The Characterization of the Limit Correlated Equilibrium Payoff Set with General Monitoring

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The Mediator in the Real Example

- AC-Treuhand AG
  - organizing meetings of the cartel members;
  - distributing agreed market shares;
  - calculating deviations;
  - collecting and verifying data;
  - acting as a modelator in case of tensions between cartel members;
  - reshaping the arrangement
- Trade Associations perform the role of a third-party facilitator.

A Big Picture

- Sugaya (2014) establishes the folk theorem with private monitoring if identifiability conditions are satisfied.
- The remaining question is what happens if a sufficient condition for the folk theorem is violated:
  - Discount factor is less than 1.
  - Identifiability conditions are not satisfied.

A Big Picture

- A general case is very hard to analyze…
- Special cases:
  - An upper bound with a fixed discount factor (but monitoring is not fixed): Sugaya and Wolitzky (2015).
  - A tight characterization of correlated equilibria with cheap talk with a fixed monitoring (but $\delta \to 1$).

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- Common discount factor $\delta < 1$.

Correlated Equilibrium

- We consider the correlated equilibrium in the repetition of this stage game. The reason is …

Correlated Equilibrium

- Possibly correlated private signals offer endogenous correlation device to the players in Nash equilibrium.
- What correlation is possible depends on monitoring structure and possible equilibrium strategy (<- this is what we want to characterize first of all!).

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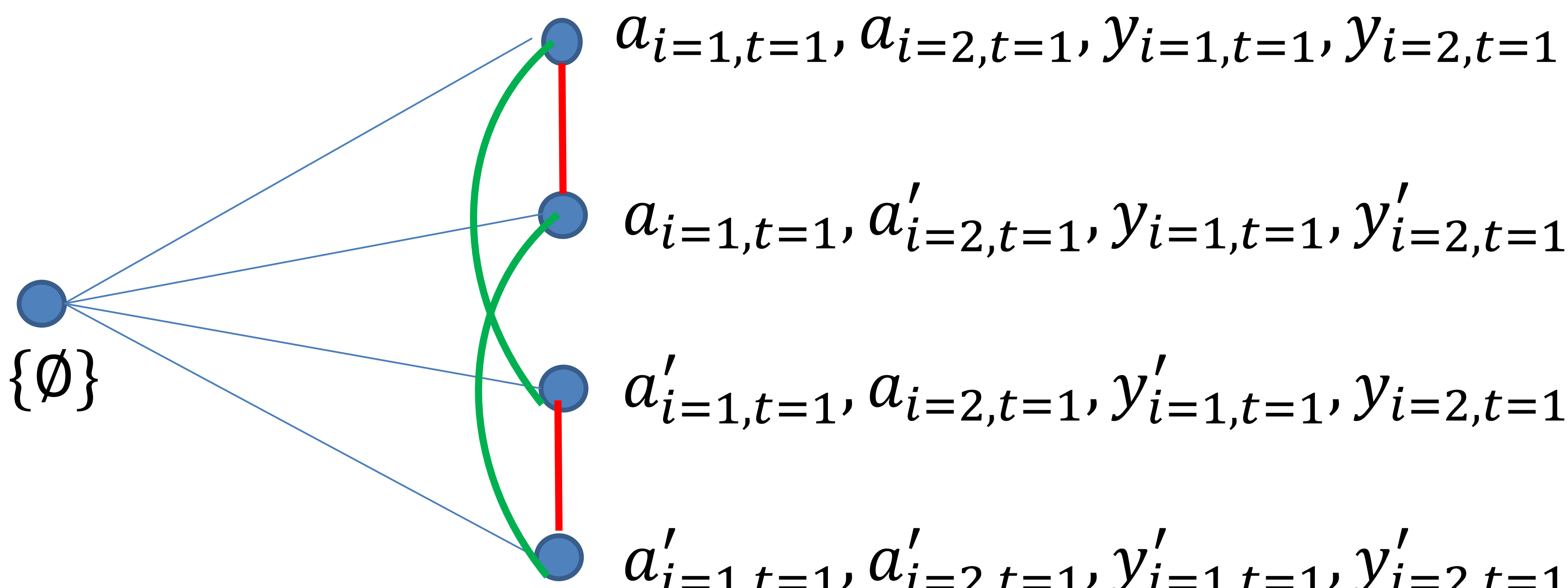
- What correlation is possible depends on monitoring structure and possible equilibrium strategy (<- this is what we want to characterize first of all!).
- We consider correlated equilibrium directly.

Difficulty: No Recursive Structure

- The continuation strategy profile from a history profile does not have to be a correlated equilibrium of the original game:



Main Result

The set of limit sequential correlated equilibria with cheap talk

$$\lim_{\delta \to 1} E^{\mathrm{corr}}(\delta)$$

|| if observable realized own payoffs

The set of limit sequential equilibria with a mediator $\lim_{\delta \to 1} E^{\mathrm{med}}(\delta)$

∪ if observable realized own payoffs

Characterizing a set Q

∪: always include

The set of sequential equilibria with a mediator $E^{\mathrm{med}}(\delta)$

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Timing of the Game with a Mediator

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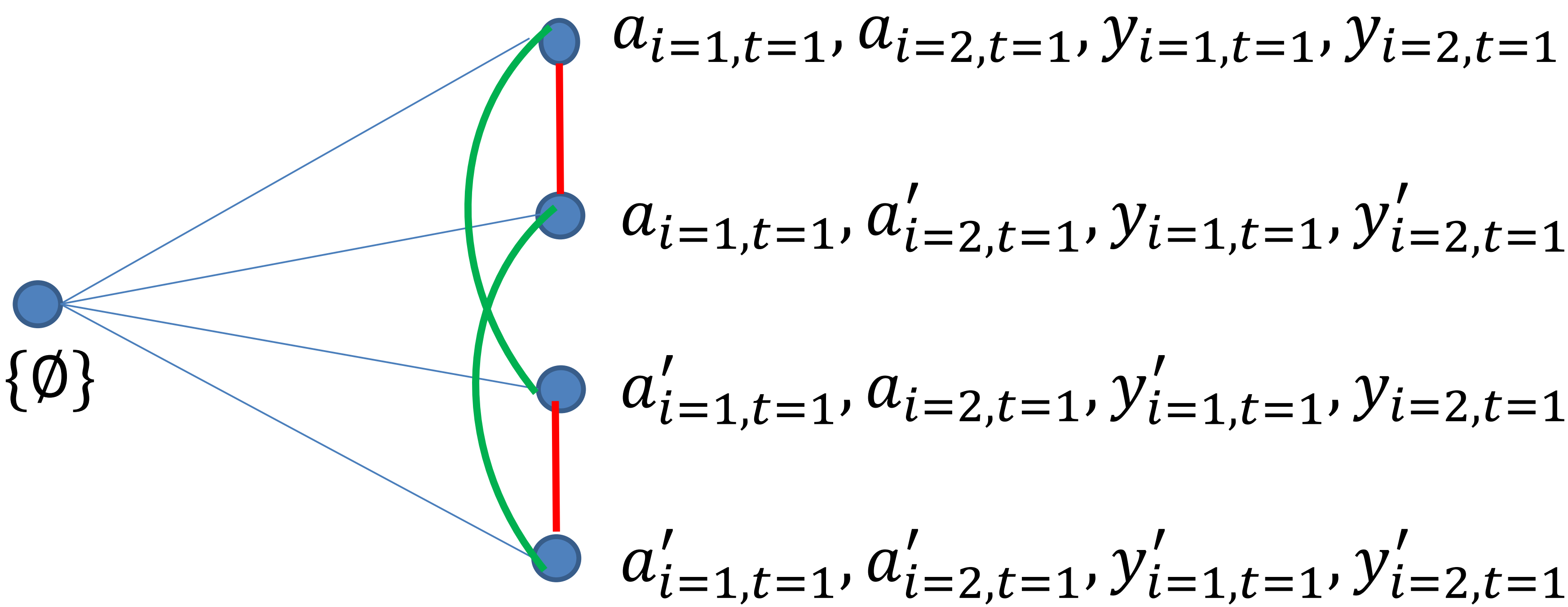
4. Player i reports $m_{i,t}$ ($= y_{i,t}$ on eqm path).

