Approachability of Convex Sets in "Some" Absorbing Games

Rida Laraki
CNRS, LAMSADE and École Polytechnique, Paris

Joint work with

Janos Flesch and Vianney Perchet

Singapor, December 4, 2015

Blackwell Approachability

- Many sequential decision problems can be reduced to a repeated game between a decision maker against Nature (or advisory).

Blackwell Approachability

- Many sequential decision problems can be reduced to a repeated game between a decision maker against Nature (or advisory).
- At each stage t, the DM chooses an element $i_t \in I$ and nature chooses a state $j_t \in J$, generating a sequence of outcomes $\{g_t = g(i_t, j_t)\}_{t=1}^{\infty}$.

Blackwell Approachability

- Many sequential decision problems can be reduced to a repeated game between a decision maker against Nature (or advisory).
- At each stage t, the DM chooses an element $i_t \in I$ and nature chooses a state $j_t \in J$, generating a sequence of outcomes $\{g_t = g(i_t, j_t)\}_{t=1}^{\infty}$.
- Blackwell assumed that outcomes are vectorial payoffs $g_t \in \mathbf{R}^d$ and considers the problem where the DM would like to garantee that the average outcome $\frac{1}{T}\sum_{t=1}^{T} g(i_t, j_t)$ belongs to some target set $\mathcal{C} \subset \mathbf{R}^d$ as $T \to \infty$ irrespective of nature moves.

Blackwell Approachability

- Many sequential decision problems can be reduced to a repeated game between a decision maker against Nature (or advisory).
- At each stage t, the DM chooses an element $i_t \in I$ and nature chooses a state $j_t \in J$, generating a sequence of outcomes $\{g_t = g(i_t, j_t)\}_{t=1}^{\infty}$.
- Blackwell assumed that outcomes are vectorial payoffs $g_t \in \mathbf{R}^d$ and considers the problem where the DM would like to garantee that the average outcome $\frac{1}{T}\sum_{t=1}^{T} g(i_t, j_t)$ belongs to some target set $\mathcal{C} \subset \mathbf{R}^d$ as $T \to \infty$ irrespective of nature moves.
- He proved that a necessary and sufficient condition for a convex set $\mathcal{C}$ to be approachable is:

$$\forall y \in \Delta(I) \;\; \exists x \in \Delta(J) : \quad g(x, y) \in \mathcal{C}.$$

Blackwell Approachability

- Many sequential decision problems can be reduced to a repeated game between a decision maker against Nature (or advisory).
- At each stage t, the DM chooses an element $i_t \in I$ and nature chooses a state $j_t \in J$, generating a sequence of outcomes $\{g_t = g(i_t, j_t)\}_{t=1}^{\infty}$.
- Blackwell assumed that outcomes are vectorial payoffs $g_t \in \mathbf{R}^d$ and considers the problem where the DM would like to garantee that the average outcome $\frac{1}{T}\sum_{t=1}^{T} g(i_t, j_t)$ belongs to some target set $\mathcal{C} \subset \mathbf{R}^d$ as $T \to \infty$ irrespective of nature moves.
- He proved that a necessary and sufficient condition for a convex set $\mathcal{C}$ to be approachable is:

$$\forall y \in \Delta(I) \;\; \exists x \in \Delta(J) : \quad g(x, y) \in \mathcal{C}.$$

- Blackwell also proved that a convex set is either approchable or excludable.

Applications and Extensions of Blackwell Approachability

- The first game theory application of Blackwell approchability is due to Aumann and Maschler.

Applications and Extensions of Blackwell Approachability

- The first game theory application of Blackwell approchability is due to Aumann and Maschler.
- They use it to construct an optimal strategy for the uninformed player in repeated games with incomplete information.

Applications and Extensions of Blackwell Approachability

- The first game theory application of Blackwell approchability is due to Aumann and Maschler.
- They use it to construct an optimal strategy for the uninformed player in repeated games with incomplete information.
- Approachability gained a lot of attention since then in economics, game theory, and machine learning.

Applications and Extensions of Blackwell Approachability

- The first game theory application of Blackwell approchability is due to Aumann and Maschler.
- They use it to construct an optimal strategy for the uninformed player in repeated games with incomplete information.
- Approachability gained a lot of attention since then in economics, game theory, and machine learning.
- It is used, for example, to construct non regret or calibrated algorithms.

Applications and Extensions of Blackwell Approachability

- The first game theory application of Blackwell approchability is due to Aumann and Maschler.
- They use it to construct an optimal strategy for the uninformed player in repeated games with incomplete information.
- Approachability gained a lot of attention since then in economics, game theory, and machine learning.
- It is used, for example, to construct non regret or calibrated algorithms.
- There is a formal equivalence between approachability, non-regret and calibration algorithms (Vianney Perchet's survey in JDG).

Applications and Extensions of Blackwell Approachability

- The first game theory application of Blackwell approchability is due to Aumann and Maschler.
- They use it to construct an optimal strategy for the uninformed player in repeated games with incomplete information.
- Approachability gained a lot of attention since then in economics, game theory, and machine learning.
- It is used, for example, to construct non regret or calibrated algorithms.
- There is a formal equivalence between approachability, non-regret and calibration algorithms (Vianney Perchet's survey in JDG).
- Here is a list of papers that uses or extends approachability:

Applications and Extensions of Blackwell Approachability

- The first game theory application of Blackwell approchability is due to Aumann and Maschler.
- They use it to construct an optimal strategy for the uninformed player in repeated games with incomplete information.
- Approachability gained a lot of attention since then in economics, game theory, and machine learning.
- It is used, for example, to construct non regret or calibrated algorithms.
- There is a formal equivalence between approachability, non-regret and calibration algorithms (Vianney Perchet's survey in JDG).
- Here is a list of papers that uses or extends approachability:

  Vieille, [Hart & Mas-Colell], Spinat, Lehrer, Dawid, Renault & Tomala
  [As Soulaimani, Quincampoix & Sorin], Perchet, [Lehrer & Solan]
  Rakhlin, [Sridharan & Tewari], [Perchet & Quincampoix], Lovo, Horner & Tomala
  [Foster & Vohra], [Fudenberg & Levine], [Sandroni, Smorodinsky & Vohra]
  [Hart & Mas-Colell], [Cesa-Bianchi & Lugosi], [Benaim, Hofbauer & Sorin]

Examples

Our paper aims to extend Blackwell conditions to “some” absorbing games.

Examples

Our paper aims to extend Blackwell conditions to “some” absorbing games.

Big Match game of type I

	L	R
T	a^*	b^*
B	c	d

Examples

Our paper aims to extend Blackwell conditions to “some” absorbing games.

Big Match game of type I

	L	R
T	a^*	b^*
B	c	d

Big Match game of type II

	L	R
T	a^*	b
B	c^*	d

Examples

Our paper aims to extend Blackwell conditions to “some” absorbing games.

Big Match game of type I

$$\begin{array}{c|c|c|} & L & R \\ \cline{2-3} T & a^* & b^* \\ \cline{2-3} B & c & d \\ \cline{2-3} \end{array}$$

Big Match game of type II

$$\begin{array}{c|c|c|} & L & R \\ \cline{2-3} T & a^* & b \\ \cline{2-3} B & c^* & d \\ \cline{2-3} \end{array}$$

Quitting Games

$$\begin{array}{c|c|c|} & L & R \\ \cline{2-3} T & a^* & b^* \\ \cline{2-3} B & c^* & d \\ \cline{2-3} \end{array}$$

Generalized Quitting Games

Generalized Quitting Games

Sets of actions:

Pure actions of player 1 (the decision maker): $\mathbf{I} = \mathcal{I} \times \mathcal{I}^*$
Pure actions of player 2 (nature or advisory): $\mathbf{J} = \mathcal{J} \times \mathcal{J}^*$.
Mixed actions of P1 $\mathbf{x} \in \Delta(\mathcal{I} \times \mathcal{I}^*)$, $x \in \Delta(\mathcal{I})$, $x^* \in \Delta(\mathcal{I}^*)$,
Mixed actions of P2 $\mathbf{y} \in \Delta(\mathcal{J} \times \mathcal{J}^*)$, $y \in \Delta(\mathcal{I})$, $y^* \in \Delta(\mathcal{I}^*)$.
Positive measures $\alpha \in \mathcal{M}(\mathbf{I})$ and $\beta \in \mathcal{M}(\mathbf{I})$.

