Dynamic Atomic Congestion Games with Seasonal Flows

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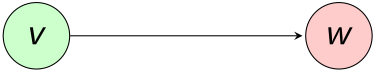
Dynamic congestion games

- Most models of congestion games are static.
- The static game represents the steady state of a dynamic model with constant flow over time.
- Even if the flow of travellers is constant, how is the steady state reached?
- In real life traffic flows are rarely constant, although often (nearly) periodic. How does this affect the steady state?

Edge dynamics

Each edge had a travel time and a capacity. For example, $\tau_e = 2$ and $\gamma_e = 2$.



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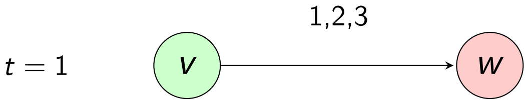

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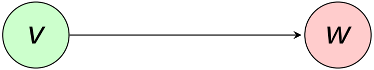

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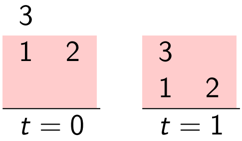

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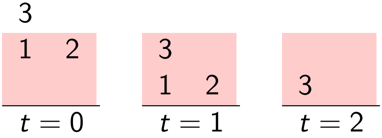

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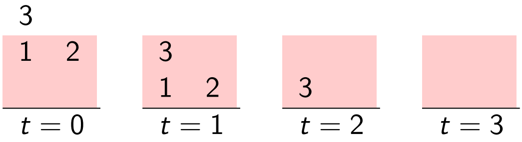


Related literature

Continuous time and flows

- Koch and Skutella (2011) provide a characterization of Nash flows over time via a sequence of thin flows with resetting.
- Cominetti, Correa and Larré (2011) prove existence and uniqueness of Nash flows over time.
- Macko, Larson and Steskal (2013) analyse Braess's paradox for flows over time.

Discrete time and flows

- Werth, Holzhauser and Krumke (2014).

Model

- A directed network $\mathscr{N} = (V, E, (\tau_e)_{e \in E}, (\gamma_e)_{e \in E})$ with a single source and sink, where
  - $\tau_e \in \mathbb{N}$ is the travel time,
  - $\gamma_e \in \mathbb{N}$ is the capacity.
- Time is discrete and players are atomic.
- Inflow is deterministic, but is allowed to be periodic.

Model

- At each stage t, a generation G_t of δ_t players departs from the source. Players are ordered by priority $\lhd$.
- At time t, player $[it]$ observes the choices of players $[js] \lhd [it]$ and chooses an edge $e = (s, v) \in E$.
- Player $[it]$ arrives at time $t + \tau_e$ at the exit of e.

Model

- At this exit a queue might have formed by
  1. players who entered e before $[it]$,
  2. players who entered e at the same time as $[it]$, but have higher priority.

  Recall at most γ_e players can exit e simultaneously.
- When exiting edge $e = (s, v)$, player $[it]$ chooses an outgoing edge $e' = (v, v')$. This is repeated until player $[it]$ arrives at the destination.

This defines a game with perfect information $\Gamma(\mathscr{N}, K, \delta)$.

Latencies

- $c_{it}(\sigma) = \sum_{e \in r_{it}(\sigma)} \tau_e$ is the travel time of player $[it]$,
- $w_{it}(\sigma)$ is the waiting time of player $[it]$,
- $\ell_{it}(\sigma)$ is the total latency suffered by player $[it]$:

$$\ell_{it}(\sigma) = c_{it}(\sigma) + w_{it}(\sigma).$$

- $\ell_t(\sigma) = \sum_{[it] \in G_t} \ell_{it}(\sigma)$ is the total cost of generation G_t.

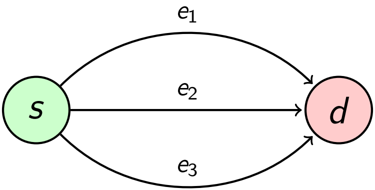
Solution concepts

- Equilibrium. Each player minimizes her own total latency given the queues in the system.
  - Exists: multiple equilibria
  - Subgame perfect Markov equilibrium
- Optimum. A social planner controls all players and seeks to minimize the long-run total costs, averaged over a period.

Overview

Uniform inflow

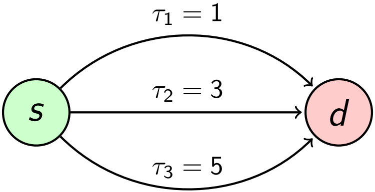
In a parallel network each route is made of a single edge. The capacity of the network is $\gamma = \sum_e \gamma_e$.



We assume that $\delta_t = \gamma$ for all $t \in \mathbb{N}$.

