Topics on strategic learning

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Topics on strategic learning IV:

Tools via continuous time

1. Fictitious play

Discrete fictitious play

Consider a finite game with I players having pure strategy sets S^i and mixed strategy sets $X^i = \Delta(S^i)$. The evaluation fonction is G from $S = \prod_i S^i$ to $\mathbb{R}^I$.
The game is played repeatedly in discrete time and the moves are announced.
Given an n-stage history $h_n = (x_1 = \{x_1^i\}_{i\in I}, x_2, \ldots, x_n) \in S^n$, the move x_{n+1}^i of player i at stage $n+1$ is a best reply to the “time average moves" of her opponents.

$$x_{n+1}^i \in BR^i(\overline{x}_n^{-i}) \tag{1}$$

where BR^i is the best reply correspondence of player i, from $\Delta(S^{-i})$ to X^i, with $S^{-i} = \prod_{j\neq i} S^i$.

Since one deals with time averages one has

$$\bar{x}^i_{n+1} = \frac{n\bar{x}^i_n + x^i_{n+1}}{n+1}$$

hence the stage difference is expressed as

$$\bar{x}^i_{n+1} - \bar{x}^i_n = \frac{x^i_{n+1} - \bar{x}^i_n}{n+1}$$

so that (1) can also be written as :

$$\bar{x}^i_{n+1} - \bar{x}^i_n \in \frac{1}{(n+1)}[BR^i(\bar{x}^{-i}_n) - \bar{x}^i_n]. \tag{2}$$

Definition

Brown (1949, 1951)
A sequence $\{x_n\}$ of moves in S satisfies discrete fictitious play (DFP) if (2) holds.

Remark. x^i_n does not appear explicitly any more in (2): the natural state variable of the process is the empirical average $\bar{x}^i_n \in X^i$.

Continuous fictitious play and best reply dynamics

The continuous (formal) counterpart of the above difference inclusion is the differential inclusion, called continuous fictitious play (CFP):

$$\dot{X}_t^i \in \frac{1}{t}[BR^i(X_t^{-i}) - X_t^i]. \tag{3}$$

The change of time $Z_s = X_{e^s}$ leads to

$$\dot{Z}_s^i \in [BR^i(Z_s^{-i}) - Z_s^i] \tag{4}$$

called continuous best reply (CBR) and studied by Gilboa and Matsui (1991).
We will deduce properties of the initial discrete time process from the analysis of the continuous time counterpart.

Zero-sum case

Harris (1998); Hofbauer (1995).
Let $x = x^1$, $y = x^2$. Then $F^1 = -F^2 = F(x, y) = xAy$.

Define $a(y) = \max_{x \in X} F(x, y)$ and $b(x) = \min_{y \in Y} F(x, y)$, then the duality gap at (x, y) is

$$W(x, y) = a(y) - b(x) \geq 0.$$

Moreover (x^*, y^*) belongs to the set of optimal strategies, $X_F \times Y_F$, iff $W(x^*, y^*) = 0$.
W will play the rôle of a potential (distance) for the set $X_F \times Y_F$.

Proposition

The "duality gap" criteria converges uniformly to 0 in (CBR) and (CFP).

Proof

Let (x_t, y_t) be a solution of (CBR) (4) and introduce

$$\alpha_t = x_t + \dot{x}_t \in BR^1(y_t), \beta_t = y_t + \dot{y}_t \in BR^2(x_t).$$

Consider the evaluation of the duality gap along a trajectory: $w_t = W(x_t, y_t)$. Note that $a(y_t) = F(\alpha_t, y_t)$ hence

$$\frac{d}{dt}a(y_t) = D_1F(\alpha_t, y_t)\dot{\alpha}_t + D_2F(\alpha_t, y_t)\dot{y}_t$$

but the first term is 0 (envelope theorem). As for the second one $D_2F(\alpha_t, y_t)\dot{y}_t = F(\alpha_t, \dot{y}_t)$, by linearity. Thus:

$$\begin{aligned}\dot{w}_t &= \frac{d}{dt}a(y_t) - \frac{d}{dt}b(x_t) = F(\alpha_t, \dot{y}_t) - F(\dot{x}_t, \beta_t) = F(x_t, \dot{y}_t) - F(\dot{x}_t, y_t) \\ &= F(x_t, \beta_t) - F(\alpha_t, y_t) = b(x_t) - a(y_t) = -w_t.\end{aligned}$$

It follows that exponential convergence holds: $w_t = e^{-t}w_0$, hence convergence at a rate $1/t$ in the original (CFP). ∎

Extension: Hofbauer and Sorin (2006)
F is a continuous, concave/convex real function defined on a product of two compact convex subsets of an euclidean space.

Proposition

Under (H), any solution w_t of (CBR) satisfies

$$\dot{w}_t \leq -w_t \; a.e.$$

$X_F \times Y_F$ *is a global attractor.*

To deduce results for the discrete time case from results in the continuous time one, we introduce a discrete deterministic approximation.

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Discrete deterministic approximation

Consider a differential inclusion, where Φ is u.s.c. with convex-compact values:

$$\dot{z}_t \in \Phi(z_t). \tag{5}$$

Let α_n a sequence of positive real numbers with $\sum \alpha_n = +\infty$. Given $a_0 \in Z$, define inductively $\{a_n\}$ through the difference equation:

$$a_{n+1} - a_n \in \alpha_{n+1}\Phi(a_n). \tag{6}$$

Definition

$\{a_n\}$ following (6) is a discrete deterministic approximation (DDA) of (5).

