = 0$ where we define

$$L_k(t) \coloneqq F_k(t) + \lambda G_k(t),$$

and F_k and G_k are defined as follows:

$$F_k(t) \coloneqq \alpha_k U_0 e^{-rt} + \beta_k V_0 e^{-t},$$

and

$$\begin{aligned} G_k(t) \coloneqq{} & 2\sigma_k^2 u_k(t) + \gamma^2 \beta_k \int_0^\infty U(j) e^{-|j-t|} \,\mathrm{d}j \\ & + (1-r)\sigma_k^2 \int_0^\infty e^{-r\Delta} \left[u_k(t+\Delta) + u_k(t-\Delta)\right] \mathrm{d}\Delta \\ & + (1-r)\frac{\gamma^2\beta_k}{2} \int_0^\infty e^{-r\Delta} \int_0^\infty U(j) e^{-|j+\Delta-t|} \,\mathrm{d}j \,\mathrm{d}\Delta \\ & + (1-r)\frac{\gamma^2\beta_k}{2} \int_0^\infty e^{-r\Delta} \int_0^\infty U(i) e^{-|t+\Delta-i|} \,\mathrm{d}i \,\mathrm{d}\Delta. \end{aligned}$$

Throughout the proof, any function defined on the nonnegative real line is extended to the entire real line with the convention that these functions assign value zero to any negative input. Let some function $h : \mathbb{R}_+ \to \mathbb{R}$ be twice differentiable and such that h, h', h'' are all integrable. In the differentiation of our functions, we use the following arguments.

First, if $H(t) = \int_0^\infty h(i) e^{-|t+\Delta-i|} \,\mathrm{d}i$ for some $\Delta \geq 0$, then

$$H(t) = \int_0^{t+\Delta} h(i) e^{-(t+\Delta-i)} \,\mathrm{d}i + \int_{t+\Delta}^\infty h(i) e^{t+\Delta-i} \,\mathrm{d}i,$$

so that

$$H''(t) = H(t) - 2h(t+\Delta).$$

Similarly, if instead $H(t) = \int_0^\infty h(j)e^{-|j+\Delta-t|}\,\mathrm{d}j$ then if $t > \Delta$,

$$H(t) = \int_0^{t-\Delta} h(j)e^{(j+\Delta-t)}\,\mathrm{d}j + \int_{t-\Delta}^\infty h(j)e^{-(j+\Delta-t)}\,\mathrm{d}j,$$

and for every t,

$$H''(t) = H(t) - 2h(t-\Delta).$$

Finally, if $H(t) = \int_0^\infty e^{-r\Delta}\left[h(t+\Delta) + h(t-\Delta)\right]\mathrm{d}\Delta$, then

$$H'(t) = e^{-rt}h(0) + \int_0^\infty e^{-r\Delta}\left[h'(t+\Delta) + h'(t-\Delta)\right]\mathrm{d}\Delta,$$

and

$$H''(t) = -re^{-rt}h(0) + e^{-rt}h'(0) + \int_0^\infty e^{-r\Delta}\left[h''(t+\Delta) + h''(t-\Delta)\right]\mathrm{d}\Delta.$$

(Since we have extended h to the real line, we should add that by convention, the derivatives of h at 0 is defined to be the right-derivative of h at 0, which by the smoothness assumption imposed on h is well-defined.)

We then have

$$\begin{aligned}
L_k(t) - L_k''(t) = {} & \alpha_k U_0(1-r^2)e^{-rt} \\
& + 2\lambda\sigma_k^2[u_k(t) - u_k''(t)] + 2\lambda\gamma^2\beta_k U(t) \\
& + \lambda(1-r)\sigma_k^2 \int_0^\infty e^{-r\Delta}[u_k(t+\Delta) + u_k(t-\Delta)]\,\mathrm{d}\Delta \\
& - \lambda(1-r)\sigma_k^2 \int_0^\infty e^{-r\Delta}[u_k''(t+\Delta) + u_k''(t-\Delta)]\,\mathrm{d}\Delta \\
& - \lambda(1-r)\sigma_k^2\left[-re^{-rt}u_k(0) + u_k'(0)e^{-rt}\right] \\
& + \lambda(1-r)\gamma^2\beta_k \int_0^\infty e^{-r\Delta}U(t-\Delta)\,\mathrm{d}\Delta \\
& + \lambda(1-r)\gamma^2\beta_k \int_0^\infty e^{-r\Delta}U(t+\Delta)\,\mathrm{d}\Delta.
\end{aligned}$$