Generalized Quitting Games

Sets of actions:

Pure actions of player 1 (the decision maker): $\mathbf{I} = \mathcal{I} \times \mathcal{I}^*$
Pure actions of player 2 (nature or advisory): $\mathbf{J} = \mathcal{J} \times \mathcal{J}^*$.
Mixed actions of P1 $\mathbf{x} \in \Delta(\mathcal{I} \times \mathcal{I}^*)$, $x \in \Delta(\mathcal{I})$, $x^* \in \Delta(\mathcal{I}^*)$,
Mixed actions of P2 $\mathbf{y} \in \Delta(\mathcal{J} \times \mathcal{J}^*)$, $y \in \Delta(\mathcal{I})$, $y^* \in \Delta(\mathcal{I}^*)$.
Positive measures $\alpha \in \mathcal{M}(\mathbf{I})$ and $\beta \in \mathcal{M}(\mathbf{I})$.

Vectorial payoffs

$g(i,j) \in \mathbb{R}^d$, $\forall (ij) \in (\mathbf{I}, \mathbf{J})$ and we use the notation

$$\frac{g^*(\alpha,\beta)}{p^*(\alpha,\beta)} := \frac{\sum_{i^* \in \mathcal{I}^*} \sum_{j^* \in \mathcal{J}^*} \alpha_i \beta_j g(i,j)}{\sum_{i^* \in \mathcal{I}^*} \sum_{j^* \in \mathcal{J}^*} \alpha_i \beta_j}$$

Generalized Quitting Games

Sets of actions:

Pure actions of player 1 (the decision maker): $\mathbf{I} = \mathcal{I} \times \mathcal{I}^*$
Pure actions of player 2 (nature or advisory): $\mathbf{J} = \mathcal{J} \times \mathcal{J}^*$.
Mixed actions of P1 $\mathbf{x} \in \Delta(\mathcal{I} \times \mathcal{I}^*)$, $x \in \Delta(\mathcal{I})$, $x^* \in \Delta(\mathcal{I}^*)$,
Mixed actions of P2 $\mathbf{y} \in \Delta(\mathcal{J} \times \mathcal{J}^*)$, $y \in \Delta(\mathcal{I})$, $y^* \in \Delta(\mathcal{I}^*)$.
Positive measures $\alpha \in \mathcal{M}(\mathbf{I})$ and $\beta \in \mathcal{M}(\mathbf{I})$.

Vectorial payoffs

$g(i,j) \in \mathbb{R}^d$, $\forall (ij) \in (\mathbf{I}, \mathbf{J})$ and we use the notation

$$\frac{g^*(\alpha,\beta)}{p^*(\alpha,\beta)} := \frac{\sum_{i^* \in \mathcal{I}^*} \sum_{j^* \in \mathcal{J}^*} \alpha_i \beta_j g(i,j)}{\sum_{i^* \in \mathcal{I}^*} \sum_{j^* \in \mathcal{J}^*} \alpha_i \beta_j}$$

Target set (to be approached by player 1)

A closed and convex set $\mathcal{C} \subset \mathbb{R}^d$.

Generalized Quitting Games

Sets of actions:

Pure actions of player 1 (the decision maker): $\mathbf{I} = \mathcal{I} \times \mathcal{I}^*$
Pure actions of player 2 (nature or advisory): $\mathbf{J} = \mathcal{J} \times \mathcal{J}^*$.
Mixed actions of P1 $\mathbf{x} \in \Delta(\mathcal{I} \times \mathcal{I}^*)$, $x \in \Delta(\mathcal{I})$, $x^* \in \Delta(\mathcal{I}^*)$,
Mixed actions of P2 $\mathbf{y} \in \Delta(\mathcal{J} \times \mathcal{J}^*)$, $y \in \Delta(\mathcal{I})$, $y^* \in \Delta(\mathcal{I}^*)$.
Positive measures $\alpha \in \mathcal{M}(\mathbf{I})$ and $\beta \in \mathcal{M}(\mathbf{I})$.

Vectorial payoffs

$g(i,j) \in \mathbb{R}^d$, $\forall (ij) \in (\mathbf{I}, \mathbf{J})$ and we use the notation

$$\frac{g^*(\alpha,\beta)}{p^*(\alpha,\beta)} := \frac{\sum_{i^* \in \mathcal{I}^*} \sum_{j^* \in \mathcal{J}^*} \alpha_i \beta_j g(i,j)}{\sum_{i^* \in \mathcal{I}^*} \sum_{j^* \in \mathcal{J}^*} \alpha_i \beta_j}$$

Target set (to be approached by player 1)

A closed and convex set $\mathcal{C} \subset \mathbb{R}^d$.

Restrictions:

Any action in $\mathcal{I}^*$ or $\mathcal{J}^*$ is absorbing with probability 1.
If $\mathcal{J}^* = \emptyset$ then the game is a *Big-match of type I*.
If $\mathcal{I}^* = \emptyset$ then the game is a *Big-match of type II*.

The Game

- The game is played in discrete time $t = 1, 2, ...$

The Game

- The game is played in discrete time $t = 1, 2, \ldots$
- At each stage $t = 1$, after observing past moves, simultaneously, player 1 chooses $i_t \in \mathbf{I}$ and player 2 chooses $j_t \in \mathbf{J}$.

The Game

- The game is played in discrete time $t = 1, 2, \ldots$
- At each stage $t = 1$, after observing past moves, simultaneously, player 1 chooses $i_t \in \mathbf{I}$ and player 2 chooses $j_t \in \mathbf{J}$.
- If $i_t \in \mathcal{I}^*$ or $j_t \in \mathcal{J}^*$, the game is absorbed:
  from stage t on, the vector payoff is $g_t = g(i_t, j_t)$.

The Game

- The game is played in discrete time $t = 1, 2, \ldots$
- At each stage $t = 1$, after observing past moves, simultaneously, player 1 chooses $i_t \in \mathbf{I}$ and player 2 chooses $j_t \in \mathbf{J}$.
- If $i_t \in \mathcal{I}^*$ or $j_t \in \mathcal{J}^*$, the game is absorbed: from stage t on, the vector payoff is $g_t = g(i_t, j_t)$.
- If $i_t \in \mathcal{I}$ and $j_t \in \mathcal{J}$, the game is not absorbed: the payoff of stage t is g_t, and we move to stage $t + 1$.

The Game

- The game is played in discrete time $t = 1, 2, \ldots$
- At each stage $t = 1$, after observing past moves, simultaneously, player 1 chooses $i_t \in \mathbf{I}$ and player 2 chooses $j_t \in \mathbf{J}$.
- If $i_t \in \mathcal{I}^*$ or $j_t \in \mathcal{J}^*$, the game is absorbed:
  from stage t on, the vector payoff is $g_t = g(i_t, j_t)$.
- If $i_t \in \mathcal{I}$ and $j_t \in \mathcal{J}$, the game is not absorbed:
  the payoff of stage t is g_t, and we move to stage $t + 1$.
- Player 1 wants to approach the set $\mathcal{C}$, player 2 wants to avoid $\mathcal{C}$.

Approachability Notions Studied

Uniform Approachability

$\forall \varepsilon > 0$, player 1 has a strategy such that after some stage $T \in \mathbb{N}$, the expected payoff $\overline{g}_T = \frac{1}{T} = \sum_{t=1}^{T} g_t$ is ε-close to $\mathcal{C}$, no matter the strategy of player 2.

$\mathcal{C}$ is excludable if player 2 can approach the complement of some δ neighborhood of $\mathcal{C}$.

Approachability Notions Studied

Uniform Approachability

$\forall \varepsilon > 0$, player 1 has a strategy such that after some stage $T \in \mathbb{N}$, the expected payoff $\overline{g}_T = \frac{1}{T} = \sum_{t=1}^{T} g_t$ is ε-close to $\mathcal{C}$, no matter the strategy of player 2.
$\mathcal{C}$ is excludable if player 2 can approach the complement of some δ neighborhood of $\mathcal{C}$.

Weak Approachability

$\forall \varepsilon > 0$, there exists $\theta_\varepsilon \in \mathbb{R}$ such that for every $\theta = \{\theta_s\}_{s \in \mathbb{N}^*} \in \Delta(\mathbb{N}^*)$ satisfying $\|\theta\|_2 = \sqrt{\sum_{s=1}^{\infty} \theta_s^2} \leq \theta_\varepsilon$, player 1 has a strategy such that the expected θ-averaged payoff $\overline{g}_\theta = \sum_{t=1}^{\infty} \theta_t g_t$ is ε-close to $\mathcal{C}$, $\forall$ strategy of player 2.

Approachability Notions Studied

Uniform Approachability

$\forall \varepsilon > 0$, player 1 has a strategy such that after some stage $T \in \mathbb{N}$, the expected payoff $\overline{g}_T = \frac{1}{T} = \sum_{t=1}^{T} g_t$ is ε-close to $\mathcal{C}$, no matter the strategy of player 2.

