Dynamic Games with Almost Perfect Information

Wei He, University of Iowa

Yeneng Sun, National University of Singapore

December 2, 2015

Outline

Dynamic games with almost perfect information

In real life situations, most decisions are made in a dynamic context in the sense that multi-period decisions influence the final outcomes.

The players take turns to move in each period (alternating-move game) or move simultaneously in each period.

The information is assumed to be complete and the history is completely observed.

Zero-sum games

Zermelos Theorem (1913)

In any two-person zero-sum game of in which the players observe the history and payoffs, and move alternatingly with **finitely** many choices (like chess and go) in finitely many stages, either one of the players has a winning strategy, or both players can individually ensure a draw.

Literature: finite-action games

1. Selten (1965): finite action, finite horizon
2. Fudenberg and Levine (1983): finite action, infinite horizon

Dynamic games with general action spaces

How much do we know?

Continuous dynamic games with perfect information

For deterministic games with perfect information (**no Nature**), the existence of pure-strategy subgame-perfect equilibria was shown in Harris (1985), Hellwig and Leininger (1987), Borgers (1989, 1991) and Hellwig-Leininger-Reny-Robson (1990) with the model parameters being continuous in actions.

If nature is present, non-existence: Luttmer and Mariotti (2003)

A nonexistence example: Luttmer and Mariotti (2003)

- Five stages.
- The first stage: player 1 chooses $a_1 \in [0, 1]$.
- The second stage: player 2 chooses $a_2 \in [0, 1]$.
- In stage 3, Nature chooses x by randomizing uniformly over the interval $[-2 + a_1 + a_2, 2 - a_1 - a_2]$.
- After this, players 3 and 4 move sequentially.

Payoff

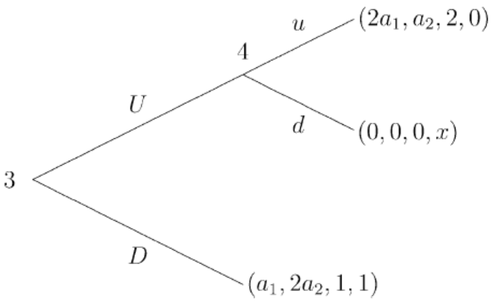

Figure : Payoff

Analysis

1. Let α and β denote the probabilities with which players 3 and 4 choose U and u, respectively.
2. Consider the subgame defined by $a_1 = a_2 = 1$. Nature's move is degenerate in this subgame: $x = 0$.
3. The set of all equilibrium expected payoffs for players 1 and 2 is given by

$$\{(1, 2-1.5\alpha) : \alpha \in [0, 1]\} \cup \{(2\beta, \beta) : \beta \in [0.5, 1]\} \tag{1}$$

Analysis

1. By choosing a_1 and a_2 with $a_1 + a_2 < 1$, players 1 and 2 can achieve the respective payoffs of $\frac{3}{2}a_1$ and $\frac{3}{2}a_2$.
2. Any selection from (1) yields a payoff of no more than 1 for at least one of the players 1 and 2.
3. Some player has no best response.

Continuous dynamic games with simultaneous moves

A nonexistence example of Harris, Reny and Robson (1995)
Players 1 and 2 move simultaneously in stage 1;
Players 3 and 4 move simultaneously in stage 2.
Player 1 has action space $[-1, 1]$, while players 2, 3 and 4 have action space $\{L, R\}$.

Harris, Reny and Robson (1995) showed the existence of subgame-perfect correlated equilibrium for continuous dynamic games with alternating or simultaneous moves.

Basic questions

Little is known about general dynamic games with alternating or simultaneous moves beyond the existence of subgame-perfect correlated equilibrium in the continuous case with public randomization!

1. Continuous dynamic games with simultaneous moves, no exogenous public randomization device: existence? upper hemicontinuity of equilibrium payoff set?
2. Continuous dynamics games with alternating moves, presence of Nature: existence of pure-strategy SPE? upper hemicontinuity of equilibrium payoff set?
3. Discontinuous dynamics games: can one drop the continuity requirement in the state variable?

Model

- The set of players: $I_0 = \{0, 1, \ldots, n\}$. The players in $I = \{1, \ldots, n\}$ are active and player 0 is Nature.
- All players move simultaneously in each period.
- Time is discrete, indexed by $t = 0, 1, 2, \ldots$.
- The set of starting points is a closed set $H_0 = X_0 \times S_0$, where X_0 is a compact metric space and S_0 is a Polish space.
- At stage $t \geq 1$, player i's action will be chosen from a Polish space X_{ti} for each $i \in I$.
- Nature's action is chosen from a Polish space S_t. Let $X^t = \prod_{0 \leq k \leq t} X_k$ and $S^t = \prod_{0 \leq k \leq t} S_k$.

Action correspondence

- Given $t \geq 0$, a history up to the stage t is a vector

  $$h_t = (x_0, s_0, x_1, s_1, \ldots, x_t, s_t) \in X^t \times S^t.$$

  The set of all such possible histories is denoted by H_t. For any $t \geq 0$, $H_t \subseteq X^t \times S^t$.
- For any $t \geq 1$ and $i \in I$, let $A_{ti} \colon H_{t-1} \to X_{ti}$ be the action correspondence
  - $A_{ti}(h_{t-1})$ is the set of available actions for player $i \in I$ given the history h_{t-1};
  - A_{ti} is nonempty, compact valued, and continuous.