Next, we let $p_k(t) \coloneqq L_k(t) - L_k''(t)$, as well as

$$J_k(t) \coloneqq \int_0^\infty e^{-r\Delta} \left[u_k(t+\Delta) + u_k(t-\Delta)\right] \mathrm{d}\Delta,$$
$$J(t) \coloneqq \sum_{k=1}^{K} \beta_k J_k(t).$$

We observe that $J_k''(t) = -2ru_k(t) + r^2 J_k(t)$. Plugging in J_k in the expression for p_k:

$$\begin{aligned} p_k(t) = \alpha_k U_0(1-r^2)e^{-rt} + 2\lambda\sigma_k^2 \left[u_k(t) - u_k''(t)\right] + 2\lambda\gamma^2\beta_k U(t) \\ + 2r\lambda(1-r)\sigma_k^2 u_k(t) + \lambda(1-r)(1-r^2)\sigma_k^2 J_k(t) + \lambda(1-r)\gamma^2\beta_k J(t). \end{aligned}$$

After differentiation, we get

$$\begin{aligned} p_k''(t) = r^2\alpha U_0(1-r^2)e^{-rt} + 2\lambda\sigma_k^2 \left[u_k''(t) - u_k''''(t)\right] + 2\lambda\gamma^2\beta U''(t) \\ + 2r\lambda(1-r)\sigma_k^2 u_k''(t) + \lambda(1-r)(1-r^2)\sigma_k^2 \left[-2ru_k(t) + r^2 J_k(t)\right] \\ + \lambda(1-r)\gamma^2\beta_k \left[-2rU(t) + r^2 J(t)\right]. \end{aligned}$$

Finally, we let $q_k(t) \coloneqq p_k''(t) - r^2 p_k(t)$. We have

$$\begin{aligned} q_k(t) = 2\lambda\sigma_k^2 \left[u_k''(t) - u_k''''(t)\right] - r^2 2\lambda\sigma_k^2 \left[u_k(t) - u_k''(t)\right] \\ + 2\lambda\gamma^2\beta_k U''(t) - 2r^2\lambda\gamma^2\beta_k U(t) \\ + 2r\lambda(1-r)\sigma_k^2 u_k''(t) - 2r^3\lambda(1-r)\sigma_k^2 u_k(t) \\ - 2r\lambda(1-r)(1-r^2)\sigma_k^2 u_k(t) \\ - 2r\lambda(1-r)\gamma^2\beta_k U(t). \end{aligned}$$

We must have $q_k(t) = 0$ for all k and all t. In particular, and since $\lambda \neq 0$,

$$\frac{1}{2\lambda} \sum_{k=0}^{n} \frac{\beta_k}{\sigma_k^2} q_k(t) = 0,$$

hence

$$\begin{aligned} U'' - U'''' - r^2(U - U'') + \gamma^2 m_\beta U'' - r^2\gamma^2 m_\beta U \\ + r(1-r)U'' - r(1-r)U - r(1-r)\gamma^2 m_\beta U = 0. \end{aligned}$$

The characteristic polynomial associated to this homogeneous linear differential equation has roots $\pm\sqrt{1+\gamma^2 m_\beta} = \pm\kappa$ and $\pm\sqrt{r}$. As we have assumed that the solution to the optimization problem is admissible, it follows that U must be bounded, and we discard the positive roots. Thus U must have the form

$$U(t) = C_1 e^{-\sqrt{r}t} + C_2 e^{-\kappa t}, \tag{39}$$

for some constants C_1 and C_2.

Next, pick an arbitrary pair (i,j) with $i \neq j$, and define $Z_{ij}(t) \coloneqq \beta_i \sigma_j^2 u_j(t) - \beta_j \sigma_i^2 u_i(t)$. That $(\beta_i q_j(t) - \beta_j q_i(t))/(2\lambda) = 0$ yields, after simplification, the following differential equation for Z_{ij}:

$$Z_{ij}'' - Z_{ij}'''' - r^2(Z_{ij} - Z_{ij}'') + r(1-r)(Z_{ij}'' - Z_{ij}) = 0.$$

The characteristic polynomial associated to this homogeneous linear differential equation has roots ± 1 and $\pm\sqrt{r}$. As Z_{ij} must be bounded, we get that Z_{ij} has the form

$$Z_{ij}(t) = C_1' e^{-\sqrt{r}t} + C_2' e^{-t}, \tag{40}$$

for some constants C_1' and C_2'.

