Zero-sum Revision Games

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- Players have to prepare their actions in a pre-play phase preceding the payoff-relevant play in a one shot game,
- During the pre-play phase:
  - prepared actions are commonly observed.
  - Prepared actions can be change only at the bell of a Poisson clock.
- Only the last prepared action profile matters for the payoff.

Some examples

- Preopening in the stock market (Nasdaq, Euronext, Toronto SE, daily from 7a.m. to 9 a.m.)
- Interaction through internet servers (e-bay auctions).
- Preparatory meetings to negotiate the terms of a treaty.
- Armies deploying their troops on the ground
- ...

Component game: Zero-sum game

- 2 players.
- X_i: player i's finite set of actions.
- $U : X_1 \times X_2 \to \mathbb{R}$: player 1's payoff matrix (generic).
- $$BR_1^U(x) := \arg\max_{y_1} U(y_1, x_2) \ ; \ BR_2^U(x) := \arg\min_{y_2} U(x_1, y_2)$$
- Stackelberg payoff where 1 plays first:
$$S_1 = \max_{x_1 \in X_1} \min_{x_2 \in X_2} U(x_1, x_2)$$
- Stackelberg payoff where 2 plays first
$$S_2 = \min_{x_2 \in X_2} \max_{x_1 \in X_1} U(x_1, x_2)$$
- Value of the game: V
$$S_1 \leq V \leq S_2$$
with equality if pure Nash.

Revision game

- At $t = 0$, starting prepared action $x(0) \in X$ exogenous.
- Between time 0 and T, Poisson arrivals of revision times independent for each player (Asynchronous moves).
- Each player can change his prepared actions only at his revision times.
- At T players get their only payoff and this results from players playing, in the component game, their last prepared actions.

Revision game as a stochastic game

- Finite time horizon $[0, T]$.
- Game $\Gamma_{[\tau,T]}(x)$ with $\tau < T$ and $x \in X$.
- Time η is drawn from an exponential distribution with parameter λ.

- If $\eta + \tau > T$, then the game is over and players' payoff is $\{U(x), -U(x)\}$

- if $\eta + \tau < T$, then
  - With Pr $q \in (0, 1)$ player 1 chooses an action $y_1 \in X_1$ and the game $\Gamma_{[\tau+\eta,T]}(y_1, x_2)$ starts.
  - With Pr $1 - q$ player 2 chooses an action $y_2 \in X_2$ and the game $\Gamma_{[\tau+\eta,T]}(x_1, y_2)$ starts.
- Initial game: $\Gamma_{[0,T]}(x(0))$

Related literature

- **Non-zero sum revision game:** Kamda and Kandori 2009; Lovo and Tomala (2015).

  Calcagno, Kamada, Lovo and Sugaya (2014): In 2×2 conflicting interest games (generic Battle of the sexes),
  - the revision game equilibrium is unique;
  - the slow players has an advantage over the fast players;
  - revision game equilibrium payoff = component game Nash equilibrium payoff;
  - All action occurs at the beginning of the revision game.

- **Cheap talk games:** Farrell (1987), Rabin (1994), Aumann and Hart (2003), . . .
- **Switching cost games:** Lipman and Wang (2000) and Caruana and Einav (2008), . . .

Research question

What are we after?

- Under what conditions does a player prefer to play the revision game rather than the straight zero-sum game?
- Chracterization of equilibrium payoff.
- Characterization of equilibrium behavior.

Roda Map

1. Preliminaries
2. General results
3. 2×2 equilibrium chracterization

- Set of states: player who can revise and the resulting new profile of action
$$K = \{1, 2\} \times X$$
- History of past revision time and chosen actions
$$h_n = \{x, \tau_1, k_1, \dots \tau_n, k_n\} \in X \times ([0, T] \times K)^n$$
- Strategy: mapping histories and revision times into a (mixed) action
$$\sigma_i : \cup_{n \geq 0}(H_n \times [0, T]) \to \Delta X_i$$
- A Markov strategy is a measurable mapping
$$\sigma_i : X \times [0, T] \to \Delta X_i$$
- Expected payoff given σ:
$$u_\sigma(T, x) := \mathbb{E}_\sigma[U(x(T))|x(0) = x]$$
where $x(T)$ is the last prepared action profile at time T.