$\mathcal{C}$ is excludable if player 2 can approach the complement of some δ neighborhood of $\mathcal{C}$.

Weak Approachability

$\forall \varepsilon > 0$, there exists $\theta_\varepsilon \in \mathbb{R}$ such that for every $\theta = \{\theta_s\}_{s \in \mathbb{N}^*} \in \Delta(\mathbb{N}^*)$ satisfying $\|\theta\|_2 = \sqrt{\sum_{s=1}^{\infty} \theta_s^2} \leq \theta_\varepsilon$, player 1 has a strategy such that the expected θ-averaged payoff $\overline{g}_\theta = \sum_{t=1}^{\infty} \theta_t g_t$ is ε-close to $\mathcal{C}$, $\forall$ strategy of player 2.

Remark

Blackwell studied almost sure approachability. In Repeated Games, weak, uniform and almost sure notions coincide. The almost sure case was solve by Emanuel Milman (2005) for stochastic games.

Examples

In this game $\mathcal{C} = \{0\}$ is weakly approachable and

Examples

In this game $\mathcal{C} = \{0\}$ is weakly approachable and is not strongly approachable.

	L	R
T	1^*	0^*
B	0	-1

Examples

In this game $\mathcal{C} = \{0\}$ is weakly approachable and is not strongly approachable.

	L	R
T	1^*	0^*
B	0	-1

In this game $\mathcal{C} = \{0\}$ is not weakly approachable.

	L	R
T	1^*	0
B	0^*	-1

Examples

In this game $\mathcal{C} = \{0\}$ is weakly approachable and is not strongly approachable.

	L	R
T	1^*	0^*
B	0	-1

In this game $\mathcal{C} = \{0\}$ is not weakly approachable.

	L	R
T	1^*	0
B	0^*	-1

In this game $\mathcal{C} = \{0\}$ is not weakly (nor uniformly) approachable, and at the same time it is not weakly (nor uniformly) excludable.

	L	R
T	1^*	0^*
B	0^*	-1^*

Examples

In this game $\mathcal{C} = \{0\}$ is weakly approachable and is not strongly approachable.

	L	R
T	1^*	0^*
B	0	-1

In this game $\mathcal{C} = \{0\}$ is not weakly approachable.

	L	R
T	1^*	0
B	0^*	-1

In this game $\mathcal{C} = \{0\}$ is not weakly (nor uniformly) approachable, and at the same time it is not weakly (nor uniformly) excludable.

	L	R
T	1^*	0^*
B	0^*	-1^*

Blackwell condition holds:

$$\forall \mathbf{y} = qL + (1-q)R, \exists \mathbf{x} = (1-q)T + qB : g(\mathbf{x}, \mathbf{y}) = 0$$

Sufficient Condition

Condition SC

$$\forall \varepsilon,\ \forall y,\ \exists x,\ \exists \alpha,\ \forall \beta, \qquad \frac{g(x,y)+g^*(\alpha,\mathbf{y})+g^*(x,\beta)}{1+p^*(\alpha,y)+p^*(x,\beta)} \in \mathcal{C}+\varepsilon\mathcal{B}(0,1)$$

Sufficient Condition

Condition SC

$$\forall \varepsilon,\ \forall y,\ \exists x,\ \exists \alpha,\ \forall \beta, \qquad \frac{g(x,y) + g^*(\alpha,\mathbf{y}) + g^*(x,\beta)}{1 + p^*(\alpha,y) + p^*(x,\beta)} \in \mathcal{C} + \varepsilon\mathcal{B}(0,1)$$

Theorem

SC is sufficient for weak approachability in generalized quitting games.

Sufficient Condition

Condition *SC*

$$\forall \varepsilon,\ \forall y,\ \exists x,\ \exists \alpha,\ \forall \beta,\qquad \frac{g(x,y)+g^*(\alpha,\mathbf{y})+g^*(x,\beta)}{1+p^*(\alpha,y)+p^*(x,\beta)} \in \mathcal{C}+\varepsilon\mathcal{B}(0,1)$$

Theorem

SC is sufficient for weak approachability in generalized quitting games.

Lemma

Condition SC is equivalent to

(1) $\exists (x_0, x_0^*, \gamma_0) \in \Delta(\mathcal{I}) \times \Delta(\mathcal{I}^*) \times (0,1]$ *such that*

$$g(x_0^*, j) \in \mathcal{C},\ \forall j \in \mathcal{J}$$
$$\text{and } g((1-\gamma_0)x_0 + \gamma_0 x_0^*, j^*) \in \mathcal{C}, \forall j^* \in \mathcal{J}^*$$

or

(2) $\forall \varepsilon,\ \forall y \in \Delta(\mathcal{J}), \exists (x, x^*, \gamma) \in \Delta(\mathcal{I}) \times \Delta(\mathcal{I}^*) \times [0,1]$ *such that:*

$$g((1-\gamma)x + \gamma x^*, y) \in \mathcal{C} + \varepsilon\mathcal{B}(0,1)$$
$$\text{and } g(x, j^*) \in \mathcal{C} + \varepsilon\mathcal{B}(0,1), \forall j^* \in \mathcal{J}^*$$

Proof: Part 1

Suppose SC is:

$\exists(x_0, x_0^*, \gamma_0) \in \Delta(\mathcal{I}) \times \Delta(\mathcal{I}^*) \times (0,1]$ such that

$$g(x_0^*, j) \in \mathcal{C},\ \forall j \in \mathcal{J}$$
$$\text{and } g((1-\gamma_0)x_0 + \gamma_0 x_0^*, j^*) \in \mathcal{C}, \forall j^* \in \mathcal{J}^*$$

Proof: Part 1

Suppose SC is:

$\exists(x_0, x_0^*, \gamma_0) \in \Delta(\mathcal{I}) \times \Delta(\mathcal{I}^*) \times (0,1]$ such that

$$g(x_0^*, j) \in \mathcal{C},\ \forall j \in \mathcal{J}$$
$$\text{and } g((1-\gamma_0)x_0 + \gamma_0 x_0^*, j^*) \in \mathcal{C}, \forall j^* \in \mathcal{J}^*$$

- Player 1 play i.i.d according to $(1-\gamma_0)x_0 + \gamma_0 x_0^* \in \Delta(\mathbf{I})$.

Proof: Part 1

Suppose SC is:

$\exists(x_0, x_0^*, \gamma_0) \in \Delta(\mathcal{I}) \times \Delta(\mathcal{I}^*) \times (0,1]$ such that

$$\begin{array}{l} g(x_0^*, j) \in \mathcal{C},\ \forall j \in \mathcal{J} \\ \text{and } g((1-\gamma_0)x_0 + \gamma_0 x_0^*, j^*) \in \mathcal{C}, \forall j^* \in \mathcal{J}^* \end{array}$$

- Player 1 play i.i.d according to $(1-\gamma_0)x_0 + \gamma_0 x_0^* \in \Delta(\mathbf{I})$.
- The game is absorbed at each stage with proba γ_0 or 1 (depending on P2).

Proof: Part 1

Suppose SC is:

$\exists(x_0, x_0^*, \gamma_0) \in \Delta(\mathcal{I}) \times \Delta(\mathcal{I}^*) \times (0,1]$ such that

$$g(x_0^*, j) \in \mathcal{C}, \ \forall j \in \mathcal{J}$$
$$\text{and } g((1-\gamma_0)x_0 + \gamma_0 x_0^*, j^*) \in \mathcal{C}, \forall j^* \in \mathcal{J}^*$$

- Player 1 play i.i.d according to $(1-\gamma_0)x_0 + \gamma_0 x_0^* \in \Delta(\mathbf{I})$.
- The game is absorbed at each stage with proba γ_0 or 1 (depending on P2).
- By condition SC, if the game is absorbed, the payoff is necessarily in $\mathcal{C}$.