Putting together (39) and (40), we get that

$$u_k(t) = D_{1,k} e^{-\sqrt{r}t} + D_{2,k} e^{-\kappa t} + D_{3,k} e^{-t}, \tag{41}$$

for some constants $D_{1,k}$, $D_{2,k}$, $D_{3,k}$.

Determination of the constants. As in the proof of the confidential setting, here we have established that the solution belongs to a family of functions that are sums of some given scaled time exponentials. We now solve for the constant factors $D_{1,k}$, $D_{2,k}$, $D_{3,k}$, $k \geq 1$.

We first note that, since the term e^{-t} vanishes in Equation (39) that gives the general form of the function U, the equality

$$\sum_{k=1}^{K} \beta_k D_{3,k} = 0 \tag{42}$$

obtains.

Using (42), we plug (41) in the equation for $L_k(t)$ and get that

$$L_k(t) = L_{1,k} e^{-rt} + L_{2,k} e^{-\kappa t} + L_{3,k} e^{-t},$$

where $L_{1,k}$, $L_{2,k}$, $L_{3,k}$ are scalar factors that will be expressed as a function of the primitives of the model and the constants $D_{1,k}$, $D_{2,k}$, $D_{3,k}$. Note that the exponential $e^{-\sqrt{r}t}$, that exists in the general form of $u_k(t)$ given in (41), vanishes after simplification, while instead an exponential e^{-rt} appears that is not present in $u_k(t)$.

We observe that

$$\begin{aligned} L_{2,k} &= \frac{2\sigma_k^2\lambda(r-\kappa^2)}{(r-\kappa)(\kappa+r)}D_{2,k} + \frac{2\gamma^2\lambda\beta_k\,(r-\kappa^2)}{(\kappa-1)(\kappa+1)(\kappa-r)(\kappa+r)}\sum_{i=1}^{K}\beta_i D_{2,i} \\ &= \frac{2\lambda\sigma_k^2\,(r-\kappa^2)\,D_{2,k}}{(r-\kappa)(\kappa+r)} + \frac{2\lambda\beta_k\,(\kappa^2-r)}{m_\beta(r-\kappa)(\kappa+r)}\sum_{i=1}^{K}\beta_i D_{2,i}, \end{aligned}$$

using that $\gamma^2 = (\kappa^2-1)/m_\beta$. That $L_{2,k}=0$ for all k implies

$$D_{2,k} = a\frac{\beta_k}{\sigma_k^2}, \tag{43}$$

for some constant a. It can be seen that if $a=0$, then $D_{1,k}=D_{2,k}=D_{3,k}=0$ for all k, in which case $u_k=0$ and the variance normalization constraint is violated. Hence, in the remainder of the proof, we will assume $a\neq 0$. (As it turns out, as ratings yield the same market belief up to a scalar, the precise value of a will be irrelevant as long as it is non-zero.) In particular,

$$\sum_{k=1}^{K}\alpha_k D_{2,k} = a m_{\alpha\beta},$$

and

$$\sum_{k=1}^{K}\beta_k D_{2,k} = a m_\beta.$$

Using (42), (43), and $\gamma^2=(\kappa^2-1)/m_\beta$ we get

$$\begin{aligned} L_{3,k} = {}& \frac{(\kappa^2-1)\,\lambda\beta_k}{(\sqrt{r}-1)\,(r+1)m_\beta}\sum_{i=1}^{K}\beta_i D_{1,i} + \frac{\beta_k}{r+\sqrt{r}}\sum_{i=1}^{K}\alpha_i D_{1,i} \\ &+ \frac{\beta_k}{r+1}\sum_{i=1}^{K}\alpha_i D_{3,i} + \frac{2\lambda\sigma_k^2}{r+1}D_{3,k} + \frac{a\beta_k m_{\alpha\beta}}{\kappa+r} + \frac{a(\kappa+1)\lambda\beta_k}{r+1}. \end{aligned} \tag{44}$$

As $L_{3,k}=0$ for all k, we can multiply (44) by β_k/σ_k^2, sum over k, and use (42) to get that $D_{3,k} = 0$. In addition, after plugging $D_{3,k} = 0$, the term $L_{3,k}$ simplifies to

$$\frac{(\kappa^2 - 1)\,\lambda\beta_k}{(\sqrt{r} - 1)\,(r+1)m_\beta} \sum_{i=1}^{K} \beta_i D_{1,i} + \frac{\beta_k}{r + \sqrt{r}} \sum_{i=1}^{K} \alpha_i D_{1,i} + \frac{a\beta_k m_{\alpha\beta}}{\kappa + r} + \frac{a(\kappa+1)\lambda\beta_k}{r+1} = 0, \tag{45}$$

which we will use to determine λ.