Example

Inflow$=(3, 3, 3, \ldots)$. What happens in the equilibrium?



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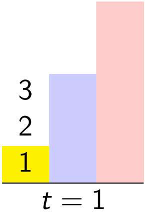

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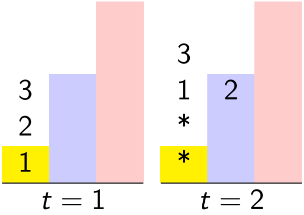

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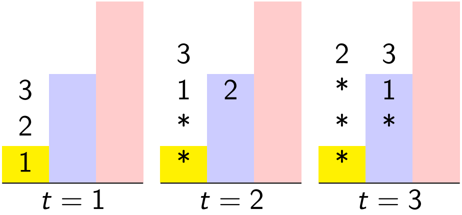

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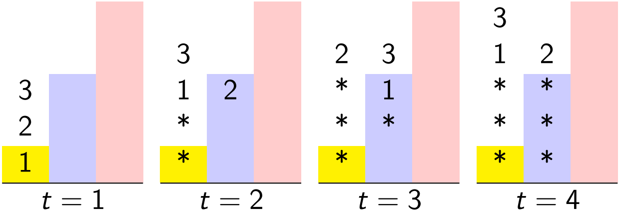

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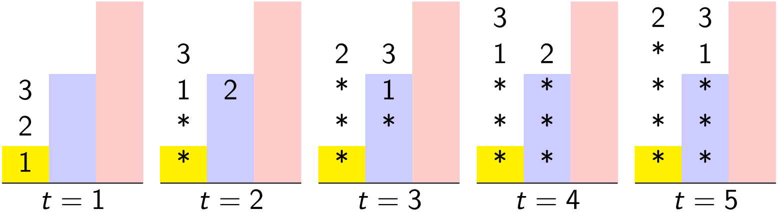

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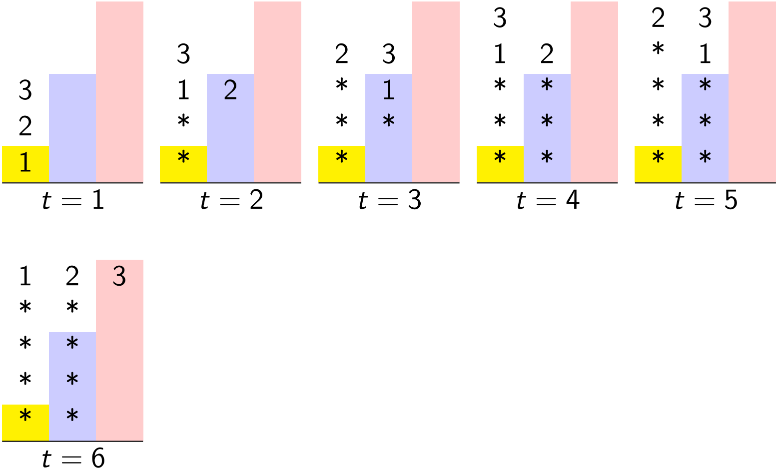

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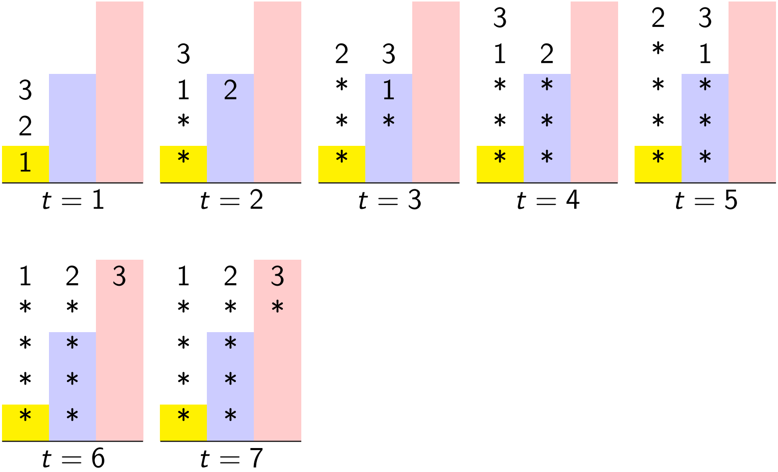

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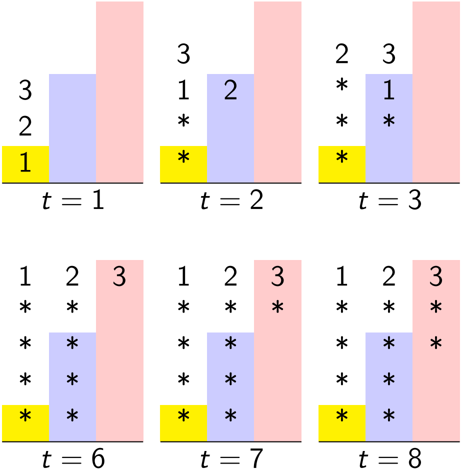

. . . Total costs=3·5=15.