The associated continuous time trajectory $\mathbf{A} : \mathbb{R}^+ \to Z$ is constructed in two stages.
First define a sequence of times $\{\tau_n\}$ by: $\tau_0 = 0, \tau_{n+1} = \tau_n + \alpha_{n+1}$; then let $A_{\tau_n} = a_n$ and extend the trajectory by linear interpolation on each intervall $[\tau_n, \tau_{n+1}]$:

$$A_t = a_n + \frac{(t - \tau_n)}{(\tau_{n+1} - \tau_n)}(a_{n+1} - a_n).$$

Since $\sum \alpha_n = +\infty$ the trajectory is defined on $\mathbb{R}^+$.

To compare $\mathbf{A}$ to a solution of (5) we use the next approximation property which states that two differential inclusions defined by correspondences having graphs close one to the other will have sets of solutions close, on a given compact time interval.

Notations

$\mathscr{A}(\Phi, T, z) = \{\mathbf{z};\ \mathbf{z}$ is a solution of (5) on $[0, T]$ with $z_0 = z\}$,
$D_T(\mathbf{y}, \mathbf{z}) = \sup_{0 \leq t \leq T} \|y_t - z_t\|$,
G_Φ is the graph of Φ and G_Φ^ε is an ε-neighborhood of G_Φ.

Proposition

$\forall T \geq 0, \forall \varepsilon > 0, \exists \delta > 0$ *such that:*

$$\inf\{D_T(\mathbf{y}, \mathbf{z}); \mathbf{z} \in \mathscr{A}(\Phi, T, z)\} \leq \varepsilon$$

for any $\mathbf{y}$ *solution of* $\dot{y}_t \in \tilde{\Phi}(y_t)$, *with* $y_0 = z$ *and* $d(G_\Phi, G_{\tilde{\Phi}}) \leq \delta$.

Assume α_n decreasing to 0. Then the set $L(\{a_n\})$ of accumulation points of the sequence $\{a_n\}$ coincides with the limit set of the trajectory: $L(\mathbf{A}) = \cap_{t \geq 0} \overline{A_{[t,+\infty)}}$.

Proposition

If Z is a global attractor for (5), it is also a global attractor for (6).

Proof

i) Given $\varepsilon > 0$, let T_1 such that any trajectory $\mathbf{z}$ of (5) is within ε of Z after time T_1. Given T_1 and ε, let $\delta > 0$ be defined by the previous Proposition (approximation property).

Since α_n decreases to 0, given $\delta > 0$, for $n \geq N$ large enough for a_n, hence $t \geq T_2$ large enough for A_t, one has :

$$\dot{A}_t \in \Psi(A_t) \quad \text{with} \quad G_\Psi \subset G_\Phi^\delta.$$

ii) Consider now A_t for some $t \geq T_1 + T_2$.
Starting from any position A_{t-T_1} the continuous time process $\mathbf{z}$ defined by (5) reaches Z within ε at time t (the convergence is uniform).
Since $t - T_1 \geq T_2$, the interpolated process A_s remains within ε of the former z_s on $[t - T_1, t]$, hence is within 2ε of Z at time t.
In particular this shows: $\forall \varepsilon > 0, \exists N_0$ such that $n \geq N_0$ implies

$$d(a_n, Z) \leq 2\varepsilon.$$

∎

Proposition

(DFP) converges to $X_F \times Y_F$ in the continuous saddle zero-sum case.

The initial convergence result in the discrete case (finite game) is due to Robinson (1951).

In this framework one has also:

Proposition

(Rivière, 1997)
The average of the realized payoffs along (DFP) converge to the value.

Potential games

Definition

The game (F,S) is a potential game (with potential G) if $G : S \to \mathbb{R}$ satisfies:

- discrete case:

$$F^i(s^i,s^{-i}) - F^i(t^i,s^{-i}) = G(s^i,s^{-i}) - G(t^i,s^{-i}), \quad \forall s^i \in S^i, t^i \in S^i, s^{-i} \in S^{-i}, \quad \forall$$

- continuous case:

$$\nabla_i F^i(s) = \nabla_i G(s), \quad \forall s \in S, \quad \forall i \in I.$$

In particular the best reply correspondence BR^i is the same when applied to F^i or to G.
Let $NE(F) \subset X$ be the set of Nash equilibria of F then $NE(F) = NE(G)$.

Consider a potential game where G is defined on a product X of compact convex subsets X^i of an euclidean space, $\mathscr{C}^1$ and concave in each variable.

a) Discrete time
Finite case: Monderer and Shapley (1996).

Proposition

(DFP) converges to $NE(G)$.

b) Continuous time
Finite case: Harris (1998)
Compact case: Benaim, Hofbauer and Sorin (2005).

Proposition

(CBR) (or (CFP)) converges to $NE(G)$.

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Proof

Let $W(x) = \sum_i [g^i(x) - G(x)]$ where $g^i(x) = \max_{s \in X^i} G(s, x^{-i})$.
Thus x is a Nash equilibrium iff $W(x) = 0$ (Nikaido).
Let x_t be a solution of (CBR) and consider $g_t = G(x_t)$.
Then $\dot{g}_t = \sum_i D_i G(x_t) \dot{x}_t^i$. By concavity one obtains:

$$G(x_t^i, x_t^{-i}) + D_i G(x_t^i, x_t^{-i})\, \dot{x}_t^i \geq G(x_t^i + \dot{x}_t^i, x_t^{-i})$$

which implies

$$\dot{g}_t \ \geq \ \sum_i [G(x_t^i + \dot{x}_t^i, x_t^{-i}) - G(x_t)] = W(x_t) \geq 0$$

hence g is increasing but bounded.
g is thus constant on the limit set $L(\mathbf{x})$.
By the previous majoration, for any accumulation point x^* one has $W(x^*) = 0$ and x^* is a Nash equilibrium. ■

In this framework also, one can deduce the convergence of the discrete time process from the properties of the continuous time analog.

