Network congestion games with player-specific costs

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CERMICS, Optimisation et Systèmes

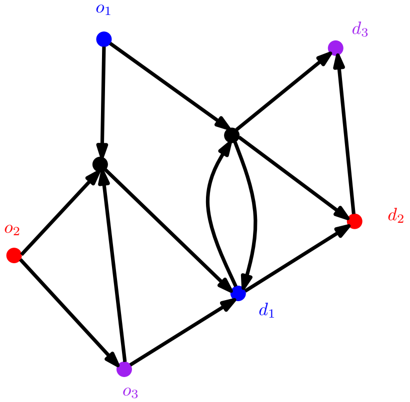
Model: non-atomic network congestion game

$\star$ $D = (V, A)$: directed graph.

$\star$ $\mathcal{L} \subseteq V^2$: set of origin-destination pairs.

$\star$ $b^{od} \in \mathbb{R}_+$: number of players going from o to d (the demand). Continuum of players: they are negligible.

$\star$ On each arc $a \in A$, there is a continuous cost $c_a(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$.



x_a = number of players choosing arc a.

$\sum_{a \in P} c_a(x_a)$ = cost of path P.

Players choose minimal cost paths.

Equilibrium

Input.	$D = (V, A)$ network	$\mathcal{L} \subseteq V^2$ origin-destination pairs	$(b^{od})_{od \in \mathcal{L}}$ demand	$(c_a(\cdot))_{a \in A}$ costs

Output.
For every $a \in A$ and every $od \in \mathcal{L}$:

- x_a^{od} = number of players choosing arc a at equilibrium among those going from o to d.

Equilibrium: mathematical description

(x_a^{od}) is an equilibrium if for every $(o,d) \in \mathcal{L}$

$$\begin{cases} (x_a^{od})_{a\in A} = o\text{-}d \text{ flow of value } b^{od} & \\ x_a = \sum_{(o,d)\in\mathcal{L}} x_a^{od} & a \in A \\ \sum_{a\in P} c_a(x_a) \leq \sum_{a\in Q} c_a(x_a) & P, Q \in \mathcal{P}^{od},\ P \text{ is used} \end{cases}$$

$\mathcal{P}^{od}$ = set of o-d paths
P used if $x_a^{od} > 0$ for all $a \in P$

Practical interest

- This model $=$ good indication of what happens in practice
  - ⋆ used in transport engineering, telecoms,...
- Useful since the phenomena are nonintuitive
  - ⋆ Braess paradox $=$ opening a new road may increase all travel times
  - ⋆ paradox recovered by the model

Questions

Does a Nash equilibrium exist?

Is it unique? *(i.e. are the x_a unique?)*

Is the equilibrium efficiently computable?

How far from social optimum?

Questions

Does a Nash equilibrium exist?

- ⋆ Yes (fixed point theorem).

Is it unique? *(i.e. are the x_a unique?)*

- ⋆ $c_a(\cdot)$ increasing $\Rightarrow$ uniqueness

Is the equilibrium efficiently computable?

- ⋆ $c_a(\cdot)$ nondecreasing $\Rightarrow$ convex optimization

How far from social optimum?

- ⋆ Price of Anarchy = cost at equilibrium/optimal social cost

Existence and uniqueness of the equilibrium

Theorem (Beckman, 1956)

An equilibrium always exists and it is “unique” when the cost functions $c_a(\cdot)$ are increasing.

“Unique” means there are unique x_a’s solutions of the system

$$\begin{cases} (x_a^{od})_{a\in A} = o\text{-}d \text{ flow of value } b^{od} & (o,d)\in\mathcal{L} \\ x_a = \displaystyle\sum_{(o,d)\in\mathcal{L}} x_a^{od} & a\in A \\ \displaystyle\sum_{a\in P} c_a(x_a) \leq \sum_{a\in Q} c_a(x_a) & P, Q \in \mathcal{P}^{od},\ P \text{ is used},\ (o,d)\in\mathcal{L} \end{cases}$$

Computation of an equilibrium

Theorem (Beckman, 1956)