Optimum

Can we do better in the long-run than 15 per generation?

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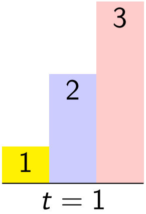

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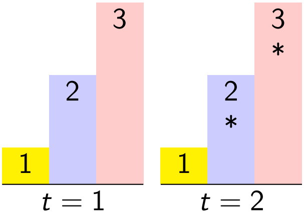

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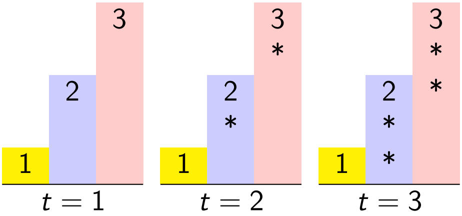

... Total costs=1+3+5=9.

Steady state

Proposition

Let $\mathscr{N}$ be a parallel network. Then

$$WEq(\mathscr{N}, \gamma) = \gamma \cdot \max_{e \in E} \tau_e,$$

$$Opt(\mathscr{N}, \gamma) = \sum_{e \in E} \gamma_e \cdot \tau_e.$$

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Equilibrium flows eventually coincide with optimal flows, but equilibrium costs are higher.

Price of anarchy

- Let $\mathscr{N}$ be a parallel network. Then

$$PoA(\mathscr{N}, \gamma) = \frac{WEq(\mathscr{N}, \gamma)}{Opt(\mathscr{N}, \gamma)} \leq \frac{\max_e \tau_e}{\min_e \tau_e}.$$

- The price of anarchy is unbounded over the class of parallel networks.

Example Bad network: $\tau_1 = 1$, $\gamma_1 = N$, $\tau_2 = N$, $\gamma_2 = 1$.

$$PoA(\mathscr{N}, \gamma) = \frac{(N+1) \cdot N}{2N}.$$

Periodic departures

- Inflow is a K-periodic vector:

$$\delta = (\delta_1, \ldots, \delta_K) \in \mathbb{N}^K$$

  such that $\sum_{k=1}^{K} \delta_k = K \cdot \gamma$. We denote $\mathbb{N}_K(\gamma)$ the set of such sequences.
- When δ is not-uniform, queues have to be created when there is a surge of players.

Example

Equilibrium for inflow (4,2,3).

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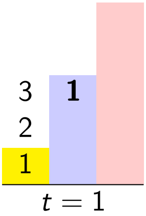

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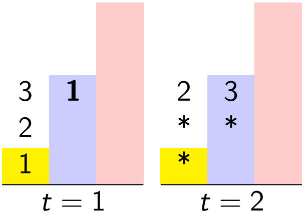

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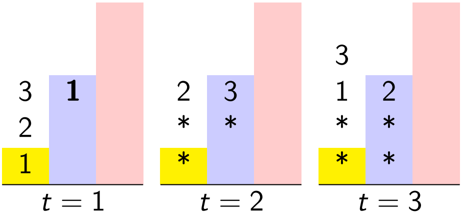

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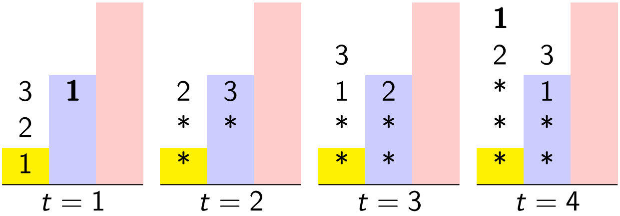

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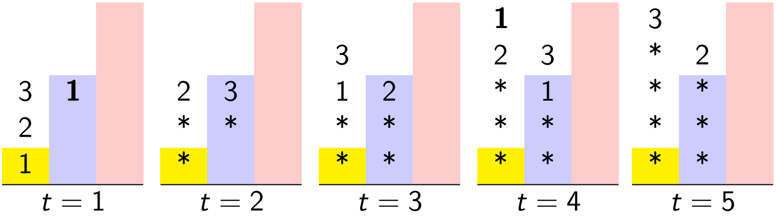