Proposition

Assume $G(NE(G))$ with non empty interior. Then (DFP) converges to $NE(G)$.

Contrary to the zero-sum case where the attractor was global, the proof uses here the tools of stochastic approximation.

Remarks. Note that one cannot expect uniform convergence. Consider the standard symmetric coordination game:

$(1,1)$	$(0,0)$
$(0,0)$	$(1,1)$

The only attractor that contains $NE(F)$ is the diagonal. In particular convergence of (CFP) does not imply directly convergence of (DFP). Note that the equilibrium $(1/2,1/2)$ is unstable but the time to go from $(1/2^{+},1/2^{-})$ to $(1,0)$ is not bounded.

2. Stochastic Approximation for Differential Inclusions

We summarize here results from Benaïm, Hofbauer and Sorin (2005), following the approach for ODE by Benaïm (1996, 1999), Benaïm and Hirsch (1996).

1. Differential inclusions
Given a correspondence F from $\mathbb{R}^m$ to itself, consider the differential inclusion

$$\dot{\mathbf{x}} \in F(\mathbf{x}). \qquad (I)$$

It induces a set-valued dynamical system $\{\Phi_t\}_{t\in\mathbb{R}}$ defined by

$$\Phi_t(x) = \{\mathbf{x}(t) : \mathbf{x} \text{ is a solution to } (I) \text{ with } \mathbf{x}(0) = x\}.$$

We also write $\mathbf{x}(t) = \varphi_t(x)$ and define $\Phi_A(B) = \cup_{t\in A,\, x\in B}\Phi_t(x)$.

2. Attractors

Definition

1) C is invariant if for any $x \in C$ there exists a complete solution: $\varphi_t(x) \in C$ for all $t \in \mathbb{R}$.
2) C is attracting if it is compact and there exist a neighborhood U, $\varepsilon_0 > 0$ and a map $T : (0, \varepsilon_0) \to \mathbb{R}^+$ such that: for any $y \in U$, any solution φ, $\varphi_t(y) \in C^\varepsilon$ for all $t \geq T(\varepsilon)$, i.e.

$$\Phi_{[T(\varepsilon),+\infty)}(U) \subset C^\varepsilon, \quad \forall \varepsilon \in (0, \varepsilon_0).$$

U is a uniform basin of attraction of C and we write $(C;U)$ for the couple.
3) C is an attractor if it is attracting and invariant.
4) The ω-limit set of C is defined by

$$\omega_\Phi(C) = \cap_{s\geq 0}\overline{\cup_{y\in C}\cup_{t\geq s}\Phi_t(y)} = \cap_{s\geq 0}\overline{\Phi_{[s,+\infty)}(C)}. \qquad (7)$$

5) Given a closed invariant set L, the induced set-valued dynamical system is denoted by Φ^L. L is attractor free if Φ^L has no proper attractor.

3. Lyapounov functions

We describe here practical criteria for attractors.

Proposition

Let A be a compact set, U be a relatively compact neighborhood and V a function from $\overline{U}$ to $\mathbb{R}^+$. Assume:
i) $\Phi_t(U) \subset U$ *for all* $t \geq 0$.
ii) $V^{-1}(0) = A$
iii) V *is continuous and strictly decreasing on trajectories on* $\overline{U} \setminus A$:

$$V(x) > V(y), \quad \forall x \in U \setminus A, \forall y \in \Phi_t(x), \quad \forall t > 0.$$

Then:
a) A *is Lyapounov stable and* $(A;U)$ *is attracting.*
b) $(B;U)$ *is an attractor for some* $B \subset A$.

Definition

A real continuous function V on U open in $\mathbb{R}^m$ is a Lyapunov function for (A,U), $A \subset U$ if :
$V(y) < V(x)$ for all $x \in U \setminus A, y \in \Phi_t(x), t > 0,$
$V(y) \leq V(x)$ for all $x \in A, y \in \Phi_t(x)$ and $t \geq 0.$

Proposition

Suppose V is a Lyapunov function for (A,U). Assume that $V(A)$ has empty interior. Let L be a non empty, compact, invariant and attractor free subset of U. Then L is contained in A and $V_{|L}$ is constant.

3. Asymptotic pseudo-trajectories

Definition

A continuous function $\mathbf{z} : \mathbb{R}^+ \to \mathbb{R}^m$ is an asymptotic pseudo-trajectory (APT) for (I) if for all T

$$\lim_{t\to\infty} \inf_{\mathbf{x}\in S_{\mathbf{z}(t)}} \sup_{0\le s\le T} \|\mathbf{z}(t+s) - \mathbf{x}(s)\| = 0. \tag{8}$$

where S_x denotes the set of solutions of (I) starting from x at 0.