Ex Ante Recursive Structure



$\mu_1 \in \mathrm{M}_1$: Set of rec. dist. supportable in equilibrium in period 1.

$\mu_2 \in \mathrm{M}_2(a_{i=1,t=1}, a_{i=2,t=1}, y_{i=1,t=1}, y_{i=2,t=1})$: Set of rec. dist. supportable in equilibrium in period 2.

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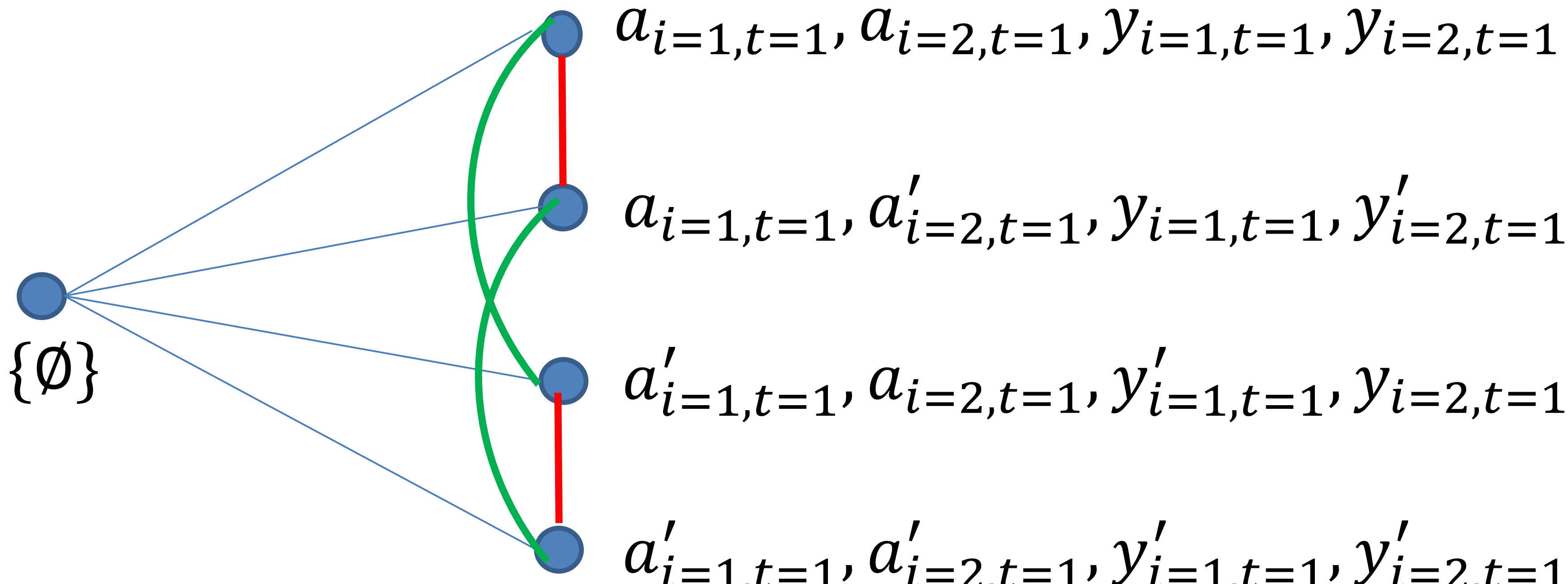

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$M_1 \neq M_2(a_{i=1,t=1}, a_{i=2,t=1}, y_{i=1,t=1}, y_{i=2,t=1})$

$\mu_2 \in \mathrm{M}_2(a_{i=1,t=1}, a_{i=2,t=1}, y_{i=1,t=1}, y_{i=2,t=1})$: Set of rec. dist. supportable in equilibrium in period 2.

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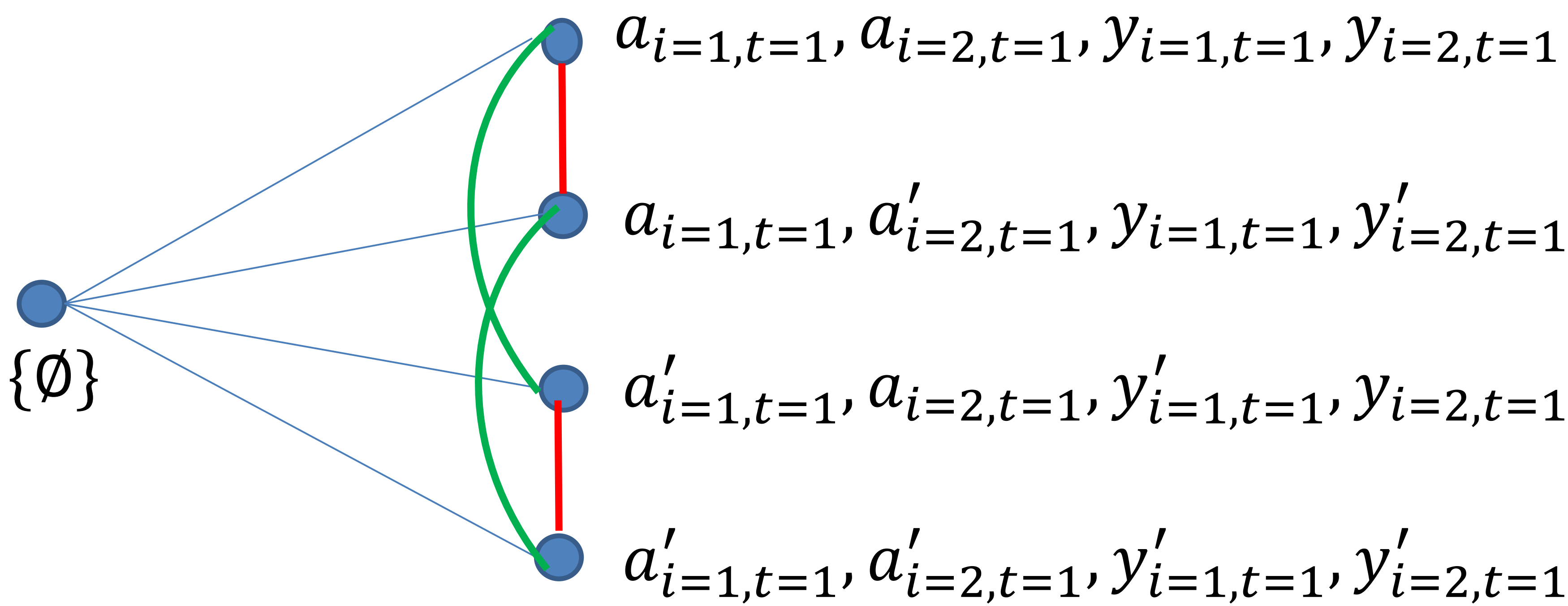

$\mu_1 \in \mathrm{M}_1$: Set of rec. dist. supportable in equilibrium in period 1.