Theorem

(Lovo and Tomala (2015)) The revision game has a Markov perfect equilibrium. With

- t *is the remaining time.*
- $u(x, t)$*: equilibrium payoff of the game of length* t *with starting action profile* x*.*
- $u(t) := \{u(t, x)\}_{x \in X}$
- $\sigma_i(t, x) \in BR_i^{u(t)}(x_{-i})$

Remark that $u(t,x)$ is Lipschitz.

Let

$$u^+(t,x) := \max_{y_1 \in X_1} u(t, y_1, x_2) \quad ; \quad u^-(t,x) := \min_{y_2 \in X_2} u(t, x_1, y_2);$$
$$\lambda_1 := \lambda q \quad ; \quad \lambda_2 := \lambda(1-q)$$

Then

$$\begin{aligned} u(t,x) &= U(x)e^{-\lambda t} + \int_{s=0}^{t} e^{-\lambda(t-s)} \left(\lambda_1 u^+(s,x) + \lambda_2 u^-(s,x)\right) ds, \\ \frac{\partial u(t,x)}{\partial t} &= \lambda_1(u^+(t,x) - u(t,x)) + \lambda_2(u^-(t,x) - u(t,x)), \\ u(0,x) &= U(x). \end{aligned}$$

Proposition

1. *The revision game has a Markov perfect equilibrium in pure strategy.*
2. *The equilibrium payoff $u(t, x)$ is Lipschitz in t, U and is continuous in $(q, \lambda) \in (0, 1) \times (0, \infty)$.*
3. *The equilibrium payoff $u(t, x)$ is non-decreasing in q.*

Take a MPE and suppose that for some time t, $\sigma_i(t,x)$ is not pure. For this t replace $\sigma_i(t,x)$ by a pure action in $\sigma'_i(t,x) \in BR_i^{u(t)}(x)$. Observe that u^+ and u^- do not chaneg with σ or σ'. Hence

$$u(t,x) = u(x)e^{-\lambda t} + \int_{s=0}^{t} e^{-\lambda(t-s)} \left(\lambda_1 u^+(s,x) + \lambda_2 u^-(s,x)\right) ds$$

is the same under σ and σ'.
Zero sum structure is crucial.

- 1-Lipschitz in t:

$$|u(t,x) - u(t+\varepsilon,x)| \leq ||u(t)||(1 - e^{-\lambda\varepsilon})$$

- 1-Lipschitz in U: Take $U' \neq U$, then

$$|u(t,x) - u'(t,x)| \leq \max_{y \in X} |U(y) - U'(y)|$$

- Continuous in λ, take $\lambda' \neq \lambda$, then

$$u(t,x)|_{\lambda} = u\left(\frac{\lambda'}{\lambda}t', x\right)\Big|_{\lambda'}$$

- Continuous in q: Payoff continuously depends on the distribution of revision time that is continuous in q.

Let $q' < q$.

$$\frac{\partial u(t,x)}{\partial t} = \lambda(qu^{+}(t,x) + (1-q)u^{-}(t,x) - u(t,x))$$

If for some t, $u(t,x)|_{q'} = u(t,x)|_{q}$, then

$$\left.\frac{\partial u(t,x)}{\partial t}\right|_{q'} \leq \left.\frac{\partial u(t,x)}{\partial t}\right|_{q}$$

implying $u(\tau,x)|_{q} \leq u(\tau,x)|_{q'}$, for some $\varepsilon > 0$, and all $\tau \in (t, t+\varepsilon)$.
but

$$u(0,x)|_{q'} = u(0,x)|_{q} = U(x)$$

so it can never be that

$$u(\tau,x)|_{q'} > u(\tau,x)|_{q}$$

Consider a revision game where the starting action profile is x and let

$$\underline{R}(x) := \liminf_{t\to\infty} u(t,x) \text{ and } \overline{R}(x) := \limsup_{t\to\infty} u(t,x)$$

