A probabilistic representation for continuous-time zero-sum games with incomplete information on both sides.

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Plan

1. Settings, preliminary game
2. Incomplete information on one side
3. Incomplete information on both sides

Settings

Are given

- a finite time horizon $T > 0$,
- two metric, compact control spaces U, V,
- for some $I \in \mathbb{N}^*$ and all $i \in \{1, \dots, I\}$ a continuous map $\ell_i : [0, T] \times U \times V \to \mathbb{R}$,
- $\Delta(I)$ denotes the set of probabilities on $\{1, \dots, I\}$
  $\simeq$ simplex in $\mathbb{R}^I$.

Isaac's assumption: For all $(t, p) \in [0, T] \times \Delta(I)$,

$$\min_{u \in U} \max_{v \in V} \sum_{i=1}^{I} p_i \ell_i(t, u, v) = \max_{v \in V} \min_{u \in U} \sum_{i=1}^{I} p_i \ell_i(t, u, v) := H(t, p)$$

A preliminary game

Fix $(t, p) \in [0, T] \times \Delta(I)$.
For any $u_\cdot : [t, T] \to U$ and $v_\cdot : [t, T] \to V$ measurable, consider the mean payoff

$$J(t, p, u_\cdot, v_\cdot) = \sum_i p_i \int_t^T \ell_i(s, u_s, v_s)ds.$$

P1 plays $u_\cdot$ and aims to minimize $J(t, p, u, v)$,
P2 plays $v_\cdot$ and aims to maximize $J(t, p, u, v)$.

Interpretation: The index i is chosen according to p, but no player is informed : both optimize the average game.

Proposition: Under Isaacs assumption, the game has a value

$$V_0(t, p) = \int_t^T H(s, p)ds = J(t, p, u_\cdot^*, v_\cdot^*),$$

with $u_s^* \in \mathrm{Argmin}_u \max_v \sum_{i=1}^I p_i \ell_i(s, u, v)$
and $v_s^* \in \mathrm{Argmax}_v \min_u \sum_{i=1}^I p_i \ell_i(s, u, v)$.

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Payoff: $J(t,p,u,v) = \sum_i p_i \int_t^T \ell_i(s, u_s, v_s) ds$.

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Proposition: Under Isaacs assumption, the game has a value

$$V_0(t,p) = \int_t^T H(s,p) ds,$$

with $H(s,p) = \min_{u \in U} \max_{v \in V} \sum_{i=1}^I p_i \ell_i(s,u,v)$.

Remark:
Obviously, V_0 satisfies

$$\begin{cases} \frac{\partial V_0}{\partial t} + H(t,p) = 0, \ (t,p) \in [0,T] \times \Delta(I), \\ V_0|_{t=T} = 0. \end{cases}$$

Game with incomplete information on one side

Given $(t, p) \in [0, T] \times \Delta(I)$, suppose now that,

- at time t, $i \in \{1, \dots, I\}$ is chosen randomly according to p and shown to P1 (not to P2),
- P2 knows p,
- both player observe the actions of their opponent.

P1 still wants to minimize, P2 to maximize the running cost

$$J(t, p, u., v.) = \sum_{i=1}^{I} p_i \int_t^T \ell_i(s, u_s, v_s) ds.$$

Strategies for P1:
$\alpha : (i, (v_r, t \leq r \leq s)) \mapsto u_s$ random (+technical assumptions)
Strategies for P2:
$\beta : (u_r, t \leq r \leq s) \mapsto v_s$ random (+technical assumptions) .

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Values

$J(t,p,u.,v.) = \sum_{i=1}^{I} p_i \int_t^T \ell_i(s,u_s,v_s)ds$
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Payoff: (p,α,β) induces a probability $P_{p,\alpha,\beta}$ on $\{1,\dots,I\} \times \mathcal{U}_t \times \mathcal{V}_t$.
Set $J(t,p,\alpha,\beta) = E_{p,\alpha,\beta}[\int_t^T \ell_i(s,u_s,v_s)ds]$.