Proof: Part 1

Suppose SC is:

$\exists (x_0, x_0^*, \gamma_0) \in \Delta(\mathcal{I}) \times \Delta(\mathcal{I}^*) \times (0,1]$ such that

$$g(x_0^*, j) \in \mathcal{C},\ \forall j \in \mathcal{J}$$
$$\text{and } g((1-\gamma_0)x_0 + \gamma_0 x_0^*, j^*) \in \mathcal{C}, \forall j^* \in \mathcal{J}^*$$

- Player 1 play i.i.d according to $(1-\gamma_0)x_0 + \gamma_0 x_0^* \in \Delta(\mathbf{I})$.
- The game is absorbed at each stage with proba γ_0 or 1 (depending on P2).
- By condition SC, if the game is absorbed, the payoff is necessarily in $\mathcal{C}$.
- Consequently,

$$d\left(\mathbb{E}\left[\overline{g}_\theta\right], \mathcal{C}\right) \leq \sum_{s=1}^{\infty} (1-\gamma_0)^s \theta_s M \leq \frac{1-\gamma_0}{\sqrt{2\gamma_0 - \gamma_0^2}} \|\theta\|_2 M$$

Proof: Part 2

Suppose SC is:

$\forall \varepsilon,\ \forall y \in \Delta(\mathcal{J}), \exists (x, x^*, \gamma) \in \Delta(\mathcal{I}) \times \Delta(\mathcal{I}^*) \times [0,1]$ such that:

$$(1-\gamma) g(x,y) + \gamma g(x^*, y) \in \mathcal{C} + \varepsilon \mathcal{B}(0,1)$$
$$\text{and } g(x, j^*) \in \mathcal{C} + \varepsilon \mathcal{B}(0,1), \forall j^* \in \mathcal{J}^*$$

Proof: Part 2

Suppose SC is:

$\forall \varepsilon,\ \forall y \in \Delta(\mathcal{J}), \exists (x, x^*, \gamma) \in \Delta(\mathcal{I}) \times \Delta(\mathcal{I}^*) \times [0,1]$ such that:

$$(1-\gamma) g(x,y) + \gamma g(x^*, y) \in \mathcal{C} + \varepsilon \mathcal{B}(0,1)$$
$$\text{and } g(x, j^*) \in \mathcal{C} + \varepsilon \mathcal{B}(0,1), \forall j^* \in \mathcal{J}^*$$

- The strategy of player 1 is based on calibration (see Perchet, 2009).

Proof: Part 2

Suppose SC is:

$\forall\varepsilon,\ \forall y \in \Delta(\mathcal{J}), \exists(x, x^*, \gamma) \in \Delta(\mathcal{I}) \times \Delta(\mathcal{I}^*) \times [0,1]$ such that:

$$(1-\gamma)g(x,y) + \gamma g(x^*, y) \in \mathcal{C} + \varepsilon\mathcal{B}(0,1)$$
$$\text{and } g(x, j^*) \in \mathcal{C} + \varepsilon\mathcal{B}(0,1), \forall j^* \in \mathcal{J}^*$$

- The strategy of player 1 is based on calibration (see Perchet, 2009).
- Player 1 predicts, stage by stage, y and plays a response using SC.

Proof: Part 2

Suppose SC is:

$\forall \varepsilon,\ \forall y \in \Delta(\mathcal{J}), \exists (x, x^*, \gamma) \in \Delta(\mathcal{I}) \times \Delta(\mathcal{I}^*) \times [0, 1]$ such that:

$$(1-\gamma)g(x,y) + \gamma g(x^*, y) \in \mathcal{C} + \varepsilon\mathcal{B}(0,1)$$
$$\text{and } g(x, j^*) \in \mathcal{C} + \varepsilon\mathcal{B}(0,1), \forall j^* \in \mathcal{J}^*$$

- The strategy of player 1 is based on calibration (see Perchet, 2009).
- Player 1 predicts, stage by stage, y and plays a response using SC.
- Let $\left\{y[k], k \in \{1, \ldots, K\}\right\}$ be a finite ε-discretization of $\Delta(\mathcal{J})$.

Proof: Part 2

Suppose SC is:

$\forall \varepsilon,\ \forall y \in \Delta(\mathcal{J}), \exists (x, x^*, \gamma) \in \Delta(\mathcal{I}) \times \Delta(\mathcal{I}^*) \times [0,1]$ such that:

$$(1-\gamma) g(x,y) + \gamma g(x^*, y) \in \mathcal{C} + \varepsilon \mathcal{B}(0,1)$$
$$\text{and } g(x, j^*) \in \mathcal{C} + \varepsilon \mathcal{B}(0,1), \forall j^* \in \mathcal{J}^*$$

- The strategy of player 1 is based on calibration (see Perchet, 2009).
- Player 1 predicts, stage by stage, y and plays a response using SC.
- Let $\left\{ y[k], k \in \{1, \ldots, K\} \right\}$ be a finite ε-discretization of $\Delta(\mathcal{J})$.
- By SC, for each $y[k]$, we may associate $(x[k], x^*[k], \gamma[k])$.

Proof: Part 2

Suppose SC is:

$\forall \varepsilon,\ \forall y \in \Delta(\mathcal{J}), \exists (x, x^*, \gamma) \in \Delta(\mathcal{I}) \times \Delta(\mathcal{I}^*) \times [0,1]$ such that:

$$(1-\gamma)g(x,y) + \gamma g(x^*, y) \in \mathcal{C} + \varepsilon \mathcal{B}(0,1)$$
$$\text{and } g(x, j^*) \in \mathcal{C} + \varepsilon \mathcal{B}(0,1), \forall j^* \in \mathcal{J}^*$$

- The strategy of player 1 is based on calibration (see Perchet, 2009).
- Player 1 predicts, stage by stage, y and plays a response using SC.
- Let $\left\{y[k], k \in \{1, \ldots, K\}\right\}$ be a finite ε-discretization of $\Delta(\mathcal{J})$.
- By SC, for each $y[k]$, we may associate $(x[k], x^*[k], \gamma[k])$.
- The strategy of player 1 at stage τ (history dependent) is defined as:

$$\gamma_\tau[k_\tau] x^*[k_\tau] + (1 - \gamma_\tau[k_\tau]) x[k]$$

where:

$$\gamma_\tau[k_\tau] := \frac{\gamma[k_\tau]\theta_\tau}{(1-\gamma[k_\tau])\sum_{s=\tau}^{\infty} \theta_s + \gamma[k_\tau]\theta_\tau}$$

Necessary Condition

Condition *SC*

$$\forall \varepsilon,\ \forall y,\ \exists x,\ \exists \alpha,\ \forall \beta, \qquad \frac{g(x,y)+g^*(\alpha,y)+g^*(x,\beta)}{1+p^*(\alpha,y)+p^*(x,\beta)} \in \mathcal{C}+\varepsilon\mathcal{B}(0,1)$$

Necessary Condition

Condition *SC*

$$\forall \varepsilon,\ \forall y,\ \exists x,\ \exists \alpha,\ \forall \beta, \qquad \frac{g(x,y)+g^*(\alpha,y)+g^*(x,\beta)}{1+p^*(\alpha,y)+p^*(x,\beta)} \in \mathcal{C}+\varepsilon\mathcal{B}(0,1)$$

Condition *NC*

$$\forall \varepsilon,\ \forall y,\ \forall \beta\ \exists x,\ \exists \alpha, \qquad \frac{g(x,y)+g^*(\alpha,y)+g^*(x,\beta)}{1+p^*(\alpha,y)+p^*(x,\beta)} \in \mathcal{C}+\varepsilon\mathcal{B}(0,1)$$

Necessary Condition

Condition *SC*

$$\forall \varepsilon,\ \forall y,\ \exists x,\ \exists \alpha,\ \forall \beta,\quad \frac{g(x,y)+g^*(\alpha,y)+g^*(x,\beta)}{1+p^*(\alpha,y)+p^*(x,\beta)} \in \mathcal{C}+\varepsilon\mathcal{B}(0,1)$$

Condition *NC*

$$\forall \varepsilon,\ \forall y,\ \forall \beta\ \exists x,\ \exists \alpha,\quad \frac{g(x,y)+g^*(\alpha,y)+g^*(x,\beta)}{1+p^*(\alpha,y)+p^*(x,\beta)} \in \mathcal{C}+\varepsilon\mathcal{B}(0,1)$$

Theorem

NC is necessary for weak approachability in generalized quitting games.

Necessary Condition

Condition *SC*

$$\forall\varepsilon,\ \forall y,\ \exists x,\ \exists\alpha,\ \forall\beta,\qquad \frac{g(x,y)+g^*(\alpha,y)+g^*(x,\beta)}{1+p^*(\alpha,y)+p^*(x,\beta)}\in\mathcal{C}+\varepsilon\mathcal{B}(0,1)$$

Condition *NC*

$$\forall\varepsilon,\ \forall y,\ \forall\beta\ \exists x,\ \exists\alpha,\qquad \frac{g(x,y)+g^*(\alpha,y)+g^*(x,\beta)}{1+p^*(\alpha,y)+p^*(x,\beta)}\in\mathcal{C}+\varepsilon\mathcal{B}(0,1)$$

Theorem

NC is necessary for weak approachability in generalized quitting games.