When the $c_a(\cdot)$'s are nondecreasing, (x_a^{od}) is an equilibrium if and only if it is an optimal solution of

$$
\begin{array}{lll}
\min & \displaystyle\sum_{a\in A}\int_0^{x_a} c_a(t)dt & \\
s.c. & \displaystyle\sum_{(o,d)\in\mathcal{L}} x_a^{od} = x_a & \forall a \in A \\
 & \displaystyle\sum_{a\in\delta^+(o)} x_a^{od} - \sum_{a\in\delta^-(o)} x_a^{od} = b^{od} & \forall (o,d) \in \mathcal{L} \\
 & \displaystyle\sum_{a\in\delta^+(v)} x_a^{od} = \sum_{a\in\delta^-(v)} x_a^{od} & \forall (o,d) \in \mathcal{L}, \forall v \in V \setminus \{o,d\} \\
 & x_a^{od} \geq 0 & \forall (o,d) \in \mathcal{L}, \forall a \in A.
\end{array}
$$

Convex optimization! $\Longrightarrow$ computation is easy when the $c_a(\cdot)$'s are nondecreasing.

The multiclass case

i.e. with player-specific costs

Model – multiclass case

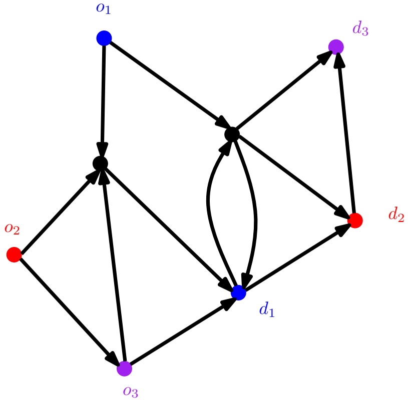
⋆ $D = (V, A)$: directed graph.

⋆ $\mathcal{L} \subseteq V^2$: set of origin-destination pairs.

⋆ $b^{od,k} \in \mathbb{R}_+$: number of class k players going from o to d (the demand). Continuum of players: they are negligible.

⋆ On each arc $a \in A$ and for each class k, there is a continuous cost $c_a^k(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$.

Players choose minimal cost paths.



x_a = number of players choosing arc a.

$\sum_{a \in P} c_a^k(x_a)$ = cost of path P experienced by a class k player.

Equilibrium

Input.	$D = (V, A)$ network	$\mathcal{L} \subseteq V^2$ or.-dest. pairs	$(b^{od,k})_{od \in \mathcal{L}, k \in K}$ demand	$(c_a^k(\cdot))_{a \in A, k \in K}$ costs

Output.
For every $a \in A$, every $od \in \mathcal{L}$, and every $k \in K$:

- $x_a^{od,k}$ = number of class k players choosing arc a at equilibrium, among those going from o to d.

Equilibrium: mathematical description

$(x_a^{od,k})$ is an equilibrium if for every $(o,d) \in \mathcal{L}$ and every $k \in K$

$$\begin{cases} (x_a^{od,k})_{a\in A} = o\text{-}d \text{ flow of value } b^{od,k} & \\ x_a = \displaystyle\sum_{(o,d)\in\mathcal{L},k\in K} x_a^{od,k} & a \in A \\ \displaystyle\sum_{a\in P} c_a^k(x_a) \le \sum_{a\in Q} c_a^k(x_a) & P,Q \in \mathcal{P}^{od},\ P \text{ is used by class } k \end{cases}$$

Questions – the multiclass case

Does a Nash equilibrium exist?

Is it unique? *(i.e. are the x_a unique?)*

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How far from social optimum?

Questions – the multiclass case

Does a Nash equilibrium exist?

- ⋆ Yes (fixed point theorem).

Is it unique? *(i.e. are the x_a unique?)*

- ⋆ $c_a(\cdot)$ increasing $\Rightarrow$ it depends

Is the equilibrium efficiently computable?

- ⋆ $c_a(\cdot)$ nondecreasing $\Rightarrow$ it depends

How far from social optimum?

- ⋆ Price of Anarchy

The multiclass case: uniqueness

The uniqueness issue

Theorem (Milchtaich, Schmeidler)

An equilibrium always exists in the multiclass setting.