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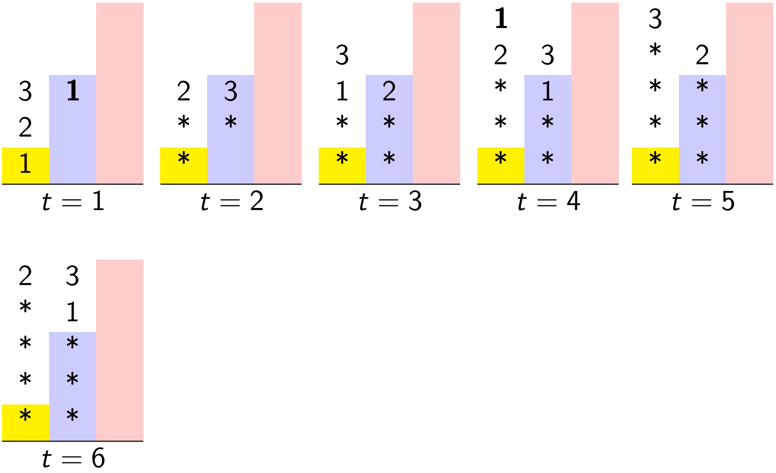

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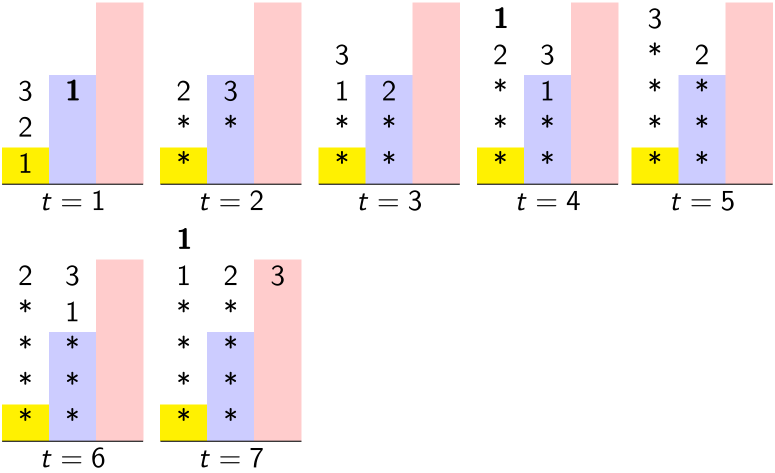

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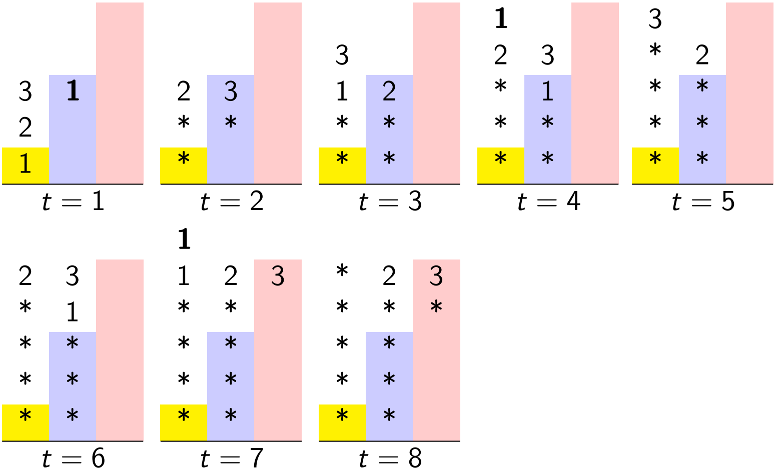

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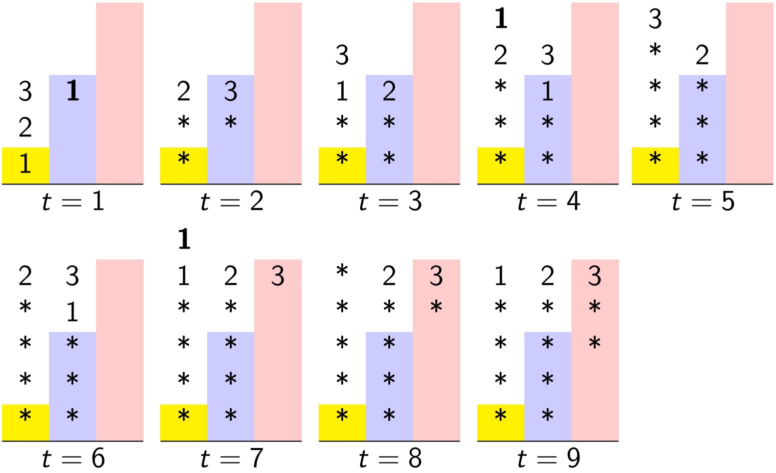

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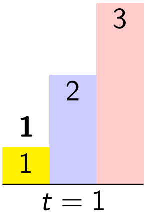