If not, player 2 just play at every stage y perturbed by β.

Necessary Condition

Condition *SC*

$$\forall \varepsilon,\ \forall y,\ \exists x,\ \exists \alpha,\ \forall \beta,\quad \frac{g(x,y)+g^*(\alpha,y)+g^*(x,\beta)}{1+p^*(\alpha,y)+p^*(x,\beta)} \in \mathcal{C}+\varepsilon\mathcal{B}(0,1)$$

Condition *NC*

$$\forall \varepsilon,\ \forall y,\ \forall \beta\ \exists x,\ \exists \alpha,\quad \frac{g(x,y)+g^*(\alpha,y)+g^*(x,\beta)}{1+p^*(\alpha,y)+p^*(x,\beta)} \in \mathcal{C}+\varepsilon\mathcal{B}(0,1)$$

Theorem

NC is necessary for weak approachability in generalized quitting games.

If not, player 2 just play at every stage y perturbed by β.

Remark: the following condition is not necessary, nor sufficient for W-approachability:

$$\forall \varepsilon,\ \forall y,\ \exists x,\ \forall \beta,\ \exists \alpha,\quad \frac{g(x,y)+g^*(\alpha,y)+g^*(x,\beta)}{1+p^*(\alpha,y)+p^*(x,\beta)} \in \mathcal{C}+\varepsilon\mathcal{B}(0,1)$$

Weak Approachability in Big Match Type 1

Lemma

In Big-Match of type I, SC and NC are equivalent to Blackwell condition:

$$\forall \mathbf{y} \in \Delta(\mathbf{J}), \exists \mathbf{x} \in \Delta(\mathbf{I}), g(\mathbf{x}, \mathbf{y}) \in \mathcal{C}$$

Weak Approachability in Big Match Type 1

Lemma

In Big-Match of type I, SC and NC are equivalent to Blackwell condition:

$$\forall \mathbf{y} \in \Delta(\mathbf{J}), \exists \mathbf{x} \in \Delta(\mathbf{I}), g(\mathbf{x}, \mathbf{y}) \in \mathcal{C}$$

which also reads, equivalently, as

$$\forall y \in \Delta(\mathcal{J}), \exists (x, x^*, \gamma) \in \Delta(\mathcal{I}) \times \Delta(\mathcal{I}^*) \times [0,1], (1-\gamma) g(x,y) + \gamma g(x^*, y) \in \mathcal{C}.$$

Weak Approachability in Big Match Type 1

Lemma

In Big-Match of type I, SC and NC are equivalent to Blackwell condition:

$$\forall \mathbf{y} \in \Delta(\mathbf{J}), \exists \mathbf{x} \in \Delta(\mathbf{I}), g(\mathbf{x}, \mathbf{y}) \in \mathcal{C}$$

which also reads, equivalently, as

$$\forall y \in \Delta(\mathcal{J}), \exists (x, x^*, \gamma) \in \Delta(\mathcal{I}) \times \Delta(\mathcal{I}^*) \times [0,1], (1-\gamma) g(x, y) + \gamma g(x^*, y) \in \mathcal{C}.$$

Against a prediction $y \in \Delta(\mathcal{J})$, play $x \in \Delta(\mathcal{I})$ "perturbed" by $x^* \in \Delta(\mathcal{I}^*)$.

Weak Approachability in Big Match Type 1

Lemma

In Big-Match of type I, SC and NC are equivalent to Blackwell condition:

$$\forall \mathbf{y} \in \Delta(\mathbf{J}), \exists \mathbf{x} \in \Delta(\mathbf{I}), g(\mathbf{x}, \mathbf{y}) \in \mathcal{C}$$

which also reads, equivalently, as

$$\forall y \in \Delta(\mathcal{J}), \exists (x, x^*, \gamma) \in \Delta(\mathcal{I}) \times \Delta(\mathcal{I}^*) \times [0, 1], (1-\gamma) g(x, y) + \gamma g(x^*, y) \in \mathcal{C}.$$

Against a prediction $y \in \Delta(\mathcal{J})$, play $x \in \Delta(\mathcal{I})$ "perturbed" by $x^* \in \Delta(\mathcal{I}^*)$.

Theorem

Blackwell condition is necessary and sufficient for weak approachability in BM games of type 1.

Strong Approachability in Big Match Type 1

Theorem

Blackwell condition is not sufficient for uniform approachability in BM of type 1.

Strong Approachability in Big Match Type 1

Theorem

Blackwell condition is not sufficient for uniform approachability in BM of type 1.

Here, Blackwell condition is satisfied for $\mathcal{C} = \{0\}$.

	L	R
T	1^*	0^*
B	0	-1

Strong Approachability in Big Match Type 1

Theorem

Blackwell condition is not sufficient for uniform approachability in BM of type 1.

Here, Blackwell condition is satisfied for $\mathcal{C} = \{0\}$.

	L	R
T	1^*	0^*
B	0	-1

But, $\forall \sigma$ for P1, $\exists \tau$ for P2 such that $u(\sigma, \tau) \notin [-\frac{1}{10}, \frac{1}{10}]$:

Strong Approachability in Big Match Type 1

Theorem

Blackwell condition is not sufficient for uniform approachability in BM of type 1.

Here, Blackwell condition is satisfied for $\mathcal{C} = \{0\}$.

	L	R
T	1^*	0^*
B	0	-1

But, $\forall\sigma$ for P1, $\exists\tau$ for P2 such that $u(\sigma,\tau) \notin [-\frac{1}{10}, \frac{1}{10}]$:

- Let τ be the stationary strategy for P2 which plays $(\frac{1}{2}, \frac{1}{2})$ at every period.

Strong Approachability in Big Match Type 1

Theorem

Blackwell condition is not sufficient for uniform approachability in BM of type 1.

Here, Blackwell condition is satisfied for $\mathcal{C} = \{0\}$.

	L	R
T	1^*	0^*
B	0	-1

But, $\forall\sigma$ for P1, $\exists\tau$ for P2 such that $u(\sigma,\tau) \notin [-\frac{1}{10}, \frac{1}{10}]$:

- Let τ be the stationary strategy for P2 which plays $(\frac{1}{2}, \frac{1}{2})$ at every period.
- If $u(\sigma,\tau) < -\frac{1}{10}$ then we are done.

Strong Approachability in Big Match Type 1

Theorem

Blackwell condition is not sufficient for uniform approachability in BM of type 1.

Here, Blackwell condition is satisfied for $\mathcal{C} = \{0\}$.

	L	R
T	1^*	0^*
B	0	-1

But, $\forall\sigma$ for P1, $\exists\tau$ for P2 such that $u(\sigma,\tau) \notin [-\frac{1}{10}, \frac{1}{10}]$:

- Let τ be the stationary strategy for P2 which plays $(\frac{1}{2}, \frac{1}{2})$ at every period.
- If $u(\sigma,\tau) < -\frac{1}{10}$ then we are done.
- Denote by q^* the probability, that play eventually absorbs. Since

$$u(\sigma,\tau) = \frac{1}{2}q^* - \frac{1}{2}(1-q^*) = q^* - \frac{1}{2},$$

we have

$$q^* \geq -\frac{1}{10} + \frac{1}{2} = \frac{4}{10}.$$

Strong Approachability in Big Match Type 1

Theorem

Blackwell condition is not sufficient for uniform approachability in BM of type 1.

Here, Blackwell condition is satisfied for $\mathcal{C} = \{0\}$.

	L	R
T	1^*	0^*
B	0	-1

But, $\forall \sigma$ for P1, $\exists \tau$ for P2 such that $u(\sigma, \tau) \notin [-\frac{1}{10}, \frac{1}{10}]$:

- Let τ be the stationary strategy for P2 which plays $(\frac{1}{2}, \frac{1}{2})$ at every period.
- If $u(\sigma, \tau) < -\frac{1}{10}$ then we are done.
- Denote by q^* the probability, that play eventually absorbs. Since

$$u(\sigma, \tau) = \frac{1}{2} q^* - \frac{1}{2}(1 - q^*) = q^* - \frac{1}{2},$$

  we have

$$q^* \geq -\frac{1}{10} + \frac{1}{2} = \frac{4}{10}.$$

- Take t large so that the proba q_t that play absorbs before t is at least $\frac{3}{10}$.

Strong Approachability in Big Match Type 1

Theorem

Blackwell condition is not sufficient for uniform approachability in BM of type 1.

Here, Blackwell condition is satisfied for $\mathcal{C} = \{0\}$.