There are examples with several equilibria, with several possible x_a's, while all $c_a^k(\cdot)$ are increasing: uniqueness of the equilibrium flows is not automatically ensured. (It contrasts with the monoclass case).

Challenge: Find necessary and/or sufficient conditions ensuring uniqueness.

Uniqueness: cost-based sufficient conditions

Proposition (Aashtiani, Magnanti, 1981)

If the players' cost functions are identical up to additive constants, then, for every two Nash equilibria, the flow on each arc in the network in the first equilibrium is equal to that in the second.

Uniqueness property

G = undirected graph, $\mathcal{L}$ = collection of o-d pairs.

$(G, \mathcal{L})$ has the uniqueness property (UP) if for any classes, demands $(b^{od,k})$, and increasing costs $(c_a^k(\cdot))$, the equilibrium flows are unique.

(on the digraph where each edge has been replaced by two opposite arcs)

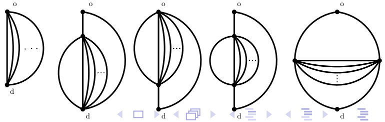
Theorem (Milchtaich, 2005)

There is only one o-d pair:

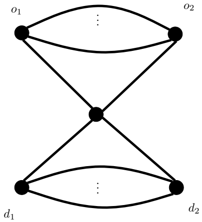
$$(G, \{(o,d)\}) \text{ has the UP} \quad \Longleftrightarrow \quad G \text{ is “nearly-parallel”}.$$

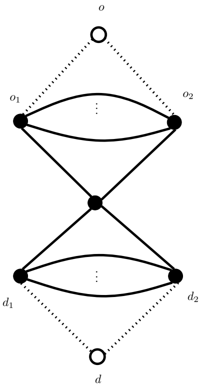
“Nearly-parallel” $=$

combination in series of



Uniqueness for general graphs using Milchtaich's theorem





- Add a fictitious origin vertex connected to every origin.
- Add a fictitious destination vertex connected to every destination.

Augmented graph has uniqueness property $\Rightarrow$ Original graph has uniqueness property.

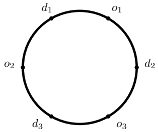
Uniqueness for general graphs using Milchtaich's theorem

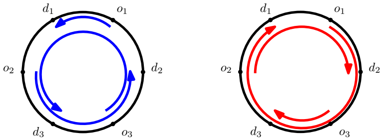
- Add a fictitious origin vertex connected to every origin.
- Add a fictitious destination vertex connected to every destination.

Augmented graph has not the uniqueness property $\Rightarrow$????

Uniqueness property on a cycle

$(G, \mathcal{L}) =$



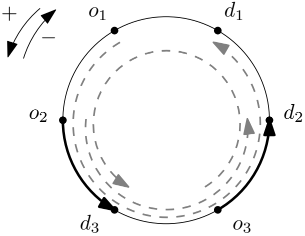


Theorem (M., Pradeau, 2014)

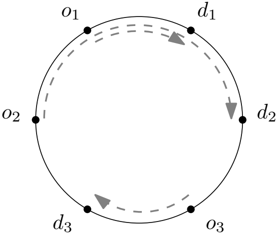
Assume that G is a cycle and let $\mathcal{L}$ be any collection of o-d pairs.

$(G, \mathcal{L})$ has the UP $\iff$ Each arc belongs to at most two o-d paths.

Example not having the uniqueness property



The positive paths.



The negative paths.

The arcs o_2d_3 and o_3d_2 are contained in three o-d paths.

Proof strategy

Step 1.

Each arc of D belongs to at most two o-d paths

$\Downarrow$

The equilibrium flows are unique whatever are the classes K, increasing costs $(c_a^k(\cdot))$, and demands $(b^{od,k})$

Step 2.

There is an arc of D belonging to at least three o-d paths

$\Downarrow$

There exist classes K, increasing costs $(c_a^k(\cdot))$, and demands $(b^{od,k})$ leading to two equilibria with distinct flows

Proof by an explicit construction of costs and demands.