Finally, given $D_{2,k} = a\beta_k/\sigma_k^2$ and $D_{3,k} = 0$, and using that $\gamma^2 = (\kappa^2 - 1)/m_\beta$, the remaining constant $L_{1,k}$ simplifies to

$$\begin{aligned} L_{1,k} = &\left(\frac{\alpha_k}{\sqrt{r}+1} - \frac{(\kappa^2 - 1)\,\lambda\beta_k}{(\sqrt{r} - 1)\,\sqrt{r}(r+1)m_\beta}\right) \sum_{i=1}^{K} \beta_i D_{1,i} + \frac{\lambda\,(\sqrt{r}+1)\,\sigma_k^2}{\sqrt{r}} D_{1,k} \\ &+ \frac{a\alpha_k m_\beta}{\kappa+1} + \frac{a\lambda\beta_k(\kappa + r)}{r+1}. \end{aligned} \tag{46}$$

As $L_{1,k} = 0$ must hold for every k, we multiply (46) by β_k/σ_k^2, sum over k, and we get an equation that the term $\sum_i \beta_i D_{1,i}$ must satisfy:

$$\begin{aligned} \left(\frac{m_{\alpha\beta}}{\sqrt{r}+1} - \frac{(\kappa^2 - 1)\,\lambda m_\beta}{(\sqrt{r} - 1)\,\sqrt{r}(r+1)m_\beta}\right) \sum_{i=1}^{K} \beta_i D_{1,i} + \frac{\lambda\,(\sqrt{r}+1)}{\sqrt{r}} \sum_{i=1}^{K} \beta_i D_{1,i} \\ + \frac{a m_{\alpha\beta} m_\beta}{\kappa+1} + \frac{a\lambda m_\beta(\kappa + r)}{r+1} = 0. \end{aligned} \tag{47}$$

As $m_{\alpha\beta} \geq 0$ and $m_\beta > 0$,

$$\frac{a m_{\alpha\beta} m_\beta}{\kappa+1} + \frac{a\lambda m_\beta(\kappa + r)}{r+1} \neq 0, \tag{48}$$

which implies that the factor of $\sum_i \beta_i D_{1,i}$ is non-zero. Similarly, if we multiply (46) by α_k/σ_k^2 and sum over k, we get an equation that the term $\sum_i \alpha_i D_{1,i}$ must satisfy:

$$\begin{aligned} \left(\frac{m_{\alpha}}{\sqrt{r}+1} - \frac{(\kappa^2 - 1)\,\lambda m_{\alpha\beta}}{(\sqrt{r} - 1)\,\sqrt{r}(r+1)m_\beta}\right) \sum_{i=1}^{K} \beta_i D_{1,i} + \frac{\lambda\,(\sqrt{r}+1)}{\sqrt{r}} \sum_{i=1}^{K} \alpha_i D_{1,i} \\ + \frac{a m_{\alpha} m_\beta}{\kappa+1} + \frac{a\lambda m_{\alpha\beta}(\kappa + r)}{r+1} = 0. \end{aligned} \tag{49}$$

Now we can solve for $\sum_i \alpha_i D_{1,i}$ and $\sum_i \beta_i D_{1,i}$ using (47) and (49). Plugging in the solutions in (45), we get a rational expression in λ, whose denominator is

$$\begin{aligned}(\kappa+1)(r+1)^2\left(\sqrt{r}-1\right)\sqrt{r}\left(\sqrt{r}+1\right)^2 m_{\alpha\beta}(\kappa+r)&\\ +(\kappa+1)\lambda(r+1)\left(\sqrt{r}+1\right)^3(r-\kappa)(\kappa+r)^2,&\end{aligned}$$

and whose numerator is

$$\begin{aligned}-a(r+1)^2\left(\sqrt{r}-\kappa\right)\left(m_\alpha m_\beta(\kappa+r)^2-(\kappa+1)m_{\alpha\beta}^2(\kappa+2r-1)\right)&\\ +4a(\kappa+1)\lambda(r+1)\sqrt{r}\left(\sqrt{r}+1\right)^2 m_{\alpha\beta}\left(\sqrt{r}-\kappa\right)(\kappa+r)&\\ +a(\kappa+1)^2\lambda^2\left(\sqrt{r}+1\right)^4\left(\sqrt{r}-\kappa\right)(\kappa+r)^2.&\end{aligned}$$