	L	R
T	1^*	0^*
B	0	-1

But, $\forall\sigma$ for P1, $\exists\tau$ for P2 such that $u(\sigma,\tau) \notin [-\frac{1}{10}, \frac{1}{10}]$:

- Let τ be the stationary strategy for P2 which plays $(\frac{1}{2}, \frac{1}{2})$ at every period.
- If $u(\sigma,\tau) < -\frac{1}{10}$ then we are done.
- Denote by q^* the probability, that play eventually absorbs. Since

$$u(\sigma,\tau) = \frac{1}{2}q^* - \frac{1}{2}(1-q^*) = q^* - \frac{1}{2},$$

  we have

$$q^* \geq -\frac{1}{10} + \frac{1}{2} = \frac{4}{10}.$$

- Take t large so that the proba q_t that play absorbs before t is at least $\frac{3}{10}$.
- Let τ' the strategy $(\frac{1}{2}, \frac{1}{2})$ at all periods before period t and L after. Then

$$u(\sigma,\tau') \geq \frac{1}{2}q_t \geq \frac{3}{20} > \frac{1}{10},$$

Approachability in Big Match of Type 2

Lemma

In Big-Match games of type II, Condition SC is equivalent to

$$\forall y \in \Delta(\mathcal{J}), \exists x \in \Delta(\mathcal{I}), g(x,y) \in \mathcal{C} \text{ and } g(x,j^*) \in \mathcal{C}, \forall j^* \in \mathcal{J}^*$$

The interpretation is: if $y \in \Delta(\mathcal{J})$ is predicted, P1 plays $x \in \Delta(\mathcal{I})$. And this strategy must be "good" if player 2 decides to quit the game.

Approachability in Big Match of Type 2

Lemma

In Big-Match games of type II, Condition SC is equivalent to

$$\forall y \in \Delta(\mathcal{J}), \exists x \in \Delta(\mathcal{I}), g(x,y) \in \mathcal{C} \text{ and } g(x,j^*) \in \mathcal{C}, \forall j^* \in \mathcal{J}^*$$

The interpretation is: if $y \in \Delta(\mathcal{J})$ is predicted, P1 plays $x \in \Delta(\mathcal{I})$. And this strategy must be "good" if player 2 decides to quit the game.

Theorem

SC is necessary and sufficient for uniform approachability in BM of type 2.

Approachability in Big Match of Type 2

Lemma

In Big-Match games of type II, Condition SC is equivalent to

$$\forall y \in \Delta(\mathcal{J}), \exists x \in \Delta(\mathcal{I}), g(x,y) \in \mathcal{C} \text{ and } g(x,j^*) \in \mathcal{C}, \forall j^* \in \mathcal{J}^*$$

The interpretation is: if $y \in \Delta(\mathcal{J})$ is predicted, P1 plays $x \in \Delta(\mathcal{I})$. And this strategy must be "good" if player 2 decides to quit the game.

Theorem

SC is necessary and sufficient for uniform approachability in BM of type 2.

Theorem

SC is is not necessary for weak approachability in BM of type 2.

A Necessary Condition

- We first restrict to BM games of type 2 where P2 has only two actions.

A Necessary Condition

- We first restrict to BM games of type 2 where P2 has only two actions.
- R is non-absorbing and L is absorbing.

A Necessary Condition

- We first restrict to BM games of type 2 where P2 has only two actions.
- R is non-absorbing and L is absorbing.
- Let g_L^* and g_R denote the corresponding payoff vectors for P1.

A Necessary Condition

- We first restrict to BM games of type 2 where P2 has only two actions.
- R is non-absorbing and L is absorbing.
- Let g_L^* and g_R denote the corresponding payoff vectors for P1.

Theorem

If $\mathcal{C}$ is weakly approchable, $\exists$ a measurable mapping $\xi : [0,1] \to \Delta(\mathbf{I})$ such that for almost every $t \in [0,1]$,

$$\int_0^t g_R(\xi(s))ds + (1-t)g_L^*(\xi(t)) \in \mathcal{C}.$$

A Necessary Condition

- We first restrict to BM games of type 2 where P2 has only two actions.
- R is non-absorbing and L is absorbing.
- Let g_L^* and g_R denote the corresponding payoff vectors for P1.

Theorem

If $\mathcal{C}$ is weakly approchable, $\exists$ a measurable mapping $\xi : [0,1] \to \Delta(\mathbf{I})$ such that for almost every $t \in [0,1]$,

$$\int_0^t g_R(\xi(s))ds + (1-t)g_L^*(\xi(t)) \in \mathcal{C}.$$

- $\forall \varepsilon > 0$, $\exists N_\varepsilon$, s.t. $\forall N \geq N_\varepsilon$, $\exists \{x^{N,\varepsilon}(k), \quad k = 1, ..., N\}$, s.t. $\forall t \in [0,1]$:

$$\sum_{k=1}^{[Nt]} \frac{g_R(x^{N,\varepsilon}(k))}{N} + (1 - \frac{[Nt]}{N})g_L^*(x^{N,\varepsilon}([Nt]+1)) \in \mathcal{C} + \varepsilon\mathcal{B}(0,1),$$

A Necessary Condition

- We first restrict to BM games of type 2 where P2 has only two actions.
- R is non-absorbing and L is absorbing.
- Let g_L^* and g_R denote the corresponding payoff vectors for P1.

Theorem

If $\mathcal{C}$ is weakly approchable, $\exists$ a measurable mapping $\xi : [0,1] \to \Delta(\mathbf{I})$ such that for almost every $t \in [0,1]$,

$$\int_0^t g_R(\xi(s))ds + (1-t)g_L^*(\xi(t)) \in \mathcal{C}.$$

- $\forall \varepsilon > 0$, $\exists N_\varepsilon$, s.t. $\forall N \geq N_\varepsilon$, $\exists \{x^{N,\varepsilon}(k), \quad k = 1, ..., N\}$, s.t. $\forall t \in [0,1]$:

$$\sum_{k=1}^{[Nt]} \frac{g_R(x^{N,\varepsilon}(k))}{N} + (1 - \frac{[Nt]}{N})g_L^*(x^{N,\varepsilon}([Nt]+1)) \in \mathcal{C} + \varepsilon\mathcal{B}(0,1),$$

- Defining $\xi^{N,\varepsilon}(s) = x^{N,\varepsilon}([sN]+1)$, we obtain that $\forall t \in [0,1]$:

$$\int_0^t g_R(\xi^{N,\varepsilon}(s))ds + (1 - \frac{[Nt]}{N})g_L^*(\xi^{N,\varepsilon}(t)) \in \mathcal{C} + \varepsilon\mathcal{B}(0,1)$$

A Necessary Condition

- We first restrict to BM games of type 2 where P2 has only two actions.
- R is non-absorbing and L is absorbing.
- Let g_L^* and g_R denote the corresponding payoff vectors for P1.

Theorem

If $\mathcal{C}$ is weakly approchable, $\exists$ a measurable mapping $\xi : [0,1] \to \Delta(\mathbf{I})$ such that for almost every $t \in [0,1]$,

$$\int_0^t g_R(\xi(s))ds + (1-t)g_L^*(\xi(t)) \in \mathcal{C}.$$

- $\forall \varepsilon > 0$, $\exists N_\varepsilon$, s.t. $\forall N \geq N_\varepsilon$, $\exists \{x^{N,\varepsilon}(k), \quad k = 1, ..., N\}$, s.t. $\forall t \in [0,1]$:

$$\sum_{k=1}^{[Nt]} \frac{g_R(x^{N,\varepsilon}(k))}{N} + (1 - \frac{[Nt]}{N})g_L^*(x^{N,\varepsilon}([Nt]+1)) \in \mathcal{C} + \varepsilon\mathcal{B}(0,1),$$

- Defining $\xi^{N,\varepsilon}(s) = x^{N,\varepsilon}([sN]+1)$, we obtain that $\forall t \in [0,1]$:

$$\int_0^t g_R(\xi^{N,\varepsilon}(s))ds + (1 - \frac{[Nt]}{N})g_L^*(\xi^{N,\varepsilon}(t)) \in \mathcal{C} + \varepsilon\mathcal{B}(0,1)$$

- We tend N to infinity and ε to zero.

A Sufficient Condition

Theorem

If there is a continuous mapping $\xi : [0,1] \to \Delta(\mathbf{I})$ such that for every $t \in [0,1]$,

$$\int_0^t g_R(\xi(s))ds + (1-t)g_L^*(\xi(t)) \in \mathcal{C},$$

then $\mathcal{C}$ is weakly approchable.