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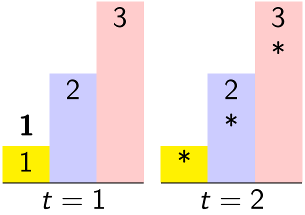

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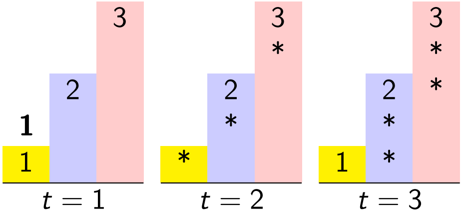

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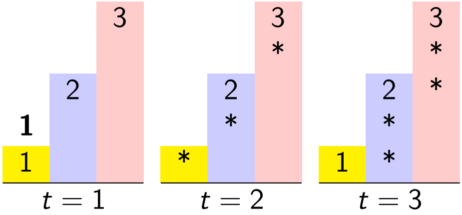

Both in the equilibrium as in the optimum, the fourth player behaves as if he was postponed by one stage.

Measure of periodicity

Definition

For any two elements $\delta, \delta' \in \mathbb{N}_K(\gamma)$, we say that δ' is obtained from δ by an elementary operation if there exist an i with $\delta_i > \gamma$ such that $\delta'_i = \delta_i - 1$, $\delta'_{i+1} = \delta_i + 1$.

Let $D(\boldsymbol{\delta})$ be the minimal number of elementary operations one has to perform to transform δ into γ_K.

Measure of periodicity

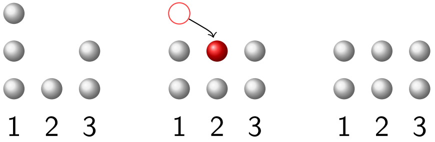

Figure: 1 operation needed to transform $(3, 1, 2)$ into $(2, 2, 2)$.

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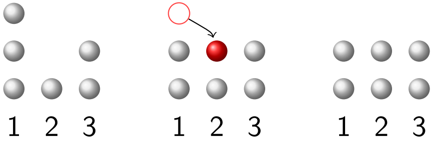

Figure: 1 operation needed to transform $(3, 1, 2)$ into $(2, 2, 2)$.

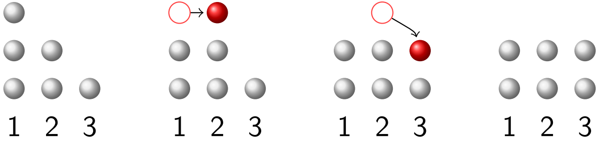

Figure: 2 operations needed to transform $(3, 2, 1)$ into $(2, 2, 2)$.

Steady state

Theorem

Let $\mathscr{N}$ *be a parallel network and* $\delta \in \mathbb{N}_K(\gamma)$*. Then*

$$WEq(\mathscr{N}, K, \delta) = K \cdot \gamma \cdot \max_{e \in E} \tau_e + D(\delta),$$

$$Opt(\mathscr{N}, K, \delta) = K \cdot \sum_{e \in E} \gamma_e \cdot \tau_e + D(\delta).$$

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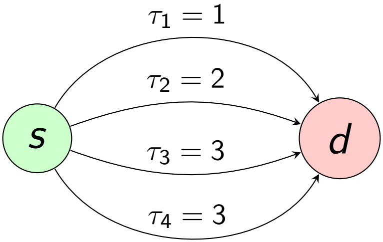
Equilibrium flows eventually coincide with optimal flows.

Overview

Parallel network below capacity



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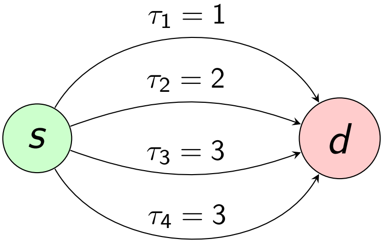

Equilibrium.

- If $\delta = 3$, then $WEq(\mathscr{N}, 1, \delta) = 9$.

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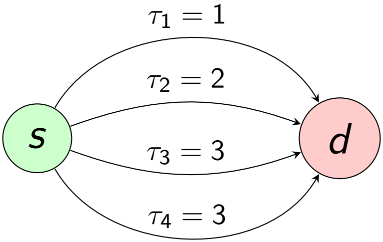


Equilibrium.

- If $\delta = 3$, then $WEq(\mathscr{N}, 1, \delta) = 9$.
- If $\delta = (6, 0)$, then $WEq(\mathscr{N}, 2, \delta) = 16 < 18$.