In other words, for each fixed T, the curve: $s \to \mathbf{z}(t+s)$ from $[0,T]$ to $\mathbb{R}^m$ shadows some trajectory for (I) of the point $\mathbf{z}(t)$ over the interval $[0,T]$ with arbitrary accuracy, for sufficiently large t.
Let

$$L(\mathbf{z}) = \bigcap_{t\ge 0} \overline{\{\mathbf{z}(s) : s \ge t\}}$$

be the limit set.

Theorem

Let $\mathbf{z}$ be a bounded APT of (I). Then $L(\mathbf{z})$ is (internally chain transitive, hence) compact, invariant and attractor free.

4. Perturbed solutions

Definition

A continuous function $\mathbf{y} : \mathbb{R}_+ = [0, \infty) \to \mathbb{R}^m$ is a perturbed solution to (I) if it satisfies the following set of conditions (II):

i) $\mathbf{y}$ is absolutely continuous.

ii) There exists a locally integrable function $t \mapsto U(t)$ such that $\lim_{t\to\infty} \sup_{0\le v\le T} \left\| \int_t^{t+v} U(s)\, ds \right\| = 0$, for all $T > 0$.

iii)

$$\dot{\mathbf{y}}(t) \in F^{\delta(t)}(\mathbf{y}(t)) + U(t),$$

for almost every $t > 0$, for some function $\delta : [0, \infty) \to \mathbb{R}$ with $\delta(t) \to 0$ as $t \to \infty$.

Here $F^{\delta}(x) := \{y \in \mathbb{R}^m : \exists z : \|z - x\| < \delta,\ d(y, F(z)) < \delta\}$.

The purpose is to investigate the long-term behavior of $\mathbf{y}$ and to describe its limit set $L(\mathbf{y})$ in terms of the dynamics induced by F.

Theorem

Any bounded solution $\mathbf{y}$ of (II) is an APT of (I).

A natural class of perturbed solutions to F arises from certain stochastic approximation processes.

Definition

A discrete time process $\{x_n\}$ with values in $\mathbb{R}^m$ is a (γ, U) discrete stochastic approximation for (I) if it verifies a recursion of the form

$$x_{n+1} - x_n \in \gamma_{n+1}[F(x_n) + U_{n+1}], \qquad (III)$$

where the characteristics $\{\gamma_n\}$ and $\{U_n\}$ satisfy
i) $\{\gamma_n\}_{n\geq 1}$ is a sequence of nonnegative numbers such that

$$\sum_n \gamma_n = \infty, \qquad \lim_{n\to\infty} \gamma_n = 0;$$

ii) $U_n \in \mathbb{R}^m$ are (deterministic or random) perturbations.

To such a process is naturally associated a continuous time interpolated (random) process $\mathbf{w}$ as usual (IV).

5. From interpolated process to perturbed solutions

The next result gives sufficient conditions on the characteristics of the discrete process (III) for its interpolation (IV) to be a perturbed solution (II).

Proposition

Assume that :

$(*)$ For all $T > 0$

$$\lim_{n\to\infty} \sup \left\{ \left\| \sum_{i=n}^{k-1} \gamma_{i+1} U_{i+1} \right\| : k = n+1, \ldots, m(\tau_n + T) \right\} = 0,$$

where $\tau_n = \sum_{i=1}^{n} \gamma_i$ and $m(t) = \sup\{k \geq 0 : t \geq \tau_k\}$;

$(**)$ $\sup_n \|x_n\| = M < \infty$.

Then the interpolated process $\mathbf{w}$ is a perturbed solution of (I).

We describe now sufficient conditions for condition $(*)$ to hold. Let $(\Omega, \mathscr{F}, P)$ be a probability space and $\{\mathscr{F}_n\}_{n\geq 0}$ a filtration of $\mathscr{F}$. A stochastic process $\{x_n\}$ satisfies the Robbins–Monro condition if:

i) $\{\gamma_n\}$ is a deterministic sequence.
ii) $\{U_n\}$ is *adapted* to $\{\mathscr{F}_n\}$,
iii) $\mathbf{E}(U_{n+1} \mid \mathscr{F}_n) = 0$.

Proposition

Let $\{x_n\}$ given by (III) be a Robbins–Monro process. Suppose that for some $q \geq 2$

$$\sup_n \mathbf{E}(\|U_n\|^q) < \infty \quad \text{and} \quad \sum_n \gamma_n^{1+q/2} < \infty.$$

Then assumption $(*)$ holds with probability 1.

Remark Typical applications are
i) U_n uniformly bounded in L^2 and $\gamma_n = \frac{1}{n}$,
ii) U_n uniformly bounded and $\gamma_n = o(\frac{1}{\log n})$.

6. Main result

Consider a random discrete process defined on a compact subset of $\mathbb{R}^K$ and satisfying the differential inclusion :

$$Y_n - Y_{n-1} \in a_n[T(Y_{n-1}) + W_n]$$

where
i) T is an u.s.c. correspondence with compact convex values
ii) $a_n \geq 0$, $\sum_n a_n = +\infty$, $\sum_n a_n^2 < +\infty$
iii) $E(W_n | Y_1, ..., Y_{n-1}) = 0$.

Theorem

The set of accumulation points of $\{Y_n\}$ is almost surely a compact set, invariant and attractor free for the dynamical system defined by the differential inclusion:

$$\dot{Y} \in T(Y).$$

A typical application is the case where:

$$Y_n - Y_{n-1} \in a_n \mathbf{T}(Y_{n-1})$$

with $\mathbf{T}$ random, where one writes

$$Y_n - Y_{n-1} \in a_n[E[\mathbf{T}(Y_{n-1})|Y_1, ..., Y_{n-1}]$$

$$+(\mathbf{T}(Y_{n-1}) - E[\mathbf{T}(Y_{n-1})|Y_1, ..., Y_{n-1}])]$$

3. Application1: Fictitious Play for potential games

Proposition

Assume $G(NE(G))$ with non empty interior. Then (DFP) converges to $NE(G)$.