A Sufficient Condition

Theorem

If there is a continuous mapping $\xi : [0,1] \to \Delta(\mathbf{I})$ such that for every $t \in [0,1]$,

$$\int_0^t g_R(\xi(s))ds + (1-t)g_L^*(\xi(t)) \in \mathcal{C},$$

then $\mathcal{C}$ is weakly approchable.

- For any $\varepsilon > 0$, let N_ε s.t. $\forall N \geq N_\varepsilon$ and $\forall s$ and $\forall t$: if $|s-t| \leq \frac{1}{N}$ then $\|\xi(s) - \xi(t)\|_1 \leq \frac{\varepsilon}{M}$.

A Sufficient Condition

Theorem

If there is a continuous *mapping* $\xi : [0,1] \to \Delta(\mathbf{I})$ *such that for every* $t \in [0,1]$,

$$\int_0^t g_R(\xi(s))ds + (1-t)g_L^*(\xi(t)) \in \mathcal{C},$$

then $\mathcal{C}$ *is weakly approchable.*

- For any $\varepsilon > 0$, let N_ε s.t. $\forall N \geq N_\varepsilon$ and $\forall s$ and $\forall t$: if $|s-t| \leq \frac{1}{N}$ then $\|\xi(s) - \xi(t)\|_1 \leq \frac{\varepsilon}{M}$.
- Define $x^N(k) = \xi(\frac{k}{N})$, then $\forall K \in \mathbb{N}^*$:

$$\sum_{k=1}^{K} \frac{g_R(x^N(k))}{N} + (1 - \frac{K}{N})g_L^*(x^N(K+1)) \in \mathcal{C} + \varepsilon$$

A Sufficient Condition

Theorem

If there is a continuous *mapping* $\xi : [0,1] \to \Delta(\mathbf{I})$ *such that for every* $t \in [0,1]$,

$$\int_0^t g_R(\xi(s))ds + (1-t)g_L^*(\xi(t)) \in \mathcal{C},$$

then $\mathcal{C}$ *is weakly approchable.*

- For any $\varepsilon > 0$, let N_ε s.t. $\forall N \geq N_\varepsilon$ and $\forall s$ and $\forall t$: if $|s-t| \leq \frac{1}{N}$ then $\|\xi(s) - \xi(t)\|_1 \leq \frac{\varepsilon}{M}$.
- Define $x^N(k) = \xi(\frac{k}{N})$, then $\forall K \in \mathbb{N}^*$:

$$\sum_{k=1}^{K} \frac{g_R(x^N(k))}{N} + (1 - \frac{K}{N})g_L^*(x^N(K+1)) \in \mathcal{C} + \varepsilon$$

- Now we use the same trick as in Vieille's weak approachability and divide each time interval of length $1/N$ on a large block of length L in which player 1 plays an i.i.d strategies $\xi(s)$.

A Sufficient Condition

Theorem

If there is a continuous *mapping* $\xi : [0,1] \to \Delta(\mathbf{I})$ *such that for every* $t \in [0,1]$,

$$\int_0^t g_R(\xi(s))ds + (1-t)g_L^*(\xi(t)) \in \mathcal{C},$$

then $\mathcal{C}$ *is weakly approchable.*

- For any $\varepsilon > 0$, let N_ε s.t. $\forall N \geq N_\varepsilon$ and $\forall s$ and $\forall t$: if $|s-t| \leq \frac{1}{N}$ then $\|\xi(s) - \xi(t)\|_1 \leq \frac{\varepsilon}{M}$.
- Define $x^N(k) = \xi(\frac{k}{N})$, then $\forall K \in \mathbb{N}^*$:

$$\sum_{k=1}^{K} \frac{g_R(x^N(k))}{N} + (1 - \frac{K}{N})g_L^*(x^N(K+1)) \in \mathcal{C} + \varepsilon$$

- Now we use the same trick as in Vieille's weak approachability and divide each time interval of length $1/N$ on a large block of length L in which player 1 plays an i.i.d strategies $\xi(s)$.
- By the law of large numbers, on the block L, the average payoff if player 2 plays always R is $g_R(\xi(s))$.

Application

For each $p \geq 1$, let us show that player 1 can weakly approach $\{0\}$ in the following game (not satisfying SC):

Application

For each $p \geq 1$, let us show that player 1 can weakly approach $\{0\}$ in the following game (not satisfying SC):

	L	R
T	1^*	p
B	0^*	-1

Application

For each $p \geq 1$, let us show that player 1 can weakly approach $\{0\}$ in the following game (not satisfying SC):

	L	R
T	1^*	p
B	0^*	-1

- Find a C^1 function ξ (where $\xi(s) =$ proba of T at time s) s.t. $\forall t$:

$$\int_0^t (\xi(s)p - (1 - \xi(s))ds + (1 - t)\xi(t) = 0,$$

Application

For each $p \geq 1$, let us show that player 1 can weakly approach $\{0\}$ in the following game (not satisfying SC):

	L	**R**
T	1^*	p
B	0^*	-1

- Find a C^1 function ξ (where $\xi(s)$ = proba of T at time s) s.t. $\forall t$:

$$\int_0^t (\xi(s)p - (1 - \xi(s))ds + (1 - t)\xi(t) = 0,$$

- This is equivalent to $\xi(0) = 0$ and for every t:

$$\xi(t)(p + 1) - 1 - \xi(t) + (1 - t)\frac{d\xi(t)}{dt} = 0,$$

Application

For each $p \geq 1$, let us show that player 1 can weakly approach $\{0\}$ in the following game (not satisfying SC):

	L	R
T	1^*	p
B	0^*	-1

- Find a C^1 function ξ (where $\xi(s)$ = proba of T at time s) s.t. $\forall t$:

$$\int_0^t (\xi(s)p - (1 - \xi(s))ds + (1-t)\xi(t) = 0,$$

- This is equivalent to $\xi(0) = 0$ and for every t:

$$\xi(t)(p+1) - 1 - \xi(t) + (1-t)\frac{d\xi(t)}{dt} = 0,$$

- Which has a unique solution $\xi(t) = \frac{1}{p}(1 - (1-t)^p))$ or:

$$(1-t)^p\mathbf{B} + (1 - (1-t)^p)(\frac{1}{p}\mathbf{T} + (1 - \frac{1}{p})\mathbf{B}),$$

Application

For each $p \geq 1$, let us show that player 1 can weakly approach $\{0\}$ in the following game (not satisfying SC):

	L	**R**
T	1^*	p
B	0^*	-1

- Find a C^1 function ξ (where $\xi(s)$ = proba of T at time s) s.t. $\forall t$:

$$\int_0^t (\xi(s)p - (1-\xi(s))ds + (1-t)\xi(t) = 0,$$

- This is equivalent to $\xi(0) = 0$ and for every t:

$$\xi(t)(p+1) - 1 - \xi(t) + (1-t)\frac{d\xi(t)}{dt} = 0,$$

- Which has a unique solution $\xi(t) = \frac{1}{p}(1-(1-t)^p))$ or:

$$(1-t)^p\mathbf{B} + (1-(1-t)^p)(\frac{1}{p}\mathbf{T} + (1-\frac{1}{p})\mathbf{B}),$$

- That is, player 1 starts at $x_0 = \mathbf{B}$ and then, with time, he increases slightly the probability of $\mathbf{T}$ until reaching $x_1 = \frac{1}{p}\mathbf{T} + (1-\frac{1}{p})\mathbf{B}$.

Application

For each $p \geq 1$, let us show that player 1 can weakly approach $\{0\}$ in the following game (not satisfying SC):

	L	R
T	1^*	p
B	0^*	-1

- Find a C^1 function ξ (where $\xi(s)$ = proba of T at time s) s.t. $\forall t$:

$$\int_0^t (\xi(s)p - (1-\xi(s))ds + (1-t)\xi(t) = 0,$$

- This is equivalent to $\xi(0) = 0$ and for every t:

$$\xi(t)(p+1) - 1 - \xi(t) + (1-t)\frac{d\xi(t)}{dt} = 0,$$

- Which has a unique solution $\xi(t) = \frac{1}{p}(1-(1-t)^p))$ or:

$$(1-t)^p\mathbf{B} + (1-(1-t)^p)(\frac{1}{p}\mathbf{T} + (1-\frac{1}{p})\mathbf{B}),$$

- That is, player 1 starts at $x_0 = \mathbf{B}$ and then, with time, he increases slightly the probability of $\mathbf{T}$ until reaching $x_1 = \frac{1}{p}\mathbf{T} + (1-\frac{1}{p})\mathbf{B}$.
- This calculus extends to any BM of type 2 with twos actions.