Upper value function: $V^+(t,p) = \inf_\alpha \sup_\beta J(t,p,\alpha,\beta)$
Lower value function: $V^-(t,p) = \sup_\beta \inf_\alpha J(t,p,\alpha,\beta)$

Theorem 1 (Cardaliaguet 2007): Under Isaac's assumption, the game has a value $V := V^+ = V^-$,

Characterization of the value V

For $(t,p) \in [0,T] \times \Delta(I)$, let $\mathcal{M}(t,p)$ be the set of càdlàg martingales (p_s) with values in $\Delta(I)$ such that $p_{t-} = p$.

Theorem 2 (Cardaliaguet,R. 2009) For all $(t,p) \in [0,T] \times \Delta(I)$,
$V(t,p) = \min_{(p_s) \in \mathcal{M}(t,p)} E\left[\int_t^T H(s,p_s)ds\right]$ $(*)$.

Comment:
For any (random) control $u.$ played by P1, set

$$p_s(i) = P[\text{ index } i \text{ has been chosen}|u_r, t \le r \le s], i \in \{1, \dots, I\}.$$

Then

- (p_s) is a $\Delta(I)$-valued martingale with $p_{t-} = p$,
- (p_s) is a martingale of belief for P2.

Optimal strategy for P1 :

- choose (p_s) optimal in (*),
- for all $s \in [t,T]$, play $u_s^* \in \text{Argmin}\max_v \sum_i p_s(i)\ell_i(s,u,v)$.

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Characterization of the value 2

$$V(t,p) = \min_{(p_s)\in\mathcal{M}(t,p)} E\left[\int_t^T H(s,p_s)ds\right] \quad (*).$$

Theorem 3 (Cardaliaguet 2007)
V is the unique continuous function such that

- $V(T,p) = 0$
- for all $t \in [0,T]$, $p \mapsto V(t,p)$ is convex,
- $\frac{\partial V}{\partial t} + H(t,p) \geq 0$,
- for all (t,p) such that $p' \mapsto V(t,p')$ is strictly convex in p',

$$\frac{\partial V}{\partial t} + H(t,p) = 0.$$

(if V is not smooth, this holds in a viscosity sense.)

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$$\left(\leq \int_t^{t+h} H(s,p)ds + V(t+h,p)\right)$$

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Idea: Let (p_s) optimal in $(*)$. Then (p_s) martingale and V strictly convex $\Rightarrow$ $\lim_{s\searrow t} p_s = p$.

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or equivalently:
V is the unique viscosity solution of:

$$\begin{cases} \min\{\frac{\partial V}{\partial t} + H;\ \lambda_{\min}(D_p^2 V)\} = 0, \\ V|_{t=T} = 0, \end{cases}$$

where $\lambda_{min}(A)$ is the smallest eigenvalue of A.

Game with incomplete information on both sides

Now let $I, J \in \mathbb{N}^*$ and $(\ell_{ij})_{(i,j)\in\{1,\dots,I\}\times\{1,\dots,J\}}$ a family of continuous functions $[0,T]\times U\times V\to\mathbb{R}$.

Given $(t,p,q)\in[0,T]\times\Delta(I)\times\Delta(J)$, suppose now that,

- at time t, $(i,j)\in\{1,\dots,I\}\times\{1,\dots,J\}$ is chosen randomly according to $p\otimes q$
- i is shown to P1, j to P2,
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P1 still wants to minimize, P2 to maximize the running cost

$$J(t,p,q,u_\cdot,v_\cdot)=\sum_{i,j}p_iq_j\int_t^T\ell_{ij}(s,u_s,v_s)ds.$$

We suppose still that Isaac's assumption holds : for all (t,p,q)

$$\min_{u\in U}\max_{v\in V}\sum_{i,j}p_iq_j\ell_{ij}(t,p,q)$$
$$=\max_{v\in V}\min_{u\in U}\sum_{i,j}p_iq_j\ell_{ij}(t,p,q):=H(t,p,q).$$

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Strategies for P1:
$\alpha : (i, (v_r, t \le r \le s)) \mapsto u_s$ random (+technical assumptions)
Strategies for P2:
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Upper Value : $W^+(t, p, q) = \inf_\alpha \sup_\beta J(t, p, q, \alpha, \beta)$,
Lower Value : $W^-(t, p, q) = \sup_\beta \inf_\alpha J(t, p, q, \alpha, \beta)$.