$\mu_2 \in \mathrm{M}_2^{ex\ ante}$: μ_2 is an ex ante dist. of rec., calculated by using the dist of $\left(a_{i=1,t=1}, a_{i=2,t=1}, y_{i=1,t=1}, y_{i=2,t=1}\right)$.

Ex Ante Recursive Structure

- $\mathrm{M}_2^{ex\ ante} \subseteq \mathrm{M}_1$.
- Each player has more information in period 2 than in period 1.
- Incentive compatibility constraint is tightened in period 2.
- This inclusion is not true in Nash equilibrium: More endogenous correlation is available in period 2.

Ex Ante Recursive Structure

- $\mathrm{M}_2^{ex\ ante} \subseteq \mathrm{M}_1$.
- Each player has more information in period 1 than in period 2.
- Incentive compatibility constraint is tightened in period 2.
- $\mu \in \mathrm{M}_1$ should satisfy the following constraint:
- For each $i \in I$ and σ_i such that

$\Pr(m|r) = \Pr(m|\sigma_i, r)$ for all $r \in \mathrm{supp}(\mu)$ and $m \in Y$, we have

$$u_i(\sigma_i, \mu) \leq u_i(\mu).$$

all support-μ undetectable deviation is non profitable

Characterization

- Given $\cdots \subseteq \mathrm{M}_3^{ex\ ante} \subseteq \mathrm{M}_2^{ex\ ante} \subseteq \mathrm{M}_1$, we have

$$E^{\mathrm{med}}(\delta) \subseteq \mathcal{M}$$

with $\mathcal{M} \equiv \{u(\mu){:}\, \mu \in \Delta(A)$, all support$-\mu$ undetectable deviation is non profitable$\}$.

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With $\mathcal{M} \equiv \{u(\mu): \mu \in \Delta(A)$, all support$-\mu$ undetectable deviation is non profitable$\}$.

- If we calculate $\mathcal{M}$ with perfect monitoring, we get the set of feasible payoff set.
- Some constraint about punishment payoffs is missing.

Characterization of an Upper Bound

Theorem:

Define $Q \equiv \mathcal{M} \cap \mathcal{P}$. We have $E(\delta) \subset Q$ for each $\delta \leq 1$.

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A constraint about punishment payoff taken from Renault and Tomala (2004), which is introduced in the context of repeated games without discounting.

How to Calculate $\mathcal{M}$

- An Example: Two public signals $y \in \{g, b\}$

	C	D
C	$2,2$	$-1,3$
D	$3,-1$	$0,0$

Payoff

	C	D
C	$.5$	$.5$
D	$.5$	$.3$

Prob of $y = g$

- (C, C) is not in $\mathcal{M}$ but as soon as it is mixed with (C, D) and (D, C), it will be in $\mathcal{M}$.
- For each $i \in I$ and σ_i such that

$$\Pr(m|r) = \Pr(m|\sigma_i, r) \text{ for all } r \in \operatorname{supp}(\mu) \text{ and } m \in Y,$$

we have

$$u_i(\sigma_i, \mu) \leq u_i(\mu).$$

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Anti Folk Theorem

Why $\lim_{\delta \to 1} E^{\text{med}}(\delta)$ is smaller than Q if the realized own payoffs are not observable?

$$y_i^1 \in \{g_i^1, b_i^1\}, y_i^2 \in \{g_i^2, b_i^2\}$$

	U	M	D	Ud	T	B
U	6,6	-10,8	-10,4	-10,6	0,0	0,0
M	8,-10	0,0	0,0	0,0	0,0	0,0
D	4,-10	0,0	0,0	0,0	0,0	0,0
Ud	6,-10	0,0	0,0	0,0	1,0	1,0
T	0,0	0,0	0,0	0,1	8,2	0,0
B	0,0	0,0	0,0	0,1	0,0	2,8

$$q(y|a) = \prod_{i \in I} q_i^1(y_i^1|a_j) \prod_{i \in I} q_i^2(y_i^2|a_i, y_j^1)$$

$$q_i^1(y_i^1 = g_i^1|a_j) = \begin{cases} \frac{3}{4} & \text{if } a_j = D \\ \frac{1}{4} & \text{if } a_j = M \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

$$q_i^2(y_i^2 = g_i^2|a_i, y_j^1) = \begin{cases} \frac{3}{4} & \text{if } a_i = U_d \text{ and } y_j^1 = g_j^1 \\ \frac{1}{4} & \text{if} a_i = U_d \text{ and } y_j^1 = b_j^1 \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

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D	4,-10	0,0	0,0	0,0	0,0	0,0
Ud	6,-10	0,0	0,0	0,0	1,0	1,0
T	0,0	0,0	0,0	0,1	8,2	0,0
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efficient

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Signal about the other player's action

$$q_i^2(y_i^2 = g_i^2|a_i, y_j^1) = \begin{cases} \frac{}{4} & \text{if } a_i = U_d \text{ and } y_j = g_j \\ \frac{1}{4} & \text{if} a_i = U_d \text{ and } y_j^1 = b_j^1 \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

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Static Nash

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(U,U) ∈ $\mathcal{M}$

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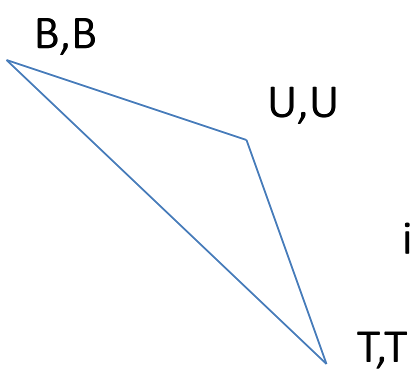

is included in Q.

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(U,U) becomes common knowledge when it is recommended since otherwise Ud is better.

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The continuation play after (U,U) is belief-free
since otherwise Ud is better
⇒(6,6) is not supportable.

A Sufficient Condition

- Problem: deviation to Ud from U is profitable and non detectable, but hurts the opponent.
- If the opponent can observe her realized own payoff, such deviation must be statistically detected by the different distribution of the realized payoff.
- In general, with observable realized own payoff, $Q \subset \lim_{\delta \to 1} E^{\mathrm{med}}(\delta)$.