If $\underline{R}(x) = \overline{R}(x) = R(x)$ then we say that the *revision game value* is $R(x)$

Proposition

1. *Irrelevance of revision when V is achieved with purestrategies:*

$$S_1 \leq \underline{R}(x) \leq \overline{R}(x) \leq S_2$$

2. *Ergodicity:*

$$\underline{R}(x) = \overline{R}(x) = R, \forall x \in X$$

   for some constant C

3. *R is* 1*-Lipschitz in U, and continuous in* $(q, \lambda) \in (0, 1) \times (0, \infty)$*.*

4. $$\lim_{q \to 1} = S_2 \text{ and } \lim_{q \to 0} = S_1$$

2×2 games

Let $X_1 = X_2 = \{\alpha, \beta\}$.

	α	β
α	$U(\alpha, \alpha)$	$U(\alpha, \beta)$
β	$U(\beta, \alpha)$	$U(\beta, \beta)$

Then the component game where U is generic, implying $x \neq x' \Rightarrow U(x) \neq U(x')$.

- **Scenario DD**: Each player has a dominant action. For example:

$$\begin{array}{ccc} & \longleftarrow & \\ \uparrow & \begin{array}{|c|c|}\hline \color{blue}{0} & 2 \\ \hline -1 & 1 \\ \hline \end{array} & \uparrow \\ & \longleftarrow & \end{array}$$

- **Scenario DN**: One player, has a dominant action whereas the other player does not. For example:

$$\begin{array}{ccc} & \longleftarrow & \\ \uparrow & \begin{array}{|c|c|}\hline \color{blue}{0} & 2 \\ \hline -1 & -5 \\ \hline \end{array} & \uparrow \\ & \longrightarrow & \end{array}$$

- **Scenario NN**: No pure Nash Eq. For example:

$$\begin{array}{ccc} & \longleftarrow & \\ \downarrow & \begin{array}{|c|c|}\hline 0 & 2 \\ \hline 3 & -5 \\ \hline \end{array} & \uparrow \\ & \longrightarrow & \end{array}$$

Proposition

Suppose U is in scenario DD, and let $\hat{x}_i$ be player i's dominant action in the component game. Then for all t,

1. *$u(t)$ is in scenario DD*
2. $$\sigma_i,(t,x) = \hat{x}_i, \forall x \in X$$
3. $$R = V$$

Intuition:

- By continuity with respect to t each player prepares his dominant action when t is close to 0.
- If for for all $\tau > t$ the other player uses a fixed action no matter what you do, then you strictly prefer preparing your dominant action at t.

Algebraic:

Solve the ODE and verify that $BR_i^{u(t)}(x)$ does not depend on t.

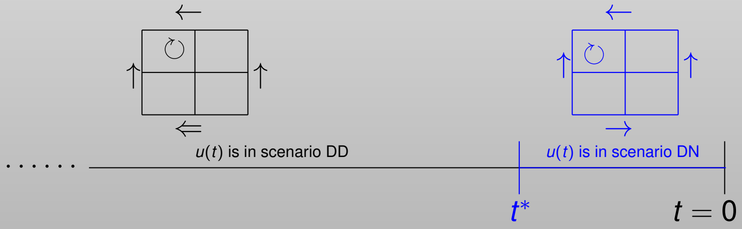
Proposition

Suppose U is in scenario DN, and $\hat{x}_1$ is player 1's dominant action in the component game. Then there is t^* finite such that

1. *For $t < t^*$:*
   - $u(t)$ *is in scenario DN*
   - $\sigma_1(t, x) = \hat{x}_1, \forall x$
   - $\sigma_2(t, x) = BR_2^U(x_1)$
2. *For $t \geq t^*$:*
   - $u(t)$ *is in scenario DD*
   - $\sigma_1(t, x) = \hat{x}_1, \forall x$
   - $\sigma_2(t, x) = BR_2^U(\hat{x}_1), \forall x$
3. $$R = V$$

	α	β
α	0	2
β	-1	-5

1. $t > t^*$: At the beginning of the revision phase players prepare the action forming the component game pure Nash equilibrium.
2. $t < t^*$: Once reached these actions they do not move.