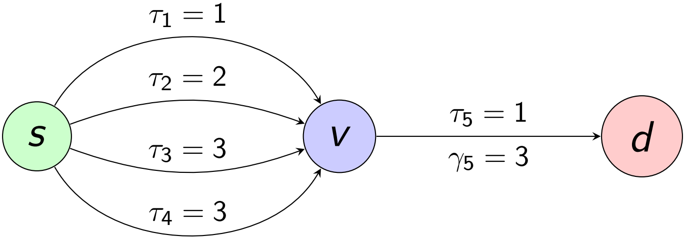
Steady state below capacity

Proposition

Let $\mathscr{N}$ be a parallel network with capacity γ and let $\delta \in \mathbb{N}_K(\gamma')$, where $\gamma' \leq \gamma$. Then

$$WEq(\mathscr{N}, K, \delta) \leq K \cdot \gamma' \cdot \max_{e \in E} \tau_e + D(\delta).$$

Chain-of-parallel network



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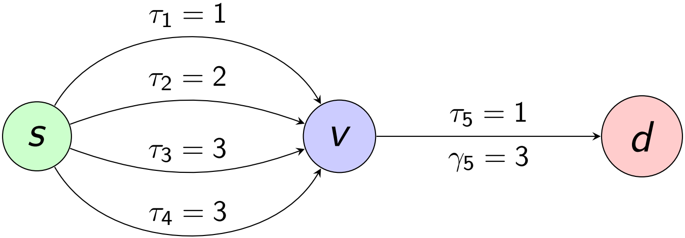
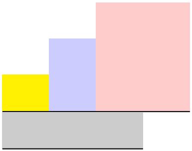


Equilibrium. If $\delta = (6, 0)$, then



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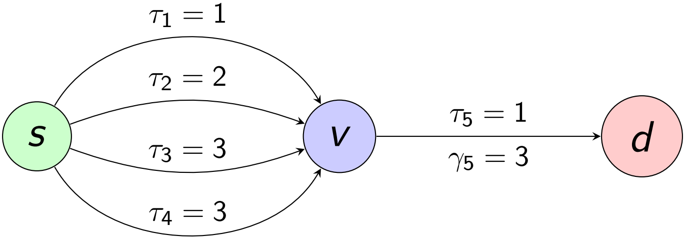

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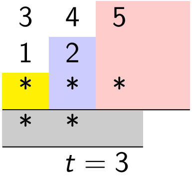

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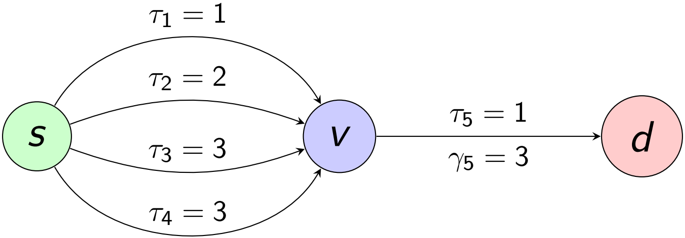

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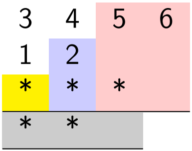

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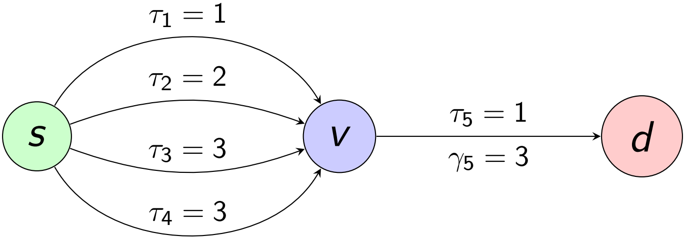

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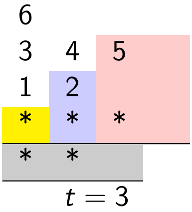

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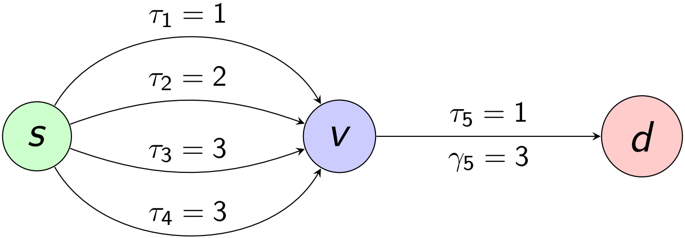

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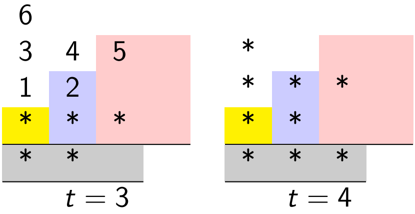