Observable Own Payoffs

- The realized payoff is $u_i(a_i, y_i)$.
- The ex ante payoff is $u_i(a) = \sum_y q(y|a)\, u_i(a_i, y_i)$.

Observable Own Payoffs

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- The ex ante payoff is $u_i(a) = \sum_y q(y|a)\, u_i(a_i, y_i)$.
- For example,
  - $y_i \in \{g_i, b_i\}$,
  - $q_i(g_i|a) = .6$ if $a_j = C_j$ and $.4$ if $a_j = D_j$,
  - $u_i(a_i, y_i) = -7 + 15 \times 1_{\{y_i = g_i\}} + 1_{\{a_i = D_i\}}$.
  - Ex ante payoff matrix is

	C_2	D_2
C_1	$2,2$	$-1,3$
D_1	$3,-1$	$0,0$

Observable Own Payoffs

Theorem:

If Q has full dimension and each player observes her own realized payoff, then we have

$$\lim_{\delta \to 1} E(\delta) = Q.$$

Intuition

- Player i's undetectable deviation from μ can be problematic because
  - (i) it may affect the other players' payoffs;
  - (ii) it may affect how player i monitors the other players.

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  - (ii) it may affect how player i monitors the other players.
- (i) observable realized own payoffs tell us that all the undetectable deviations are "harmless."
- (ii) the undetectable deviation gives the same distribution of the messages regardless of the other players' actions: works equally effective as monitoring.

Intuition

- Perturb μ so that the recommendation has a full support: $\mu^{\text{full}} = (1-\varepsilon)\mu + \varepsilon \sum_{a \in A} \frac{a}{|A|}$.
- σ_i is support-μ^{full} undetectable:

$\Pr(m|r) = \Pr(m|\sigma_i, r)$ for all $r \in \operatorname{supp}(\mu^{\text{full}}) = A$ and $m \in Y$.

Main Result

The set of limit sequential correlated equilibria with cheap talk

$$\lim_{\delta \to 1} E^{\text{corr}}(\delta)$$

|| if observable realized own payoffs

The set of limit sequential equilibria with a mediator $\lim_{\delta \to 1} E^{\text{med}}(\delta)$

∪ if observable realized own payoffs

Characterizing a set Q

∪: always include

The set of sequential equilibria with a mediator $E^{\text{med}}(\delta)$

∪: always include

The set of sequential correlated equilibria with cheap talk $E^{\text{corr}}(\delta)$

∪: always include

The set of sequential equilibria $E(\delta)$

No mediator

- Idea: initial correlation contains the information of history-contingent recommendations.
- Cryptography:
  - The information is encoded so that the players do not know future recommendations.
  - Cheap talk communication decodes the recommendation of the next period.
  - A lie in cheap talk creates a miscoordination in future: leading to minimaxing the deviator (the other players do not realize the deviation).

Conclusion

- Ultimate goal: Characterize an equilibrium payoff set with general discount factor and general monitoring.
- Toward this goal, we explore the tractable structure of the correlated equilibrium, and obtain a tight characterization of correlated equilibria as $\delta \to 1$ when realized payoffs are observable.
- More work to be done…

Appendix

Similar Deviations

- For a subset of players $J \subset N$, the set of "similar deviations"$SD(J)$ is defined as follows:
- $SD(J)$ is the set of deviations such that the mediator cannot distinguish who in J is more likely to be guilty:

Similar Deviations

- For a subset of players $J \subset N$, the set of "similar deviations"$SD(J)$ is defined as follows:
- $SD(J)$ is the set of deviations such that the mediator cannot distinguish who in J is more likely to be guilty:

$$SD(J) \equiv \left\{ (\sigma_i)_{i \in J} : \begin{array}{c} \forall i \in J, \forall j \in J \\ \Pr(m|\sigma_i, r) = \Pr(m|\sigma_j, r) \\ \text{for all } r \in A, m \in Y \end{array} \right\}.$$

Most Severe Punishment

- For λ with $\lambda_i \leq 0$ for each $i \in I$, we define the "most severe punishment payoff" as

$$l(\lambda) \equiv \min_{\tau \in \Delta(A)} \max_{\sigma \in SD(\mathrm{supp}(\lambda))} \sum_{r \in A} \tau(r) \sum_{i \in N} |\lambda_i| \, u_i(r, \sigma_i).$$

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- That is, $\lambda \cdot v \leq -l(\lambda)$.

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subject that
- For each $i \in I$ and σ_i such that
$$\Pr(m|r) = \Pr(m|\sigma_i, r) \text{ for all } r \in A \text{ and } m \in Y,$$
we have

We had $r \in \text{supp}(\mu)$ here.

$$u_i(\sigma_i, \mu) \leq u_i(\mu).$$

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	L	R	L'	R'
U	2,2	0,3	0,2	0,3
D	0,0	0,0	0,1	0,1

- We show that $u(U,L) \in E(1)$ but $u(U,L) \notin \cup_{\delta<1} E(\delta)$.

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- Player 1 has $y_1 \in \{l, r\}$, which distinguishes whether player 2 takes "L or L'" or "R or R'" only if player 1 takes D:

  $q(l|a_1, a_2) = 1$ if $a_1 = D$ and $a_2 \in \{R, R'\}$

  $q(l|a_1, a_2) = 0$ otherwise.

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- If player 2 does not take L unless $a_1 = D$ with a positive prob.
- If $a_1 = D$ with a positive prob, player 2 prefers L' to L.

Sequential Rationality of τ

- Intuitively, τ is used to punish players after reports statistically indicate a deviation.
- Do we need to require τ is sequentially rational?

$$l(\lambda) \equiv \min_{\tau \in \Delta(A)} \max_{\sigma \in SD(\mathrm{supp}(\lambda))} \sum_{r \in A} \tau(r) \sum_{i \in N} |\lambda_i| \, u_i(r, \sigma_i).$$

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- Coming up with a “proper” restriction is hard:
  - Some players may not realize that there was a deviation.
  - It also depends on whether the mediator can tremble when we construct a consistent belief system: which one is more appropriate?

Example 1: Some Players Do Not Realize Deviations

- Four-player game:

	l	r
U	$2,2,1$	$2,2,1$
D	$2,2,1$	$2,2,1$

L

	l	r
U	$1,4,1$	$0,2,-1$
D	$2,0,-1$	$2,0,-1$

R

- Players 1-3 do not observe anything.
- Player 4 is a dummy player who can monitor a perfectly.

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R

- Player 3's equilibrium payoff should be no less than 1.
- As long as $v_3 \geq 1$, player 1 can guarantee the payoff of 2 by taking D.
- Nonetheless, we can support (U, l, R).

Example 2: Trembles of the Mediator Matters

- Two-player game:

	L	R	L'	R'
U	5,5	0,6	5,5	0,6
M	1,0	1,0	1,1	1,1
D	1,6	1,6	2,6	2,6

- Player 1 has two signals $\{l, r\}$: $q_1(l|a) = 1$ if $a_2 \in \{L, L'\}$; $q_1(r|a) = 1$ if $a_2 \in \{R, R'\}$.
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- Player 2 does not observe anything.
- v with $v_1 + v_2 \geq 10$
  - cannot be supported if the mediator cannot tremble in the definition of sequential equilibrium.
  - can be supported if the mediator can.