Proposition

If U is in NN scenario, then there are $0 < t^{**} < t^*$, i^* and $x^*_{i^*}$ such that:

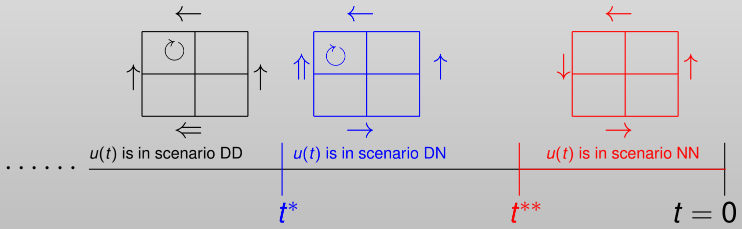
1. *For $t < t^{**}$:*
   - *$u(t)$ is in scenario NN*
   - $\sigma_i(t, x) = BR_i^U(x_{-i})$
2. *For $t^{**} < t < t^*$:*
   - *$u(t)$ is in scenario DN*
3. *For $t \geq t^*$:*
   - *$u(t)$ is in scenario DD*
4. *Generically*

$$R = u(t^{**}, x^*) \neq V$$

surplice and wrestle

0	0.2
0.3	−0.5

1. $t > t^{**}$: At the beginning of the revision phase players prepare a "surplace" action, that they keep until t^*
2. $t < t^*$: starting to t^* players actions cycle



If U is in NN scenario, then without loss of generality we have

	α	β
α	0	b
β	c	$b+c-1$

with

$$0 < b, c, < 1$$

Let $\sigma := |\lambda_1^2 - 6\lambda_1\lambda_2 + \lambda_2^2|^{\frac{1}{2}}$ and let $\hat{t}(A, B, \lambda_1, \lambda_2)$ be the smallest positive t such that

$$e^{-\frac{\lambda_1+\lambda_2}{2}t} + A\cos\left(\frac{\sigma t}{2}\right) + \frac{(\lambda_2 - \lambda_1)A + 2\lambda_2 B}{\sigma}\sin\left(\frac{\sigma t}{2}\right) = 0 \quad (1)$$

Set $\hat{t}(A, B, \lambda_1, \lambda_2)$ to infinity. Let

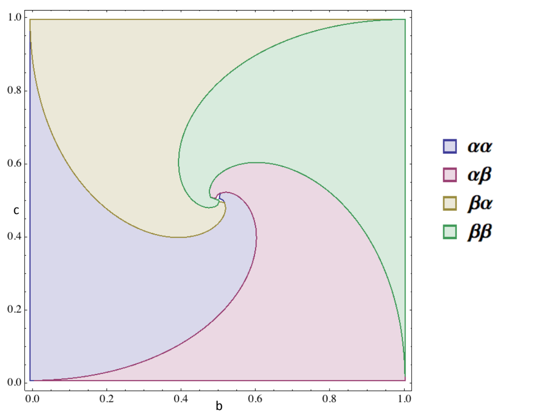
$$\begin{aligned} t_{\alpha,\alpha} &= \hat{t}(2c-1, 2b-1, \lambda_1, \lambda_1) \\ t_{\alpha,\beta} &= \hat{t}(2b-1, 1-2c, \lambda_2, \lambda_1) \\ t_{\beta,\alpha} &= \hat{t}(1-2b, 2c-1, \lambda_2, \lambda_1) \\ t_{\beta,\beta} &= \hat{t}(1-2c, 1-2b, \lambda_1, \lambda_1) \end{aligned}$$