Step 1:

Each arc of D belongs to at most two o-d paths

$\Downarrow$

The equilibrium flows are unique whatever are the classes, costs, and demands

- Let $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$ be two equilibria. Define $\Delta_{od} = x_{P^+}^{od} - \hat{x}_{P^+}^{od}$.
- Suppose $\Delta_{o_0 d_0} \neq 0$ for some o_0-d_0. There exists an o_1-d_1 s.t. $\Delta_{o_0 d_0}\Delta_{o_1 d_1} < 0$ and $\Delta_{o_0 d_0} + \Delta_{o_1 d_1} < 0$.
- We repeat this argument and get an infinite sequence $|\Delta_{o_0 d_0}| < |\Delta_{o_1 d_1}| < \cdots < |\Delta_{o_j d_j}| < \cdots$.
- Contradiction with finiteness.

Step 2:

There is an arc of D belonging to at least three o-d paths

$\Downarrow$

There exist classes K, increasing costs $(c_a^k(\cdot))$, and demands $(b^{od,k})$ leading to two equilibria with distinct flows

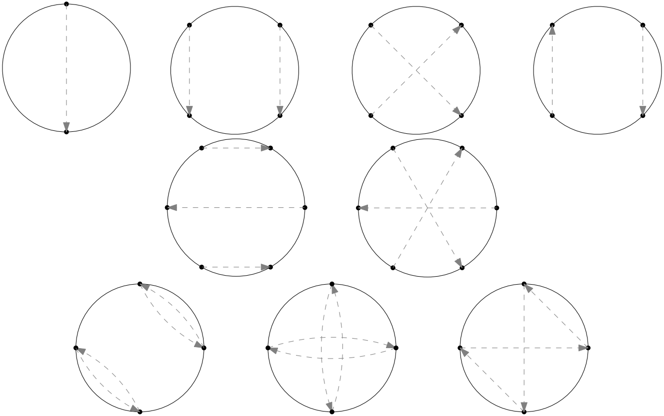
An arc in 3 *o-d* paths: explicitly building of cost functions and demands leading to two equilibria with distinct flows.

Some features:

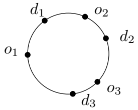
- Three classes.
- Affine cost functions.
- Explicit construction of two equilibria.
- These equilibria are strict and “single-path”.

Structural characterization

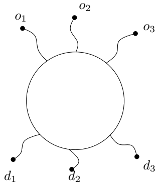
Each arc in at most two *o*-*d* paths $\Leftrightarrow$ $(G, \mathcal{L})$ homeomorphic to a minor of one of



Corollary for general graphs: examples

If 

is in $(G, \mathcal{L})$, G does not have the UP.

If 

is in $(G, \mathcal{L})$, G does not have the UP.

Having a minor without the uniqueness property

A subgraph of $(G, \mathcal{L})$ does not have the UP $\Longrightarrow$ $(G, \mathcal{L})$ does not have the UP.

A minor of $(G, \mathcal{L})$ does not have the UP:

- If the contractions involve only bridges, G does not have the UP.
- If the counterexamples have strict equilibria, G does not have the UP.
- And in general, open question.

Strong uniqueness property

G has the strong uniqueness property (SUP) $=$ $(G, \mathcal{L})$ has the UP for any collection of *o*-*d* pairs $\mathcal{L}$

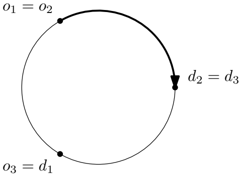
Theorem (M., Pradeau, 2014)

G has the SUP $\Longleftrightarrow$ *No cycles of length* 3 *or more.*

Graph having the SUP are thus graphs obtained from a forest by replicating some edges.

Proof

(⇒)
The graph



has one arc in three *o*-*d* paths: no UP.

(⇐) results from two easy statements:

- A graph with two vertices and parallel edges has the SUP.
- Glueing two graphs on a vertex maintains the SUP.

□

Equivalence of equilibria

Equivalence of equilibria

Two equilibria are equivalent if the contribution of each $(od, k) \in \mathcal{L} \times K$ to the flow on each arc is the same in all equilibria.

Theorem (Milchtaich 2005)

A single OD graph has the uniqueness property $\iff$ Generically, for every partition of the population into classes, all equilibria are equivalent.

Theorem (M.,Pradeau, 2014)

A ring has the uniqueness property $\iff$ Generically, for every partition of the population into classes, all equilibria are equivalent.