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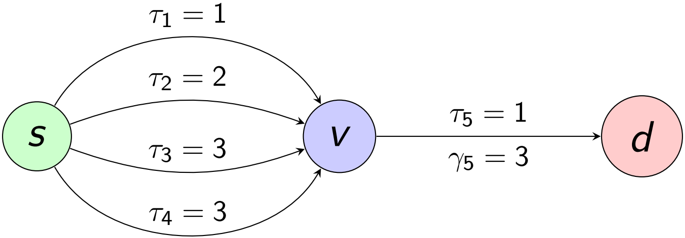

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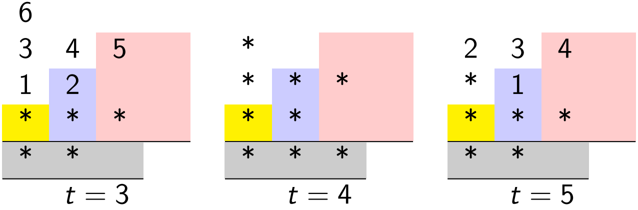
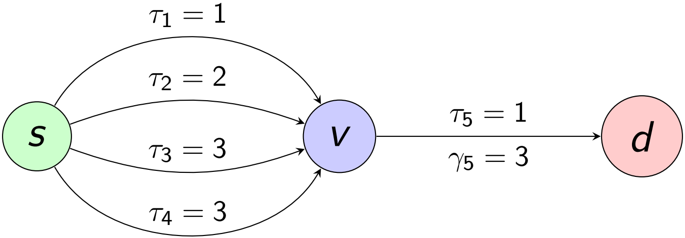
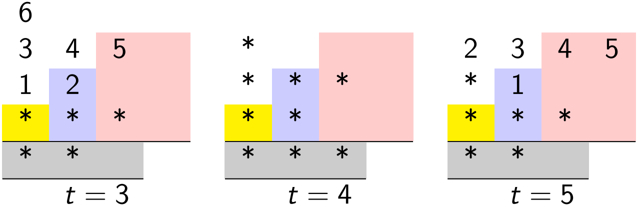


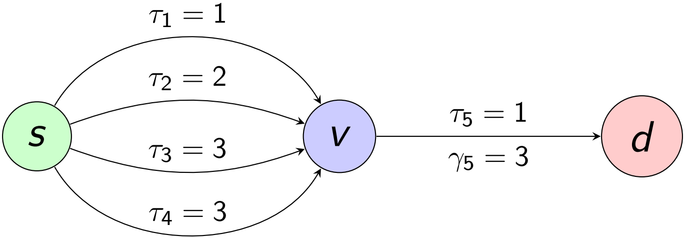
Chain-of-parallel network



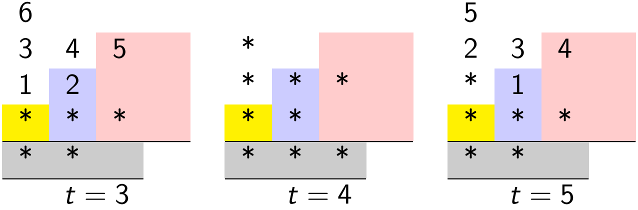
Equilibrium. If $\delta = (6, 0)$, then



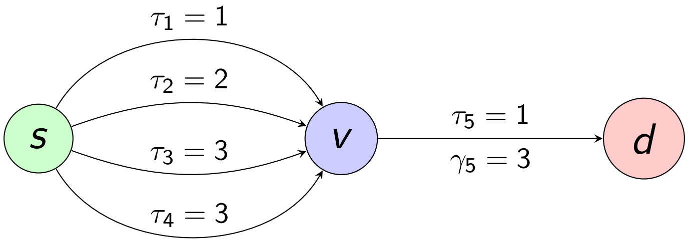
Chain-of-parallel network



Equilibrium. If $\delta = (6, 0)$, then



Chain-of-parallel network



Equilibrium. If $\delta = (6, 0)$, then

- $WEq(\mathscr{N}, 2, \delta) = 22$ (earliest-arrival property).
- $WEq^{*}(\mathscr{N}, 2, \delta) = 25$ (no overtaking).
- $WEq^{**}(\mathscr{N}, 2, \delta) = 27$ (allow overtaking).

Optimum

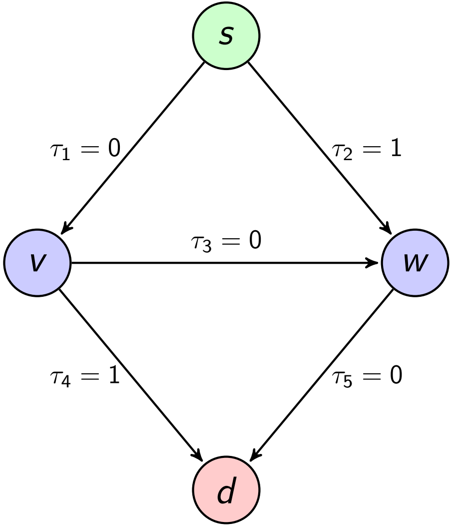
Let F^* be the (static) min-cost flow. Define $M_p^r(\sigma) = \sum_{t=pK+1}^{(p+1)K} N_p^r(\sigma)$.