Apply Proposition 2 to W with $A = NE(G)$ and $U = X$.

Recall

Proposition

Suppose V is a Lyapunov function for (A, U). Assume that $V(A)$ has empty interior. Let L be a non empty, compact, invariant and attractor free subset of U. Then L is contained in A and $V_{|L}$ is constant.

4. Application 2: No regret

Definition

P is a potential function for $D = \mathbb{R}^K_-$ if
(i) P is $\mathscr{C}^1$ from $\mathbb{R}^K$ to $\mathbb{R}^+$
(ii) $P(w) = 0$ iff $w \in D$
(iii) $\nabla P(w) \in \mathbb{R}^K_+$
(iv) $\langle \nabla P(w), w \rangle > 0, \forall w \notin D$.

Compare Hart and Mas Colell (2003).

Example: $P(w) = \sum_k ([w^k]^+)^2 = d(w,D)^2$.

1. External regret

Given a potential P for $D = \mathbb{R}^K_-$, the P-regret-based discrete procedure for player 1 is defined by

$$\sigma(h_n) \div \nabla P(\overline{R}_n) \qquad \text{if} \quad \overline{R}_n \notin D \tag{9}$$

and arbitrarily otherwise.

Discrete dynamics associated to the average regret:

$$\overline{R}_{n+1} - \overline{R}_n = \frac{1}{n+1}(R_{n+1} - \overline{R}_n)$$

By the choice of σ

$$\langle \nabla P(\overline{R}_n), \mathsf{E}(R_{n+1}|h_n)\rangle = 0.$$

(recall $\langle x, \mathsf{E}_x(R(.,U))\rangle = 0.$)
The continuous time version is expressed by the following differential inclusion in $\mathbb{R}^m$:

$$\dot{\mathbf{w}} \in N(\mathbf{w}) - \mathbf{w} \tag{10}$$

where N is a correspondence that satisfies

$$\langle \nabla P(\mathbf{w}), \mathbf{N}(\mathbf{w})\rangle = \mathbf{0}.$$

Theorem

The potential P is a Lyapounov function associated to $D = \mathbb{R}^K_-$. Hence, D contains a global attractor attractor.

Proof

For any solution $\mathbf{w}$, if $\mathbf{w}(t) \notin D$ then

$$\frac{d}{dt}P(\mathbf{w}(t)) = \langle \nabla P(\mathbf{w}(t)), \dot{\mathbf{w}}(t) \rangle$$

$$\in \langle \nabla P(\mathbf{w}(t)), N(\mathbf{w}(t)) - \mathbf{w}(t) \rangle = -\langle \nabla P(\mathbf{w}(t)), \mathbf{w}(t) \rangle < 0$$

■

Corollary

Any P-regret-based discrete dynamics satisfies internal consistency.

Proof

$D = \mathbb{R}^K_-$ contains an attractor whose basin of attraction contains the range $\mathscr{R}$ of R and the discrete process for $\bar{R}_n$ is a bounded DSA. ■

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2. Internal regret

Definition

Given a potential Q for $M = \mathbb{R}_{-}^{K^2}$, a Q-regret-based discrete procedure for player 1 is a strategy σ satisfying

$$\sigma(h_n) \in Inv[\nabla Q(\bar{S}_n)] \qquad \text{if} \quad \bar{S}_n \notin M \tag{11}$$

and arbitrarily otherwise.

The discrete process of internal regret matrices is:

$$\bar{S}_{n+1} - \bar{S}_n = \frac{1}{n+1}[S_{n+1} - \bar{S}_n]. \tag{12}$$

with the property:

$$\langle \nabla Q(\bar{S}_n), \mathsf{E}(S_{n+1}|h_n) \rangle = 0.$$

(Recall $\langle A, \mathsf{E}_\mu(S(.,U)) \rangle = 0$.)

Corresponding continuous procedure with $w \in \mathbb{R}^{K^2}$

$$\dot{\mathbf{w}}(t) \in N(\mathbf{w}(t)) - \mathbf{w}(t) \tag{13}$$

and

$$\langle \nabla Q(w), N(w \rangle = 0.$$

Theorem

The previous continuous time process satisfy:

$$\mathbf{w}_{k\ell}^{+}(t) \to_{t\to\infty} 0.$$

Corollary

The discrete process (12) satisfy:

$$[\bar{S}_n^{k\ell}]^{+} \to_{t\to\infty} 0 \qquad a.s.$$

hence conditional consistency (internal no regret) holds.

5. Application 3: Consistency with smooth fictitious play

This procedure is based only on the previous observations and not on the moves of the predictor, hence the regret cannot be used, Fudenberg and Levine (1995).