Uniqueness property: a combinatorial sufficient condition for a 'two-sided' game

Consider a nonatomic congestion game with player-specific cost functions, not necessarily played on a graph.

Proposition

Suppose that there are finite sets A^+ and A^- such that every player i has exactly two available strategies r_i^+ and r_i^- with $r_i^+ \subseteq A^+$ and $r_i^- \subseteq A^-$. Then, if all triples of pairwise distinct strategies have an empty intersection, the uniqueness property holds.

The multiclass case: computation

Affine costs

Input.

- Directed graph $D = (V, A)$,
- Or.-dest. pairs $\mathcal{L} \subseteq V^2$
- Demands $b^{od,k} \in \mathbb{R}_+$
- Costs $c_a^k(x) = \alpha_a^k x + \beta_a^k$ with $\alpha_a^k > 0$ and $\beta_a^k \geq 0$.

Output.

- An equilibrium.

The exact complexity of this computational problem remains to be determined.

Affine costs

Input.

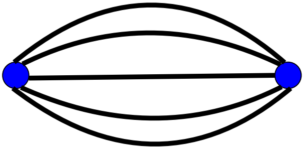
- Directed graph $D = (V, A)$,
- Or.-dest. pairs $\mathcal{L} \subseteq V^2$
- Demands $b^{od,k} \in \mathbb{R}_+$
- Costs $c_a^k(x) = \alpha_a^k x + \beta_a^k$ with $\alpha_a^k > 0$ and $\beta_a^k \geq 0$.

Theorem (M.,Pradeau, 2014)

An equilibrium can be computed in polynomial time when the number of vertices and the number of classes are fixed.

Corollary

Problem is polynomial for the parallel-arc graphs when the number of classes if fixed.



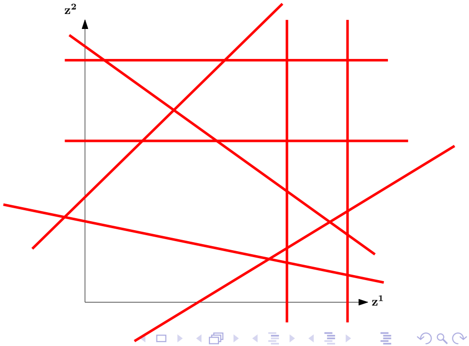
Parallel-arc graph

Consider an equilibrium and let z^k be the cost of a shortest arc for class k.

$$a \text{ in the support of class } k \Longrightarrow \begin{cases} \alpha_a^j(z^k - \beta_a^k) - \alpha_a^k(z^j - \beta_a^j) \geq 0 & \forall j \in K, \\ z_a^k - \beta_a^k \geq 0. \end{cases}$$

$\Longrightarrow$ hyperplane arrangement in $\mathbb{R}_+^K$.

- Each cell provides a candidate support S^k for class k.
- Deciding whether S^k is an equilibrium support: linear programming.



Polynomial algorithm

Let $\boldsymbol{A} = \{(S^k)_{k\in K} : \ S^k \subseteq A\}$. Algorithm consists in two steps.

A Compute a set $\mathcal{S} \subseteq \boldsymbol{A}$ of polynomial size such that for any equilibrium multiflow $(\boldsymbol{x}^k)_{k\in K}$, there is a $(S^k)_{k\in K} \in \mathcal{S}$ with $\mathrm{supp}(\boldsymbol{x}^k) \subseteq S^k$ for all k.

B Decide for every $(S^k)_{k\in K} \in \mathcal{S}$ whether there exists an equilibrium multiflow $(\boldsymbol{x}^k)_{k\in K}$ with $\mathrm{supp}(\boldsymbol{x}^k) \subseteq S^k$ for all k, and compute it if it exists.

Polynomial algorithm: first step

A Compute a set $\mathcal{S} \subseteq \boldsymbol{A}$ of polynomial size such that for any equilibrium multiflow $(\boldsymbol{x}^k)_{k\in K}$, there is a $(S^k)_{k\in K} \in \mathcal{S}$ with $\mathrm{supp}(\boldsymbol{x}^k) \subseteq S^k$ for all k.