Dispensability of Mediation

- If there exists at least five players, then we can replace mediation with private communication among players.

Dispensability of Mediation

- If there exists at least five players, then we can replace mediation with private communication among players.
- Why five?
  - To keep the result of correlation about a_{-i} secret to player i, we need to exclude player i from some step of the communication.
  - We need at least three players involved in each step of communication so that we can use the majority rule to identify a liar (if any).
  - To create a correlation between r_i and r_j, we need another player to “relate” r_i and r_j.

The Proof of $\lim_{\delta \to 1} E(\delta) = Q$

- We explain how to approximately support $v \in \operatorname{argmax}_{v' \in Q} \lambda \cdot v'$ for $\lambda = (1, \ldots, 1)$.

Easy Case

- Two-player prisoners' dilemma.
- $v \approx \left(u_i(C,C)\right)_{i \in I}$.
- For each player, there are two signals $Y_i = \{g_i, b_i\}$.
- g_i indicates more cooperation:

$$q_i\left(\, g_i \mid a_i, C_j \,\right) > q_i\left(\, g_i \mid a_i, D_j \,\right) \text{ for all } a_i.$$

- That is, individual full rank holds and $|Y_i| = |A_{-i}|$.

Modified LR

- For a small $\rho > 0$, with $\mu = (1-\rho)(C,C) + \rho \sum_{a \in A} \frac{a}{|A|}$, and $x(r,y)$ such that

1. Promise Keeping:

$$v = u(\mu) + \mathbb{E}[x(r,y)|\mu].$$

2. Strict incentive compatibility: For each i and non-faithful σ_i,

$$\mathbb{E}[u_i(r) + x_i(r,y)|\mu]$$
$$> \mathbb{E}[u_i(a_i, r_{-i}) + x_i(r, m_i, y_{-i})|\sigma_i, \mu].$$

3. Ex-ante self generation:

$$\mathbb{E}[\lambda \cdot x(r,y)|\mu] \leq 0.$$

The Proof of $\lim_{\delta \to 1} E(\delta) = Q$

- We see the repeated game as the repetition of T-period review phases.

$$\underbrace{\text{Period } 1 \quad \dots \quad \text{Period } T}_{\text{First Review Phase}} \; \underbrace{\text{Period } T+1 \;\; \dots \; \text{Period } 2T}_{\text{Second Review Phase}} \; \underbrace{\text{Period } 2T+1 \;\; \dots \; \text{Period } 3T}_{\text{Third Review Phase}} \; \dots$$

Review Phase

- We see the repeated game as the repetition of T-period review phases.

$$\underbrace{\underbrace{\text{Period } 1 \ \ \dots \ \ \text{Period } T}_{\text{First Review Phase}}}_{\text{Current Phase}} \ \underbrace{\underbrace{\text{Period } T+1 \ \ \dots \ \text{Period } 2T}_{\text{Second Review Phase}} \ \underbrace{\text{Period } 2T+1 \ \ \dots \ \text{Period } 3T}_{\text{Third Review Phase}} \ \dots}_{\text{Continuation Play}}$$

Review Phase

Period 1 … Period T

Continuation Play

The mediator recommends r_t according to μ *i.i.d.* across periods.

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Player i yields instantaneous utilities $\sum_{t=1}^{T} \delta^{t-1} u_i(a_t)$

Continuation Play

Review Phase

Period 1 … Period T Continuation Play

$$\underbrace{\hphantom{\text{Period 1 … Period } T}}$$

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Continuation Play

The mediator switches to a punishment phase with a probability contingent on $(r_t, m_t)_{t=1}^{T}$

Review Phase

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Player i yields instantaneous utilities $\sum_{t=1}^{T} \delta^{t-1} u_i(a_t)$

The mediator observes the history $(r_t, m_t)_{t=1}^{T}$.

Continuation Play

The mediator switches to a punishment phase with a probability contingent on $(r_t, m_t)_{t=1}^{T}$ to implement the decrease in player i's continuation payoff $x_i((r_t, m_t)_{t=1}^{T}) \leq 0$.

Review Phase

Period 1 … Period T Continuation Play

The mediator recommends r_t according to μ

i.i.d. across periods.

Player i maxmizes

$$\mathbb{E}[\sum_{t=1}^{T} \delta^{t-1} u_i(a_t) + \delta^T x_i((r_t, m_t)_{t=1}^{T})].$$

Review Phase

Period 1 … Period T Continuation Play

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Player i maxmizes

$$\mathbb{E}[\sum_{t=1}^{T} u_i(a_t) + x_i((r_t, m_t)_{t=1}^{T})]$$

(Player i's incentive is strict)

Sufficient Condition

- For each $\epsilon > 0$, find x such that

1. Promise Keeping:

$$\lambda \cdot \frac{1}{T} \mathbb{E}[\sum_{t=1}^{T} u(r_t) + x((r_t, m_t)_{t=1}^{T})] \geq \lambda \cdot v - \epsilon.$$

2. Incentive compatibility: For each i, for any strategy σ_i,

$$\mathbb{E}[\sum_{t=1}^{T} u_i(r) + x_i((r_t, m_t)_{t=1}^{T})]$$
$$\geq \mathbb{E}[\sum_{t=1}^{T} u_i(r) + x_i((r_t, m_t)_{t=1}^{T}) | \sigma_i].$$

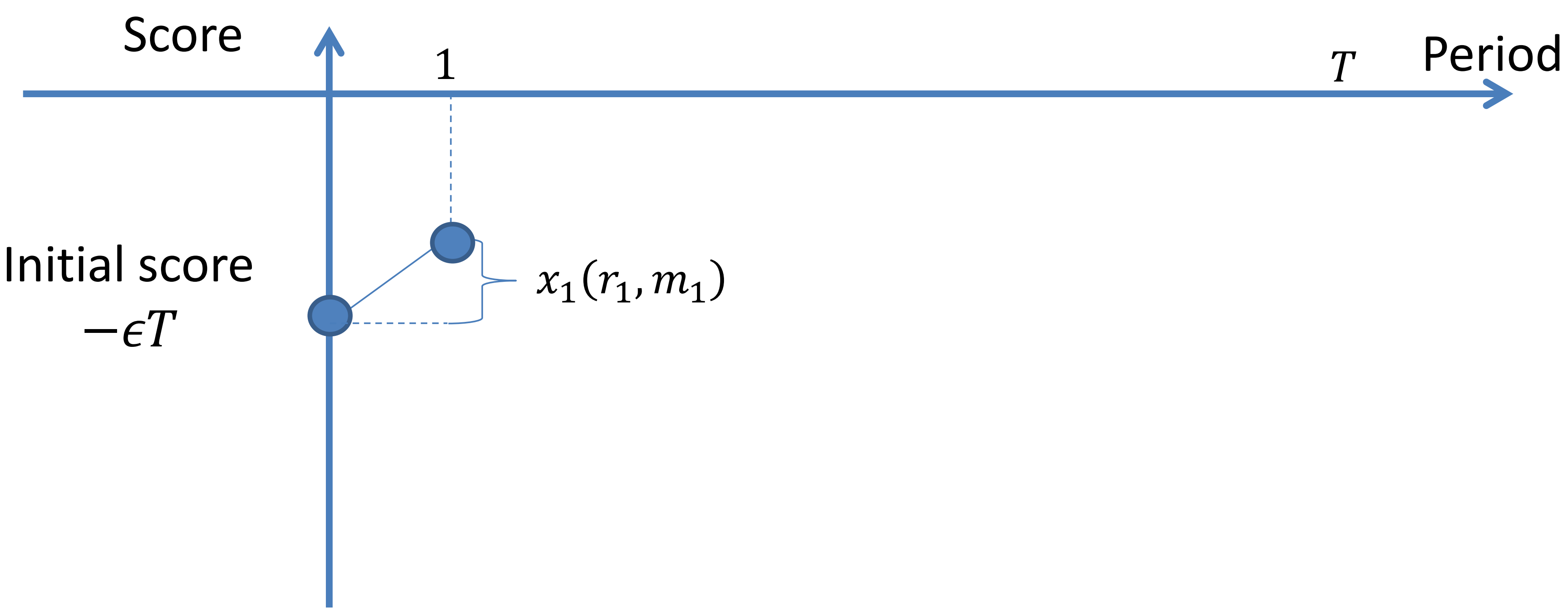
3. Self generation:

$$\lambda \cdot x((r_t, m_t)_{t=1}^{T}) \leq 0 \text{ for all } (r_t, m_t)_{t=1}^{T}.$$

Information Aggregation

- Pool $(r_t, m_t)_{t=1}^T$ during a block, and create the score

$$score_i = \sum_{t=1}^T x_i(r_t, m_t) - \epsilon T.$$



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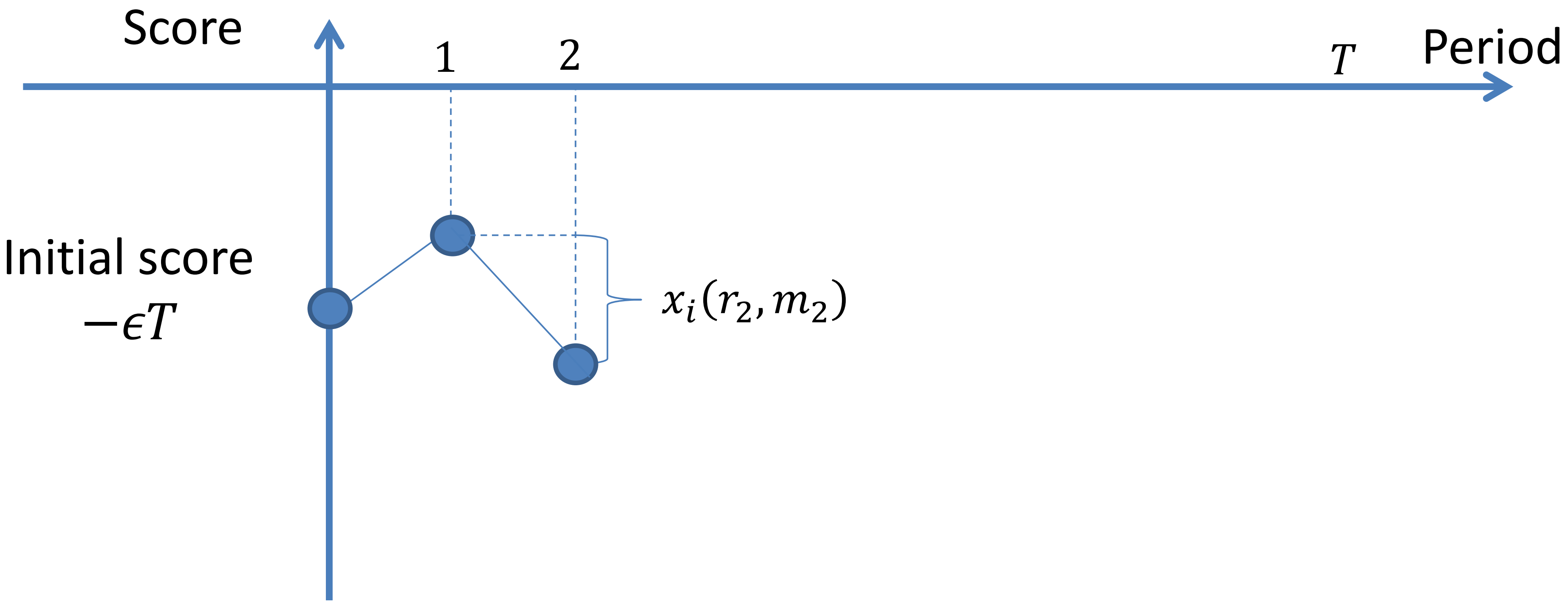

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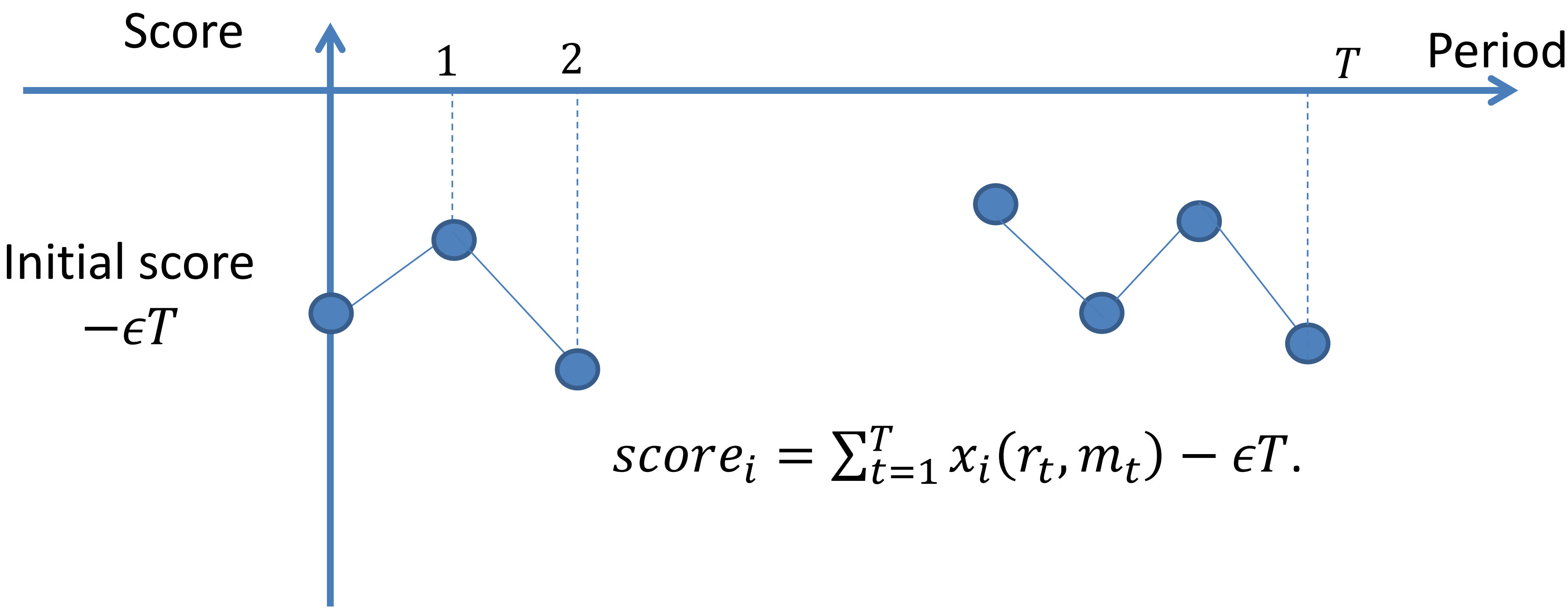