We observe that the numerator, which must equal zero, yields a quadratic equation in λ,

$$a\left(\sqrt{r}-\kappa\right)\left(A\lambda^2+B\lambda+C\right)=0, \tag{50}$$

where:

$$\begin{aligned}A &:= (\kappa+1)^2(\kappa+r)^2\left(\sqrt{r}+1\right)^4\\ B &:= 4(\kappa+1)(\kappa+r)\left(\sqrt{r}+1\right)^2\sqrt{r}(r+1)m_{\alpha\beta}\\ C &:= -(r+1)^2\left(m_\alpha m_\beta(\kappa+r)^2-(\kappa+1)m_{\alpha\beta}^2(\kappa+2r-1)\right)\\ &= -(r+1)^2\left((\kappa+r)^2\left(m_\alpha m_\beta-m_{\alpha\beta}^2\right)+(1-r)^2 m_{\alpha\beta}^2\right).\end{aligned}$$

Next, we have that $A>0$, and also that $C<0$ which owes to the Cauchy-Schwarz inequality $m_\alpha m_\beta \geq m_{\alpha\beta}^2$ and to $\kappa>1$. Hence there are two real roots of (50), one negative, and one positive. As $B>0$ and we have established that $\lambda<0$, it follows that

$$\lambda=\frac{-B+\sqrt{B^2-4AC}}{2A},$$

which, after simplification, reduces to

$$\lambda=-\frac{(r+1)\left(\sqrt{\Delta}+2\sqrt{r}m_{\alpha\beta}\right)}{(\kappa+1)\left(\sqrt{r}+1\right)^2(\kappa+r)},$$

with $\Delta := (r+\kappa)^2(m_\alpha m_\beta-m_{\alpha\beta}^2)+(1+r)^2 m_{\alpha\beta}^2$.

Finally, (46) and (47) yield a linear equation that determines $D_{1,k}$:

$$D_{1,k} = -\frac{a\sqrt{r}(r+1)m_\beta\left(\sqrt{r}-\kappa\right)(\kappa+r)}{(\kappa+1)\left(\left(r^2-1\right)\sqrt{r}m_{\alpha\beta}+\lambda\left(\sqrt{r}+1\right)^2\left(r^2-\kappa^2\right)\right)}\frac{\alpha_k}{\sigma_k^2}$$
$$-\frac{a(r+1)\sqrt{r}m_{\alpha\beta}\left(\kappa+r-\sqrt{r}-1\right)+a\lambda\left(r-\sqrt{r}\right)\left(\sqrt{r}+1\right)^2(\kappa+r)}{\left(\left(r^2-1\right)\sqrt{r}m_{\alpha\beta}+\lambda\left(\sqrt{r}+1\right)^2\left(r^2-\kappa^2\right)\right)}\frac{\beta_k}{\sigma_k^2}.$$

Letting $\widetilde{\lambda} \coloneqq (\kappa-1)\sqrt{r}(1+r)m_{\alpha\beta}+(\kappa-r)\sqrt{\Delta}$, we can make further simplifications and express the solution in a form similar to that of the confidential case: we have

$$u_k(t) = ad_k\frac{\sqrt{r}}{\widetilde{\lambda}}e^{-\sqrt{r}t} + a\frac{\beta_k}{\sigma_k^2}e^{-\kappa t},$$

with

$$d_k = \frac{\kappa-\sqrt{r}}{\kappa-r}c_k + \widetilde{\lambda}\frac{\sqrt{r}-1}{\kappa-r}\frac{\beta_k}{\sigma_k^2},$$

and as in the confidential setting

$$c_k = (\kappa^2-r^2)m_\beta\frac{\alpha_k}{\sigma_k^2} + (1-\kappa^2)m_{\alpha\beta}\frac{\beta_k}{\sigma_k^2}. \tag{51}$$

As a rating policy induces the same effort level up to a scaling, the constant multiplier a is irrelevant in the original optimization problem, and we can use, for example, $a = 1$ in the preceding expressions.

Additional References

Burns, J.A. (2014). *Introduction to The Calculus of Variations and Control with Modern Applications*, Chapman and Hall/CRC.