For fixed $|K|$ and $|V|$, can be done in polynomial time: there is a hyperplane arrangement whose cells corresponds to the possible $\mathcal{S}$'s.

Polynomial algorithm: second step

B Test for every $(S^k)_{k\in K} \in \mathcal{S}$ whether there exists an equilibrium multiflow $(\boldsymbol{x}^k)_{k\in K}$ with $\mathrm{supp}(\boldsymbol{x}^k) \subseteq S^k$ for all k, and compute it if it exists.

Can be done in polynomial time: system of linear inequalities, interior point method.

$$\begin{cases} (x_a^{od,k})_{a\in A} = o\text{-}d \text{ flow of value } b^{od,k} & \\ x_a = \displaystyle\sum_{(o,d)\in\mathcal{L},k\in K} x_a^{od,k} & a \in A \\ \displaystyle\sum_{a\in P} \alpha_a^k x_a + \beta_a^k \leq \sum_{a\in Q} \alpha_a^k x_a + \beta_a^k & P, Q \in \mathcal{P}^{od},\ P \text{ is used by class } k \end{cases}$$

A practically efficient algorithm for affine costs

Solve

$$\sum_{a\in\delta^+(v)} x_a^k = \sum_{a\in\delta^-(v)} x_a^k + b_v^k \qquad k\in K, v\in V$$

$$\alpha_{uv}^k \sum_{k'\in K} x_{uv}^{k'} + \pi_u^k - \pi_v^k - \mu_{uv}^k = -\beta_{uv}^k \qquad k\in K, (u,v)\in A$$

$$x_a^k \mu_a^k = 0 \qquad k\in K, a\in A$$

$$\pi_{s^k}^k = 0 \qquad k\in K$$

$$x_a^k \geq 0, \mu_a^k \geq 0, \pi_v^k \in \mathbb{R} \qquad k\in K, a\in A, v\in V.$$

It is a linear complementary problem.

A practically efficient algorithm for affine costs

Solve

$$\sum_{a\in\delta^+(v)} x_a^k = \sum_{a\in\delta^-(v)} x_a^k + b_v^k \qquad k\in K, v\in V$$

$$\alpha_{uv}^k \sum_{k'\in K} x_{uv}^{k'} + \pi_u^k - \pi_v^k - \mu_{uv}^k = -\beta_{uv}^k \qquad k\in K,(u,v)\in A$$

$$x_a^k \mu_a^k = 0 \qquad k\in K, a\in A$$

$$\pi_{s^k}^k = 0 \qquad k\in K$$

$$x_a^k \geq 0, \mu_a^k \geq 0, \pi_v^k \in \mathbb{R} \qquad k\in K, a\in A, v\in V.$$

It is a linear complementary problem.

A Lemke-like algorithm

Proposition (M.,Pradeau, 2014)

There is a Lemke-like algorithm solving this problem.

Theoretical consequences:

- The problem with affine costs is in the PPAD class.
- If all input parameters are rational numbers, then there always exists a rational equilibrium multiflow.

Practical consequence:

Classes	Grid	Vertices	Arcs	Pivots	Algorithm (seconds)
4	6×6	36	120	126	0.9
	8×8	64	224	249	5.4
10	6×6	36	120	322	15.0
	8×8	64	224	638	87.0
50	2×2	4	8	56	0.3
	4×4	16	48	636	105.0

The multiclass case:
price of anarchy

To be done.

Atomic splittable case

- Directed graph $D = (V, A)$
- Finite set of players I, each with demand, origin, and destination
- Unit cost for player i on arc a: $c_a^i(x) = \alpha_a^i x + \beta_a^i$
- Each player routes his/her demand in the network, splitting allowed

Define $\Delta := \sup_a \frac{\sup_i \alpha_a^i}{\inf_i \alpha_a^i}$.

$$\Delta < 3 \quad \Longrightarrow \quad \text{PoA} \leq \frac{3\Delta(|I|-1)+4}{(3-\Delta)\Delta(|I|-1)+4}.$$

One class $\Longrightarrow \Delta = 1$; bound coincides with Cominetti and al.'s result (2009).

Even if only two parallel-arcs and two players, PoA can be made arbitrarily large when $\Delta \to +\infty$.

Thank you