Extensions

If player 2 has many absorbing actions, but one non-absorbing action R, then:

Theorem

If $\exists$ *a continuous mapping* $\xi : [0,1] \to \Delta(\mathbf{I})$ *such that* $\forall t \in [0,1]$ *and* $\forall j^* \in \mathcal{J}^*$,

$$\int_0^t g_R(\xi(s))ds + (1-t)g^*(\xi(t), j^*) \in \mathcal{C},$$

then $\mathcal{C}$ *is weakly approchable.*

Extensions

If player 2 has many absorbing actions, but one non-absorbing action R, then:

Theorem

If $\exists$ *a continuous mapping* $\xi : [0,1] \to \Delta(\mathbf{I})$ *such that* $\forall t \in [0,1]$ *and* $\forall j^* \in \mathcal{J}^*$,

$$\int_0^t g_R(\xi(s))ds + (1-t)g^*(\xi(t), j^*) \in \mathcal{C},$$

then $\mathcal{C}$ *is weakly approchable. Conversely, a measurable function* ξ *must exist.*

Extensions

If player 2 has many absorbing actions, but one non-absorbing action R, then:

Theorem

If $\exists$ a continuous mapping $\xi : [0,1] \to \Delta(\mathbf{I})$ such that $\forall t \in [0,1]$ and $\forall j^* \in \mathcal{J}^*$,

$$\int_0^t g_R(\xi(s))ds + (1-t)g^*(\xi(t), j^*) \in \mathcal{C},$$

then $\mathcal{C}$ is weakly approchable. Conversely, a measurable function ξ must exist.

More generally, let $\mathcal{Y}$ (resp. $\mathcal{X}$) be the set of measurable maps from $[0,1] \to \Delta(\mathcal{J})$ (resp. $\Delta(\mathcal{I})$).

Extensions

If player 2 has many absorbing actions, but one non-absorbing action R, then:

Theorem

If $\exists$ *a continuous mapping* $\xi : [0,1] \to \Delta(\mathbf{I})$ *such that* $\forall t \in [0,1]$ *and* $\forall j^* \in \mathcal{J}^*$,

$$\int_0^t g_R(\xi(s))ds + (1-t)g^*(\xi(t), j^*) \in \mathcal{C},$$

then $\mathcal{C}$ *is weakly approchable. Conversely, a measurable function* ξ *must exist.*

More generally, let $\mathcal{Y}$ (resp. $\mathcal{X}$) be the set of measurable maps from $[0,1] \to \Delta(\mathcal{J})$ (resp. $\Delta(\mathcal{I})$).

Theorem

In any BM type 2, a necessary condition for $\mathcal{C}$ *to be weakly approachable is:*

Extensions

If player 2 has many absorbing actions, but one non-absorbing action R, then:

Theorem

If $\exists$ *a continuous mapping* $\xi : [0,1] \to \Delta(\mathbf{I})$ *such that* $\forall t \in [0,1]$ *and* $\forall j^* \in \mathcal{J}^*$,

$$\int_0^t g_R(\xi(s))ds + (1-t)g^*(\xi(t), j^*) \in \mathcal{C},$$

then $\mathcal{C}$ *is weakly approchable. Conversely, a measurable function* ξ *must exist.*

More generally, let $\mathcal{Y}$ (resp. $\mathcal{X}$) be the set of measurable maps from $[0,1] \to \Delta(\mathcal{J})$ (resp. $\Delta(\mathcal{I})$).

Theorem

In any BM type 2, a necessary condition for $\mathcal{C}$ *to be weakly approachable is:* $\forall \gamma \in \mathcal{Y}$ *continuous,* $\exists \xi \in \mathcal{X}$ *such that* $\forall t \in [0,1]$ *and* $\forall j^* \in \mathcal{J}^*$,

Extensions

If player 2 has many absorbing actions, but one non-absorbing action R, then:

Theorem

If $\exists$ *a continuous mapping* $\xi : [0,1] \to \Delta(\mathbf{I})$ *such that* $\forall t \in [0,1]$ *and* $\forall j^* \in \mathcal{J}^*$,

$$\int_0^t g_R(\xi(s))ds + (1-t)g^*(\xi(t), j^*) \in \mathcal{C},$$

then $\mathcal{C}$ *is weakly approchable. Conversely, a measurable function* ξ *must exist.*

More generally, let $\mathcal{Y}$ (resp. $\mathcal{X}$) be the set of measurable maps from $[0,1] \to \Delta(\mathcal{J})$ (resp. $\Delta(\mathcal{I})$).

Theorem

In any BM type 2, a necessary condition for $\mathcal{C}$ *to be weakly approachable is:* $\forall \gamma \in \mathcal{Y}$ *continuous,* $\exists \xi \in \mathcal{X}$ *such that* $\forall t \in [0,1]$ *and* $\forall j^* \in \mathcal{J}^*$,

$$\int_0^t g(\xi(s), \gamma(s))ds + (1-t)g^*(\xi(t), j^*) \in \mathcal{C}.$$

Extensions

If player 2 has many absorbing actions, but one non-absorbing action R, then:

Theorem

If $\exists$ a continuous mapping $\xi : [0,1] \to \Delta(\mathbf{I})$ such that $\forall t \in [0,1]$ and $\forall j^* \in \mathcal{J}^*$,

$$\int_0^t g_R(\xi(s))ds + (1-t)g^*(\xi(t), j^*) \in \mathcal{C},$$

then $\mathcal{C}$ is weakly approchable. Conversely, a measurable function ξ must exist.

More generally, let $\mathcal{Y}$ (resp. $\mathcal{X}$) be the set of measurable maps from $[0,1] \to \Delta(\mathcal{J})$ (resp. $\Delta(\mathcal{I})$).

Theorem

In any BM type 2, a necessary condition for $\mathcal{C}$ to be weakly approachable is: $\forall \gamma \in \mathcal{Y}$ continuous, $\exists \xi \in \mathcal{X}$ such that $\forall t \in [0,1]$ and $\forall j^* \in \mathcal{J}^*$,

$$\int_0^t g(\xi(s), \gamma(s))ds + (1-t)g^*(\xi(t), j^*) \in \mathcal{C}.$$

We are working on the converse.

Extensions

If player 2 has many absorbing actions, but one non-absorbing action R, then:

Theorem

If $\exists$ a continuous mapping $\xi : [0,1] \to \Delta(\mathbf{I})$ such that $\forall t \in [0,1]$ and $\forall j^* \in \mathcal{J}^*$,

$$\int_0^t g_R(\xi(s))ds + (1-t)g^*(\xi(t), j^*) \in \mathcal{C},$$

then $\mathcal{C}$ is weakly approchable. Conversely, a measurable function ξ must exist.

More generally, let $\mathcal{Y}$ (resp. $\mathcal{X}$) be the set of measurable maps from $[0,1] \to \Delta(\mathcal{J})$ (resp. $\Delta(\mathcal{I})$).

Theorem

In any BM type 2, a necessary condition for $\mathcal{C}$ to be weakly approachable is: $\forall \gamma \in \mathcal{Y}$ continuous, $\exists \xi \in \mathcal{X}$ such that $\forall t \in [0,1]$ and $\forall j^* \in \mathcal{J}^*$,

$$\int_0^t g(\xi(s), \gamma(s))ds + (1-t)g^*(\xi(t), j^*) \in \mathcal{C}.$$

We are working on the converse. Without absorption, this reduces to Vieille's characterization.

Extensions

If player 2 has many absorbing actions, but one non-absorbing action R, then:

Theorem

If $\exists$ a continuous mapping $\xi : [0,1] \to \Delta(\mathbf{I})$ such that $\forall t \in [0,1]$ and $\forall j^* \in \mathcal{J}^*$,

$$\int_0^t g_R(\xi(s))ds + (1-t)g^*(\xi(t), j^*) \in \mathcal{C},$$

then $\mathcal{C}$ is weakly approchable. Conversely, a measurable function ξ must exist.

More generally, let $\mathcal{Y}$ (resp. $\mathcal{X}$) be the set of measurable maps from $[0,1] \to \Delta(\mathcal{J})$ (resp. $\Delta(\mathcal{I})$).

Theorem

In any BM type 2, a necessary condition for $\mathcal{C}$ to be weakly approachable is: $\forall \gamma \in \mathcal{Y}$ continuous, $\exists \xi \in \mathcal{X}$ such that $\forall t \in [0,1]$ and $\forall j^* \in \mathcal{J}^*$,

$$\int_0^t g(\xi(s), \gamma(s))ds + (1-t)g^*(\xi(t), j^*) \in \mathcal{C}.$$

We are working on the converse. Without absorption, this reduces to Vieille's characterization. Thus, in general, we must combine prediction and viability.