Theorem

Let $\delta \in \mathbb{N}_K(\gamma)$. Then there exists an optimal strategy profile σ such that $M_p^r(\sigma) = K \cdot F_r^*$ for each route r and each period p, and

$$Opt(\mathscr{N}, K, \delta) = Opt(\mathscr{N}, K, \gamma) + D(\delta).$$

Braess's network



Braess's network

Worst equilibrium.

- Player [11] and [21] choose $e_1 e_3 e_5$.
- Player [12] chooses $e_1 e_3 e_5$ and [22] chooses $e_2 e_5$.
- Player [13] chooses $e_1 e_3 e_5$ and [23] chooses $e_1 e_4$.
- Player [14] chooses $e_2 e_5$ and [24] chooses $e_1 e_3 e_5$.
- For $t \geq 5$, player $[1t]$ chooses $e_1 e_4$ and $[2t]$ chooses $e_2 e_5$.

Total costs=3+3=6.

Braess's network

Best equilibrium.

- Player [11] chooses $e_1 e_3 e_5$ and [21] chooses $e_2 e_5$.
- For $t \geq 2$, player $[1t]$ chooses $e_1 e_4$ and $[2t]$ chooses $e_2 e_5$.

Total costs=1+1=2.

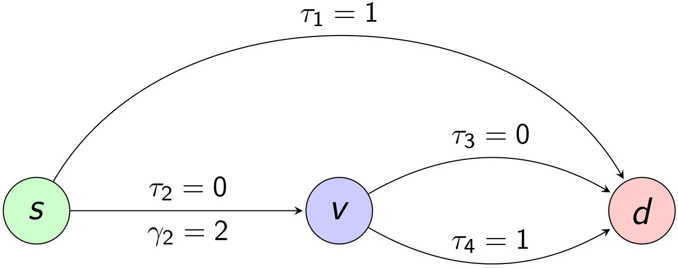
Braess's network

Proposition

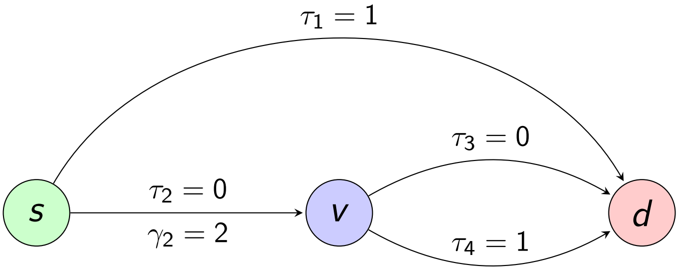
For every even integer n, there exists a network $\mathscr{N}$ in which $|V| = n$ such that

$$PoA(\mathscr{N}, \gamma) = \frac{WEq(\mathscr{N}, \gamma)}{BEq(\mathscr{N}, \gamma)} = BR(\mathscr{N}, \gamma) = n - 1.$$

Series-parallel network



Series-parallel network

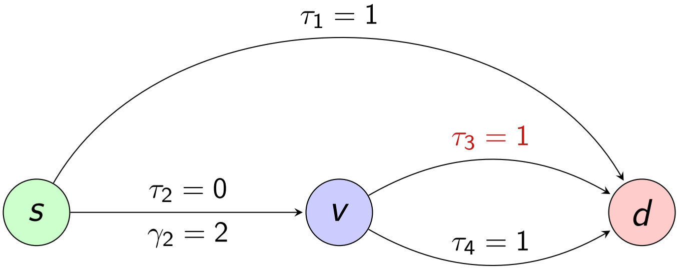


Equilibrium.

- Player [11] chooses $e_2 e_3$, [21] chooses $e_2 e_4$, [31] chooses $e_2 e_3$.
- For $t \geq 2$, [1t] chooses e_1, [2t] chooses $e_2 e_3$, [3t] chooses $e_2 e_4$.

Total costs=1+1+2=4.

Series-parallel network



- Suppose e_3 contains a queue, then total costs decrease to 3
- Another view on Braess's paradox: initial queues can improve total costs.

Overview

Summary

Two main contributions:

- We propose a measure of periodicity that characterizes the additional delay due to periodicity.
- We illustrate a new form of Braess's paradox: the presence of initial queues in a network may decrease the long-run costs in equilibrium.

Open problems

- General networks
- Multiple sources and destinations
- Connection with continuous time and flows
- Stochastic inflow

Apologies for congesting your brain.