Definition

A smooth perturbation of the payoff $U \in \mathscr{U}$ is a map
$V^{\varepsilon}(x,U) = \langle x, U \rangle - \varepsilon \rho(x), \quad 0 < \varepsilon < \varepsilon_0$, such that:
(i) $\rho : X \to \mathbb{R}$ is a $\mathscr{C}^1$ function with $\|\rho\| \leq 1$,
(ii) $\text{argmax}_{x \in X} V^{\varepsilon}(.,U)$ reduces to one point and defines a continuous map $\mathbf{br}^{\varepsilon} : \mathscr{U} \to X$, called a smooth best reply function,
(iii) $D_1 V^{\varepsilon}(\mathbf{br}^{\varepsilon}(U), U).D\mathbf{br}^{\varepsilon}(U) = 0$
(for example $D_1 U^{\varepsilon}(.,U)$ is 0 at $\mathbf{br}^{\varepsilon}(U)$).

Recall that a typical example is obtained via the entropy function

$$\rho(x) = \sum_k x_k \log x_k. \tag{14}$$

which leads to

$$[\mathbf{br}^{\varepsilon}(U)]^k = \frac{\exp(U^k/\varepsilon)}{\sum_{j\in K}\exp(U^j/\varepsilon)}. \tag{15}$$

Let

$$W^{\varepsilon}(U) = \max_x V^{\varepsilon}(x,U) = V^{\varepsilon}(\mathbf{br}^{\varepsilon}(U),U).$$

Lemma

(Fudenberg and Levine (1999))

$$DW^{\varepsilon}(U) = \mathbf{br}^{\varepsilon}(U).$$

Let us first consider external consistency.

Definition

A smooth fictitious play strategy σ^ε associated to the smooth best response function $\mathbf{br}^\varepsilon$ (in short a SFP(ε) strategy) is defined by ($\overline{U}_n$ is the average vector of regret at stage n):

$$\sigma^\varepsilon(h_n) = \mathbf{br}^\varepsilon(\overline{U}_n).$$

The corresponding discrete dynamics written in the spaces of both vectors and outcomes is

$$\overline{U}_{n+1} - \overline{U}_n = \frac{1}{n+1}[U_{n+1} - \overline{U}_n]. \tag{16}$$

$$\overline{\omega}_{n+1} - \overline{\omega}_n = \frac{1}{n+1}[\omega_{n+1} - \overline{\omega}_n]. \tag{17}$$

with

$$\mathsf{E}(\omega_{n+1}|h_n) = \langle \mathbf{br}^\varepsilon(\overline{U}_n), U_{n+1} \rangle. \tag{18}$$

Lemma

The process $(\overline{U}_n, \overline{\omega}_n)$ *is a Discrete Stochastic Approximation of the differential inclusion*

$$(\dot{\mathbf{u}}, \dot{\omega}) \in \{(U - \mathbf{u}, \langle \mathbf{br}^{\varepsilon}(\mathbf{u}), U\rangle - \omega); U \in \mathscr{U}\}. \qquad (19)$$

The main property of the continuous dynamics is given by:

Theorem

The set $\{(u,\omega) \in \mathscr{U} \times \mathbb{R} : W^{\varepsilon}(u) - \omega \leq \varepsilon\}$ *is a global attracting set for the continuous dynamics.*
In particular, for any $\eta > 0$*, there exists* $\bar{\varepsilon}$ *such that for* $\varepsilon \leq \bar{\varepsilon}$*,* $\limsup_{t\to\infty} W^{\varepsilon}(\mathbf{u}(t)) - \omega(t) \leq \eta$ *(i.e. continuous* SFP(ε) *satisfies* η*-consistency).*

Proof

Let $q(t) = W^{\varepsilon}(\mathbf{u}(t)) - \omega(t)$.

Taking time derivative one obtains, using the previous Lemma:

$$\begin{aligned}
\dot{q}(t) &= DW^{\varepsilon}(\mathbf{u}(t)).\dot{\mathbf{u}}(t) - \dot{\omega}(t) \\
&= \langle \mathbf{br}^{\varepsilon}(\mathbf{u}(t)), \dot{\mathbf{u}}(t) \rangle - \dot{\omega}(t) \\
&= \langle \mathbf{br}^{\varepsilon}(\mathbf{u}(t)), U - \mathbf{u}(t) \rangle - (\langle \mathbf{br}^{\varepsilon}(\mathbf{u}(t)), U \rangle - \omega(t)) \\
&\leq -q(t) + \varepsilon.
\end{aligned}$$

Hence

$$\dot{q}(t) + q(t) \leq \varepsilon$$

so that $q(t) \leq \varepsilon + Me^{-t}$ for some constant M and the result follows. ∎

Theorem

For any $\eta > 0$, there exists $\bar{\varepsilon}$ such that for $\varepsilon \leq \bar{\varepsilon}$, SFP$(\varepsilon)$ *is η-consistent.*

Proof

The assertion follows from the previous result and the DSA property.

∎

A similar result holds for internal no-regret procedures.

Recent advances: Benaim and Faure (2013) obtain consistency with vanishing perturbation $\varepsilon = n^{-a}, a < 1$. Process non longer autonomous.