Law of Large Numbers

- Pool $(r_t, m_t)_{t=1}^T$ during a block, and create the score

$$score_i = \sum_{t=1}^T x_i(r_t, m_t) - \epsilon T.$$

- By LLN, we have $score_i \leq 0$ after most of the histories.
- If we could use this $score_i$ directly, then we would be done.
- But we have to deal with erroneous histories with $score_i > 0$…

Irregular Rec.

- The mediator classifies her history $(r_t, m_t)_{t=1}^{T}$ as follows:

1. Irregular rec: The frequency of periods with $r_t = a$ is slightly far from $\mu(a)$ for some $a \in A$.
2. Regular rec: Otherwise.

Adjustment of $x((r_t, m_t)_{t=1}^T)$

- The mediator changes $x_i((r_t, m_t)_{t=1}^T)$ as follows:

1. If irregular rec is the case, then, for each $i \in I$,

   $$x_i((r_t, m_t)_{t=1}^T) = \sum_{t=1}^T (x_i(r_t, m_t) - X)$$

   with large X so that

   $\lambda \cdot x((r_t, m_t)_{t=1}^T) \leq 0$ after each history.

2. If regular rec is the case, then, for each $i \in I$,

   $$x_i((r_t, m_t)_{t=1}^T) = \min\{\sum_{t=1}^T x_i(r_t, m_t) - \epsilon T, 0\}.$$

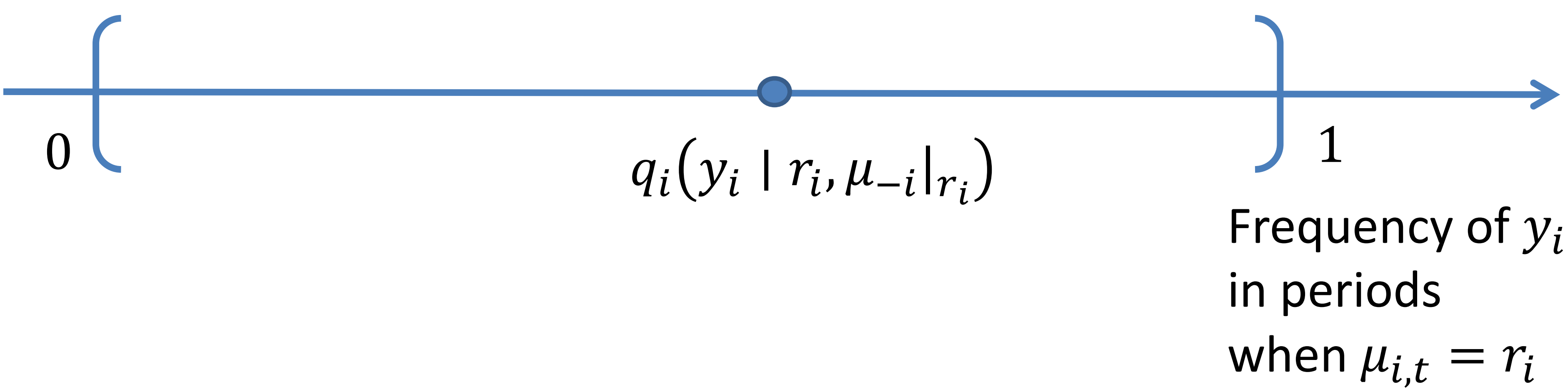
Adjustment of $x((r_t, m_t)_{t=1}^T)$

- Whether the recommendation is regular or irregular is out of player i's control: No issue of incentive.
- We need to make sure that taking minimum in
$$\min\{\sum_{t=1}^T x_i(r_t, m_t) - \epsilon T, 0\}.$$
after regular rec does not affect player i's incentive.
- We will show that, whenever player i believes that $\sum_{t=1}^T x_i(r_t, m_t) - \epsilon T$ may be positive, player i believes that irregular rec is the case.

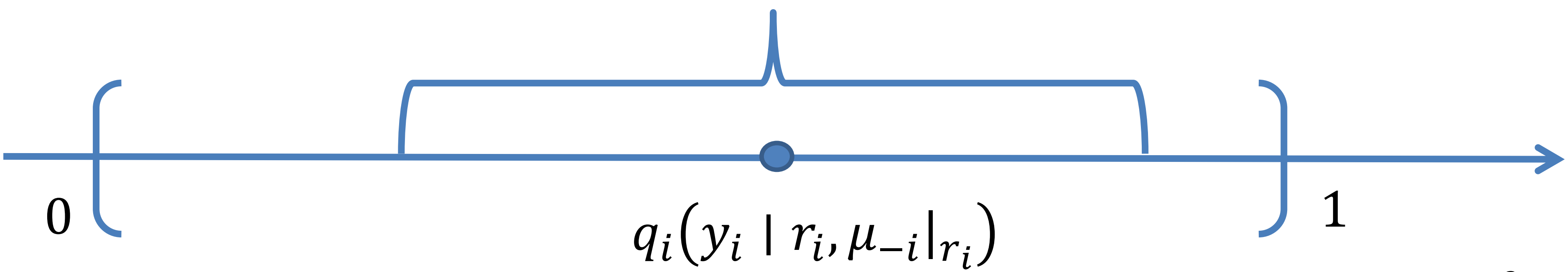
Classify Player i's History

- To verify player i's incentive, we classify her history into the following two categories:

1. If player i's recommendation is irregular, she knows that irregular rec is the case: Incentive OK.
2. If player i's recommendation is regular, …


0
$q_i(y_i \mid r_i, \mu_{-i}|_{r_i})$
1
Frequency of y_i in periods when $\mu_{i,t} = r_i$

Regions where player i believes that
the score is non positive
with probability $1 - \exp(-T)$,
if player i's signal observation is in this region
given each recommendation r_i.