Theorem 3 (Cardaliaguet 2007): Under Isaacs assumption the continuous time game with incomplete information on both sides has a value $W := W^+ = W^-$ which is the unique viscosity solution of:

$$\begin{cases} \max\{\min\{\frac{\partial W}{\partial t} + H; \lambda_{\min}(D_p^2 W)\} ; \lambda_{\max}(D_q^2 W)\} = 0, \\ W|_{t=T} = 0, \end{cases}$$

where $\lambda_{min}(A)$(resp. $\lambda_{ma}(A)$) is teh smallest (resp. largest) eigenvalue of A

Representation in terms of a martingale control-problem ?

We want to define a continuous-time zero-sum game having same value $W(t, p, q)$ as the zero-sum game with incomplete information on both sides, and such that:

- P1 controls a (càdlàg) martingale $(p_s)_{s \in [t,T]}$ with values in $\Delta(I)$ and $p_{t-} = p$.
- P2 controls a (càdlàg) martingale $(q_s)_{s \in [t,T]}$ with values in $\Delta(J)$ and $q_{t-} = q$.
- The expected payoff is $E[\int_t^T H(s, p_s, q_s) ds]$.

Repeated games, Laraki 2001 : the splitting game.

Main problem: How to define non-anticipative strategies in this context?
Chosen approach: Use the framework of stochastic differential games.

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Chosen approach: Use the framework of stochastic differential games.

A stochastic differential game.

Let (B_s^1) and (B_s^2) be two independent Brownian motions with values in $\mathbb{R}^I$, resp. $\mathbb{R}^J$.
For $t \in [0, T]$,
let $(a_s)_{t \leq s \leq T}$ be a $\mathcal{F}^{B^1, B^2}$-adapted $\mathbb{R}^{I \times I}$-valued process,
(resp. $(b_s)_{t \leq s \leq T}$ be $\mathcal{F}^{B^1, B^2}$-adapted, $\mathbb{R}^{J \times J}$-valued).

Consider the controlled stochastic differential system

$$\begin{array}{ll} (1) & X_s = p + \int_t^s \sigma(X_r, a_r) dB_r^1, \\ (2) & Y_s = q + \int_t^s \tau(Y_r, b_r) dB_r^2, \; s \in [t, T], \end{array}$$

where $\sigma(p, \cdot)$: projection on T_p, tangent space on p to $\Delta(I)$
(resp. $\tau(q, \cdot)$ on T_q, tangent space on q to $\Delta(J)$).

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Consider the controlled stochastic differential system

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where $\sigma(p, \cdot)$: projection on T_p, tangent space on p to $\Delta(I)$
(resp. $\tau(q, \cdot)$ on T_q, tangent space on q to $\Delta(J)$).

(1) $X_s^a = p + \int_t^s \sigma(X_r^a, a_r)dB_r^1$,
(2) $Y_s^b = q + \int_t^s \tau(Y_r^b, b_r)dB_r^2,\ s \in [t, T]$,
with $\sigma(p, \cdot)$: projection on T_p, tangent space on p to $\Delta(K)$
(+ analogue definition for $\tau(q, \cdot)$).

Remark. The system of controlled SDE's is highly nonstandard:

- The control spaces $\mathbb{R}^{I\times I}$ and $\mathbb{R}^{J\times J}$ are unbounded.
- The volatilities σ and τ are not continuous,

However

Theorem:

- Equations (1) and (2) have unique strong solutions (X_s^a) and (Y_s^b),
- for all $s \in [t, T]$ P-a.s., $X_s^a \in \Delta(I)$ (resp. $Y_s^a \in \Delta(J)$),
- (X_s^a) and (Y_s^b) are martingales.