Then then

$$\hat{x} = \arg \min_{y \in \{(\alpha,\alpha),(\alpha,\beta),(\beta,\alpha),(\beta,\beta)\}} t_y$$

$$t_* = t_{\hat{x}}$$

Scenario NN: sur-place actions for $q = 1/2$ and $0 < b, c < 1$



	α	β
α	0	b
β	c	$b+c-1$

Theorem

If $0 < b, c < 1$ and $q = 1/2$, then

- *The value of the game is $V = bc$*
- *the revision game value is:*

$$\begin{aligned} R &= \frac{1}{4}(2c + 2b - 1) + \frac{1}{2}(c + b - 1)(b - c)\sin(2\mu) \\ &+ \frac{1}{4}(2b - 1)(2c - 1)\cos(2\mu), \end{aligned}$$

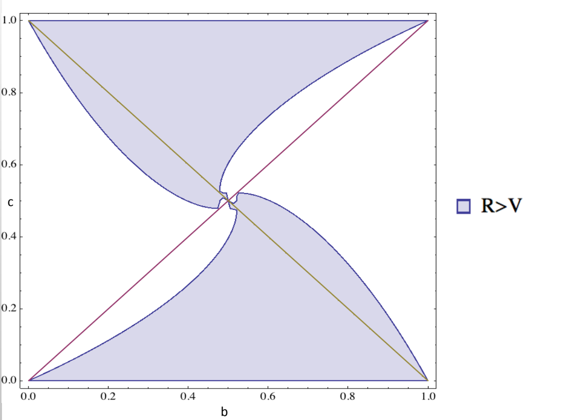
where μ is the smallest t in $\mathbb{R}_+$ satisfying:

$$e^{-t} = \max\{(1 - 2c)\cos(t) + (1 - 2b)\sin(t), (1 - 2b)\cos(t) - (1 - 2c)\sin(t),$$

$$-(1 - 2b)\cos(t) + (1 - 2c)\sin(t), -(1 - 2c)\cos(t) - (1 - 2b)\sin(t)\}.$$

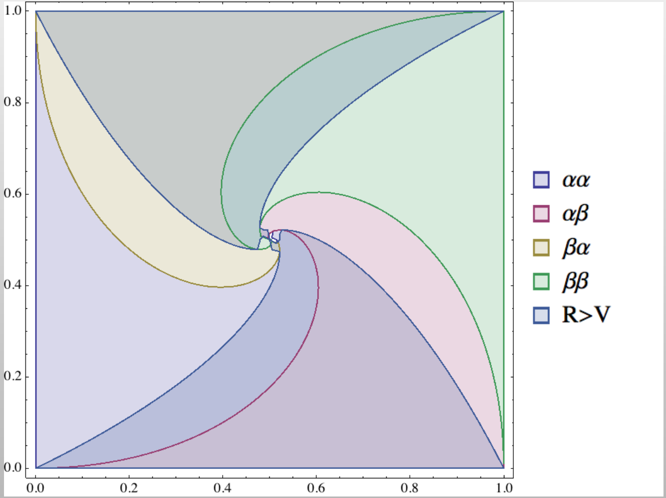
2 × 2 games

Scenario NN: R and V for $q = 1/2$ and $0 < b, c < 1$



	α	β
α	0	b
β	c	$b+c-1$

$q = 1/2$



Conclusion

- A zero-sum revision game always has a pure strategy equilibrium.
- When the component game Nash equilibrium is in pure, then players should be indifferent between paling the game with our without a (long) revision phase.
- When the component game Nash equilibrium is not pure, then
  - A player gain from being faster than the other player.
  - Generically the revision game value is different from the one-shot game value.
  - For 2×2 games, the unique equilibrium consists in players waiting on a sur-place action profile until the the deadline is close and then wrestle.