State transition & Payoff

Law of motion/state transition:
For any $t \geq 1$, Nature's action is given by a continuous mapping f_{t0} from H_{t-1} to $\triangle(S_t)$ with the topology of weak convergence.

Payoff
For each $i \in I$, the payoff function u_i is a mapping from H_∞ to $\mathbb{R}_{++}$ which is bounded and continuous.

Strategy

1. For player $i \in I$, a strategy f_i is a sequence $\{f_{ti}\}_{t\geq 1}$ such that f_{ti} is a mapping from the history H_{t-1} to $\triangle(X_{ti})$ with

$$f_{ti}(A_{ti}(h_{t-1})|h_{t-1}) = 1$$

for all histories $h_{t-1} \in H_{t-1}$.

2. A strategy profile $f = \{f_i\}_{i\in I}$ is a combination of strategies of all active players.

SPE and continuous at infinity

A **subgame-perfect equilibrium** is a strategy profile f such that for all $i \in I$, $t \geq 0$, and all $h_t \in H_t$, player i cannot improve his payoff in the subgame h_t by a unilateral change in his strategy.

1. For any $T \geq 1$, let
$$w^T = \sup_{\substack{i \in I \\ (x,s) \in H_\infty \\ (\overline{x},\overline{s}) \in H_\infty \\ x^{T-1} = \overline{x}^{T-1} \\ s^{T-1} = \overline{s}^{T-1}}} |u_i(x,s) - u_i(\overline{x},\overline{s})|. \tag{2}$$
2. A dynamic game is said to be "continuous at infinity" if $w^T \to 0$ as $T \to \infty$.
3. All discounted repeated games or stochastic games satisfy this condition.

Existence

Definition (Atomless Transition)

*A dynamic game is said to satisfy the "**atomless transition**" condition if for each $t \geq 1$, the probability $f_{t0}(\cdot|h_{t-1})$ is atomless for all $h_{t-1} \in H_{t-1}$ (the cdf is continuous).*

Theorem

If a continuous dynamic game satisfies the atomless transition condition, then it possesses a subgame-perfect equilibrium.

Continuous stochastic games

Corollary

If a continuous stochastic game satisfies the atomless transition condition, then it possesses a subgame-perfect equilibrium.

1. Mertens and Parthasarathy (2003) proved the existence of a subgame-perfect equilibrium by assuming the state transition to be norm continuous.
2. In addition to the atomless transition condition, we only require the state transition to be weakly continuous.

Dynamic games with perfect information

Proposition

If a continuous dynamic game with perfect information has atomless transitions, then it possesses a pure-strategy subgame-perfect equilibrium.

Characterizing the equilibrium payoff set

Let $E_t(h_{t-1})$ be the set of subgame-perfect equilibrium payoffs in the subgame h_{t-1}.

Proposition

If a dynamic game satisfies the atomless transition condition, then E_t is nonempty and compact valued, and upper hemicontinuous.

The Atomless Reference Measure (ARM) condition

A dynamic game is said to satisfy the "atomless reference measure (ARM)" condition if for each $t \geq 1$,

1. the probability $f_{t0}(\cdot|h_{t-1})$ is absolutely continuous with respect to λ_t on S_t with the Radon-Nikodym derivative $\varphi_{t0}(h_{t-1}, s_t)$ for all $h_{t-1} \in H_{t-1}$;
2. the mapping φ_{t0} is Borel measurable and sectionally continuous on X^{t-1}, and integrably bounded in the sense that there is a λ_t-integrable function $\phi_t \colon S_t \to \mathbb{R}_+$ such that $\varphi_{t0}(h_{t-1}, s_t) \leq \phi_t(s_t)$ for any $h_{t-1} \in H_{t-1}$ and $s_t \in S_t$.

The SPE existence in discontinuous dynamic games

Replace the continuity conditions on the action correspondences and payoffs for both the state and action variables by measurability on states and sectional continuity on actions.

The same results still hold under the ARM condition!

Question: a natural economic example of a dynamic game that must be continuous in actions but **discontinuous in states**?

Backward induction

For simplicity, we only discuss a two-stage game with continuous payoff functions.

1. Backward induction: Given the payoff correspondence in the second stage, there exists a measurable selection which serves as the payoff function in the first stage such that a Nash equilibrium exists.
2. If the payoff correspondence has good properties (nonempty, convex, compact, measurable and sectionally upper hemicontinuous in actions), then this step still holds, extending a theorem of Simon and Zame (1990).

Forward induction and infinite horizon

1. Forward induction: Given any NE payoff function in the first stage, one needs to construct an action profile and a payoff function which is the SPE payoff function in the second stage.
2. This step is difficult since one needs to be careful about the measurability. We need a deep theorem of Mertens (2003).

Infinite horizon: need to handle various subtle measurability issues

Thank you!