(1) $\quad X_s^a = p + \int_t^s \sigma(X_r^a, a_r)dB_r^1,$

(2) $\quad Y_s^b = q + \int_t^s \tau(Y_r^b, b_r)dB_r^2, \ s \in [t, T],$

with $\sigma(p, \cdot)$: projection on T_p, tangent space on p to $\Delta(K)$

(+ analogue definition for $\tau(q, \cdot)$).

Remark. The system of controlled SDE's is highly nonstandard:

- The control spaces $\mathbb{R}^{I \times I}$ and $\mathbb{R}^{J \times J}$ are unbounded.
- The volatilities σ and τ are not continuous,

However

Theorem:

- Equations (1) and (2) have unique strong solutions (X_s^a) and (Y_s^b),
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The game:

- P1 plays (a_s), P2 plays (b_s).
- Expected payoff: $E[\int_t^T H(s, X_s^a, Y_s^b)ds]$.

Value functions :

$$\tilde{W}^+(t,p,q) = \inf_\alpha \sup_\beta E[\int_t^T H(s, X_s^a, Y_s^b)ds],$$
$$\tilde{W}^-(t,p,q) = \sup_\beta \inf_\alpha E[\int_t^T H(s, X_s^a, Y_s^b)ds],$$

with $\alpha : (b_r, t \leq r \leq s) \mapsto a_r$ (resp. $\beta : (a_r, t \leq r \leq s) \mapsto b_r$) non anticipative strategies.

Proposition: $\tilde{W}^+$ and $\tilde{W}^-$ are convex-concave and Lipschitz.

Proposition: $\tilde{W}^+$ and $\tilde{W}^-$ are concave-convex and Lipschitz.

Arguments:

- Lipschitz in t: scaling property of the Brownian motion,
- Lipschitz in p: explicit computation of the projections and decomposition of the trajectory.
- Convexity:
  Splitting lemma:
  For $p^1, p^2 \in \Delta(I)$ and $p = \lambda p^1 + (1-\lambda)p^2$, let Z be a random variable with $P[Z = p^1] = \lambda, P[Z = p^2] = 1 - \lambda$.
  For all $h > 0, \epsilon > 0$, there exists a. such that
  $$E[|X^a_{t+h} - Z|] \leq \epsilon.$$
  (martingale representation property of Brownian motion)
- Concavity : Jensen inequality.

Proposition

$\tilde{W} = \tilde{W}^+ = \tilde{W}^-$ is the unique Lipschitz continuous solution in viscosity sense of

$$\begin{cases} \max\{\min\{\frac{\partial \tilde{W}}{\partial t} + H \,;\, \lambda_{\min}(D^2_p \tilde{W})\} \,;\, \lambda_{\max}(D^2_q \tilde{W})\} = 0, \\ \tilde{W}|_{t=T} = 0. \end{cases}$$

Arguments:

- $\tilde{W}^+$ is convex in p, concave in q,
- a dynamic programming principle + measurable selection theorem,
- at (t, p, q) where $\tilde{W}^+$ is strictly convex and concave, $\tilde{W}^+$ is solution to $\frac{\partial \tilde{W}^+}{\partial t} + H = 0$.
- Analogue arguments for $\tilde{W}^-$ + comparison theorem

Corollary

$\tilde{W}$ coincides with the value of the continuous time, zero-sum game with incomplete information on both sides.

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Possible extensions

- More general compact convex sets C, D instead of $\Delta(I), \Delta(J)$.
- More general PDE with the same convexity constraints

$$\max\{\min\{\frac{\partial V}{\partial t} + \mathcal{L}(V) + u\,;\, \lambda_{\min}(D^2_p V)\}\,;\, \lambda_{\max}(D^2_q V)\} = 0,$$

  Already appearing in models of continuous-time Markov games with incomplete information: see Cardaliaguet, R, Rosenberg, Vieille 2013 and Gensbittel 2013.
  (probably requires viability theory)
- PDE with different obstacles ?

Thank you for your attention!