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6. Replicator dynamics and EWA

Replicator dynamics

Evolution of a single population with K types modelized through a symmetric 2 person game with $K \times K$ payoff (fitness) matrix A
A_{ij} is the payoff of "i" facing "j".
x_t^k : frequency of type k at time t.
Replicator equation on the simplex $\Delta(K)$ of $\mathbb{R}^K$

$$\dot{x}_t^k = x_t^k \left(e^k A x_t - x_t A x_t \right), \quad k \in K \qquad (RD) \tag{20}$$

Taylor and Jonker (1978)

Replicator dynamics for I populations

$$\dot{x}_t^{ip} = x_t^{ip}[F^i(e^{ip}, x_t^{-i}) - F^i(x_t^i, x_t^{-i})], \qquad p \in S^i, i \in I$$

natural interpretation: $x_t^i = \{x_t^{ip}, p \in S^i\}$, is a mixed strategy of player i.
The model will be in the framework of an N-person game but we consider the dynamics for one player, without hypotheses on the behavior of the others.
Hence, from the point of view of this player, he is facing a (measurable) vector outcome process $\{U_t, t \geq 0\}$, with values in the cube $\mathscr{U} = [-1,1]^K$ where K is his move's set.
U_t^k is the payoff at time t if k is the choice at that time.
The $\mathscr{U}$-replicator process (RP) is specified by the following equation on $\Delta(K)$:

$$\dot{x}_t^k = x_t^k[U_t^k - \langle x_t, U_t \rangle], \qquad k \in K. \tag{21}$$

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$$\dot{x}_t^k \; = \; x_t^k[U_t^k - \langle x_t, U_t \rangle], \qquad k \in K. \tag{21}$$

Recall the logit map L from $\mathbb{R}^K$ to $\Delta(K)$ defined by

$$L^k(V) = \frac{\exp V^k}{\sum_j \exp V^j}. \tag{22}$$

Let $\mathbf{br}$ denotes the (payoff based) best reply correspondence from $\mathbb{R}^K$ to $\Delta(K)$ defined by

$$\mathbf{br}(U) = \{x \in \Delta(K); \langle x, U\rangle = \max_{y \in \Delta(K)} \langle y, U\rangle\}$$

For $\varepsilon > 0$ small, $L(V/\varepsilon)$ is a smooth approximation of $\mathbf{br}(V)$ in the following sense:
Given $\eta > 0$, let $[\mathbf{br}]^\eta$ be the correspondence from $\mathbb{R}^K$ to Δ with graph being the η-neighborhood for the uniform norm of the graph of $\mathbf{br}$. Then the L map and the $\mathbf{br}$ correspondence are related as follows:

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Proposition

There exists a function η *from* $(0,\varepsilon_0)$ *to* $\mathbb{R}^+$, *with* $\eta(\varepsilon) \to 0$ *as* $\varepsilon \to 0$, *and such that for any* $U \in C$ *and* $\varepsilon_0 > \varepsilon > 0$

$$L(U/\varepsilon) \in [\mathbf{br}]^{\eta(\varepsilon)}(U).$$

Remark

$$L(U/\varepsilon) = \mathbf{br}^{\varepsilon}(U).$$

Hofbauer, Sorin and Viossat (2009), Sorin (2009)

Define the continuous exponential weight process (CEW) on $\Delta(K)$ by:

$$x_t = L\left(\int_0^t U_s ds\right).$$

Proposition

(CEW) *satisfies* (RP).

Proof

Since $x_t = L(\int_0^t U_s ds)$

$$\dot{x}_t^k = x_t^k U_t^k - x_t^k \sum_j \frac{U_t^j \exp \int_0^t U_v^j dv}{\sum_\ell \exp \int_0^t U_v^\ell dv}$$

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The link with the best reply correspondence is the following.

Proposition

CEW satisfies

$$x_t \in [\mathbf{br}]^{\delta(t)}(\overline{U}_t)$$

with $\delta(t) \to 0$ as $t \to \infty$.

Proof

Write

$$x_t = L(\int_0^t U_s ds) = L(t\,\overline{U}_t)$$

$$= L(U/\varepsilon) \in [\mathbf{br}]^{\eta(\varepsilon)}(U)$$

with $U = \overline{U}_t$ and $\varepsilon = 1/t$. ■

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with $U = \overline{U}_t$ and $\varepsilon = 1/t$. ∎

Consider now the time average process:

$$X_t = \frac{1}{t}\int_0^t x_s ds$$

Proposition

If x_t follows (CEW) then X_t satisfies

$$\dot{X}_t \in \frac{1}{t}([\mathbf{br}]^{\delta(t)}(\overline{U}_t) - X_t) \qquad (*)$$

with $\delta(t) \to 0$ as $t \to \infty$.

Corollary

In a two person game, if both players follow (RP) the time average process (X_t, Y_t) satisfies a perturbed version of (CFP).

External consistency

Recall that a procedure satisfies external consistency (external no-regret) if for each process $\{U_t\} \in \mathscr{U}$, it produces a process $x_t \in \Delta(K)$, such that:

$$\int_0^t [U_s^k - \langle x_s, U_s \rangle] ds \leq C_t = o(t), \quad \forall k \in K$$

Proposition

(CEW) satifies external consistency.
(RP) satifies external consistency.

Proof

1. Let $S_t^k = \int_0^t U_s^k \, ds$ and $W_t = \sum_\ell \exp S_t^\ell$, so that $x_t^k = \frac{\exp S_t^k}{W_t}$,
$\dot{W}_t = \sum_k U_t^k \, \exp S_t^k = \sum_k W_t x_t^k U_t^k = \langle x_t, U_t \rangle W_t$ and

$$W_t = W_0 \exp(\int_0^t \langle x_s, U_s \rangle ds)$$

but $W_t \geq \exp S_t^k$ implies $\int_0^t \langle x_s, U_s \rangle ds \geq \int_0^t U_s^k \, ds - Log\, W_0, \forall k \in K$.

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but $W_t \geq \exp S_t^k$ implies $\int_0^t \langle x_s, U_s \rangle ds \geq \int_0^t U_s^k \, ds - Log\, W_0, \forall k \in K$.

2. By integrating:

$$\frac{\dot{x}_t^k}{x_t^k} = [U_t^k - \langle x_t, U_t \rangle], \qquad k \in K. \tag{23}$$

one obtains, on the support of x_0:

$$\int_0^t [U_s^k - \langle x_s, U_s \rangle] ds = \int_0^t \frac{\dot{x}_s^k}{x_s^k} ds = \log(\frac{x_t^k}{x_0^k}) \leq -\log x_0^k.$$

■

General property of a smoothing process:

Let x_t maximize $\langle x, \int_0^t U_s ds\rangle - \varepsilon\rho(x)$ on X.
Claim:
$A_t = \langle x_t, \int_0^t U_s ds\rangle - \varepsilon\rho(x_t)$ and $B_t = \int_0^t \langle x_s, U_s\rangle ds$ satisfy:

$$\dot{A}_t = \langle x_t, U_t\rangle = \dot{B}_t$$

(enveloppe property) hence

$$\langle x, \int_0^t U_s ds\rangle \leq \int_0^t \langle x_s, U_s ds\rangle + \varepsilon(\rho(x) + 1), \quad \forall x \in X.$$

Back to a game framework this implies that if player 1 follows (RP) the set of accumulation points of the corrrelated distribution induced by the empirical process of moves will belong to his Hannan set:

$$H^1 = \{\theta \in \Delta(S); F^1(k, \theta^{-1}) \leq F^1(\theta), \forall k \in S^1\}.$$

The example due to Viossat (2007) of a game where the limit set for the replicator dynamics is disjoint from the unique correlated equilibrium shows that (RP) does not satisfy internal consistency.

Comments

We can now compare several processes in the spirit of (payoff based) fictitious play.

The original fictitious play process (I) is defined by

$$x_t \in \mathbf{br}(\bar{U}_t)$$

The corresponding time average satisfies (CFP).

With a smooth best reply process one has (II)

$$x_t = \mathbf{br}^{\varepsilon}(\bar{U}_t)$$

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Finally the replicator process (III) satisfies

$$x_t = \mathbf{br}^{1/t}(\bar{U}_t)$$

and the time average follows a time dependent perturbation of the fictitious play process.

While in (I), the process x_t follows exactly the best reply correspondence, the induced average X_t does not have good unilateral properties.

One the other hand for (II), X_t satisfies a weak form of external consistency, with an error term $\alpha(\varepsilon)$ vanishing with ε.

In contrast, (III) satisfies exact external consistency due to a both smooth and time dependent approximation of $\mathbf{br}$.

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Comparison to the discrete time procedure

Given a discrete process $\{X_m\}$ and a corresponding EW algorithm $\{p_m\}$ the aim is to get a bound on

$$\frac{1}{n}\sum_{m=1}^{n}(X_m^k - \langle p_m, X_m\rangle)$$

from an evaluation of

$$\frac{1}{T}\int_0^T (Y_s^k ds - \langle q_s, Y_s\rangle)ds$$

where Y_t is a continuous process constructed from X_m and q_t is the $CTEW$ algorithm associated to Y_t.

Proposition

Given a discrete time process $\{X_m\} \in [0,1]^K, m = 1, ..., n$, *there exists a measurable continuous time process* $\{Y_t\} \in [0,1]^K, t \in [0,T]$, *such that*

$$\frac{1}{n}\sum_{m=1}^{n} X_m^k = \frac{1}{T}\int_0^T Y_t^k dt \qquad \text{and}$$

$$\frac{1}{n}\sum_{m=1}^{n} \langle p_m, X_m \rangle e^{-\delta} \le \frac{1}{T}\int_0^T \langle q_t, Y_t \rangle dt \le \frac{1}{n}\sum_{m} \langle p_m, X_m \rangle e^{\delta}$$

where $\{p_m\}$ *is an* $EW(T/n)$ *associated to* $\{X_m\}$, q_t *is CTEW associated to* $\{Y_t\}$ *and* $\delta = T/n$.

Alternative proof of

$$\frac{1}{n}\sum_{m=1}^{n}(X_m^k - \langle p_m, X_m\rangle) \le Mn^{-1/2}$$

Given n, choose $T=\sqrt{n}$ so that:
- the bound in the continuous version is of the order $1/T = 1/\sqrt{n}$

$$\frac{1}{T}\int_0^T (Y_t^k - \langle q_t, Y_t\rangle)dt \le \frac{\log K}{\sqrt{n}}$$

- the error term with the discrete approximation of the order of $e^\delta - 1 \sim \delta = T/n = 1/\sqrt{n}$

$$\frac{1}{n}\sum_{m=1}^{n}\langle p_m, X_m\rangle \ge \frac{1}{T}(\int_0^T \langle q_t, Y_t\rangle dt) - L/\sqrt{n}$$

Extension Kwon and Mertikopoulos (2014) to several dynamics with time varying parameters.

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7. Continuous time dynamics

Hofbauer and Sigmund (1998) *Evolutionary Games and Population Dynamics*, Cambridge U.P.
Sandholm (2010) *Population Games and Evolutionary Dynamics*, M.I.T Press.

Main results:
elimination of dominated strategies
stability of pure strict equilibria
convergence to a profile from inside implies Nash
Lyapounov implies Nash

Typical property:
MAD = Positive correlation

Main classes
0-sum games
Strategic complementarities
Potential games
Dissipative games : congestion games

Several frameworks:
Population games
Congestion games
Extension to composite cases