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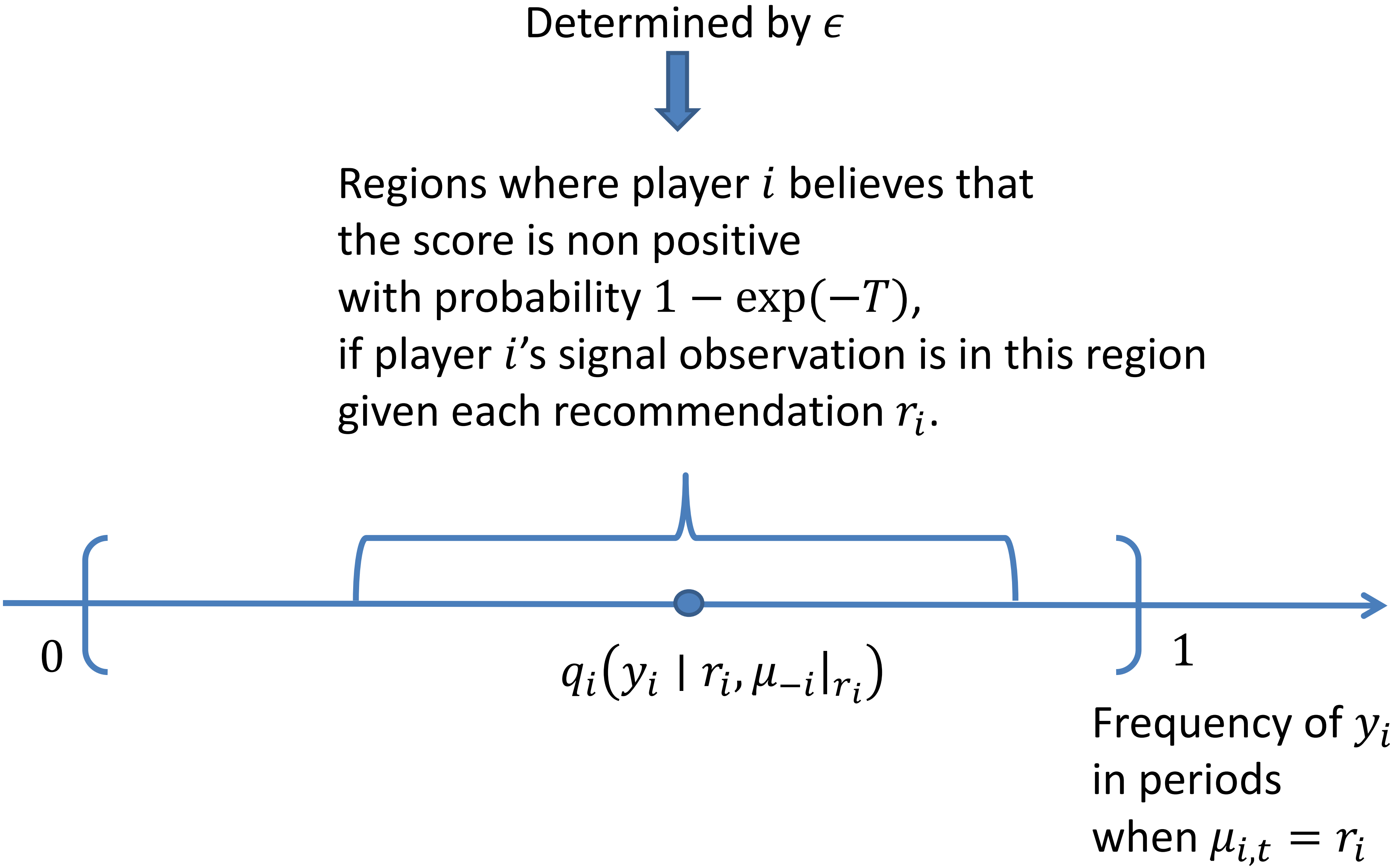
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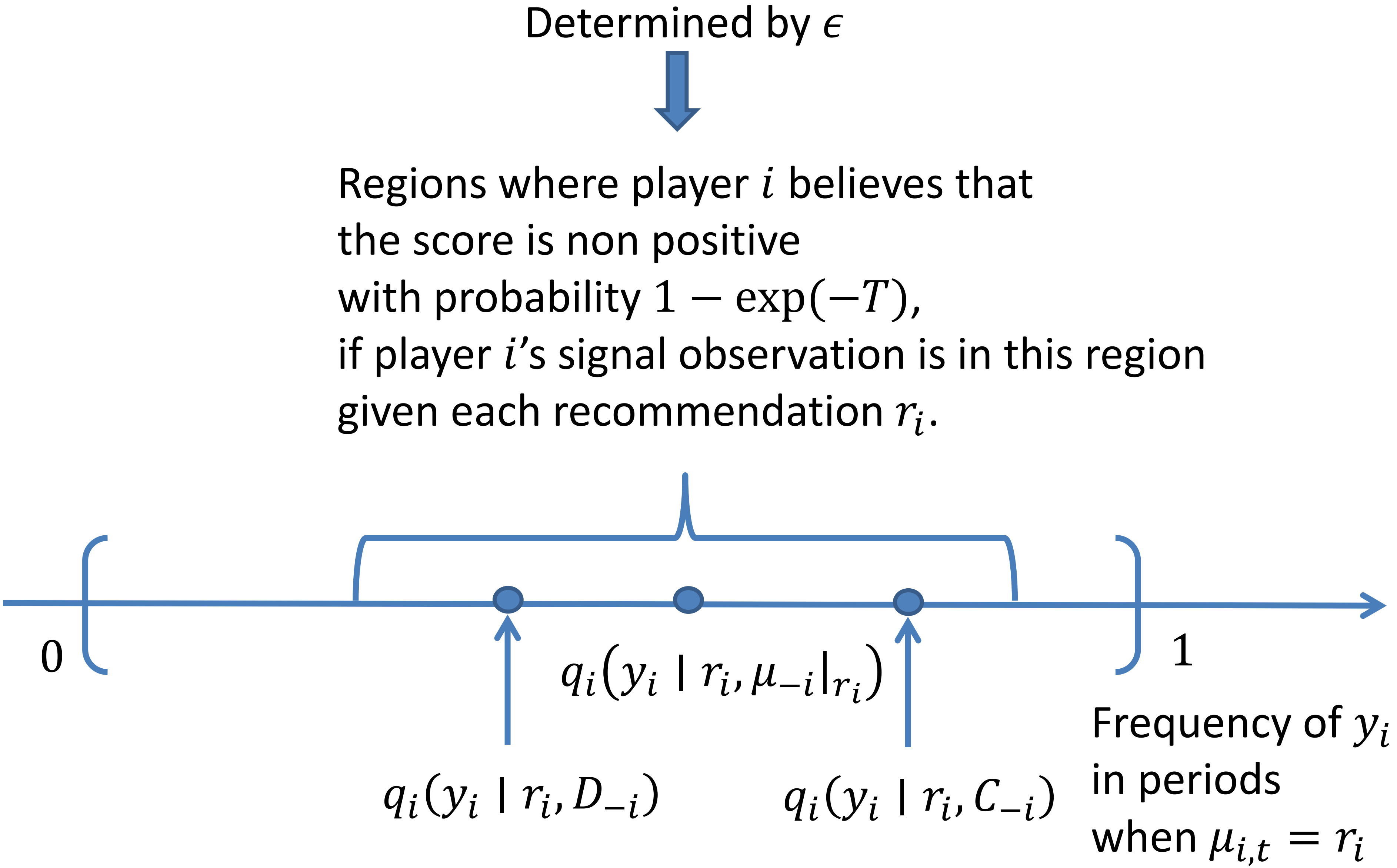
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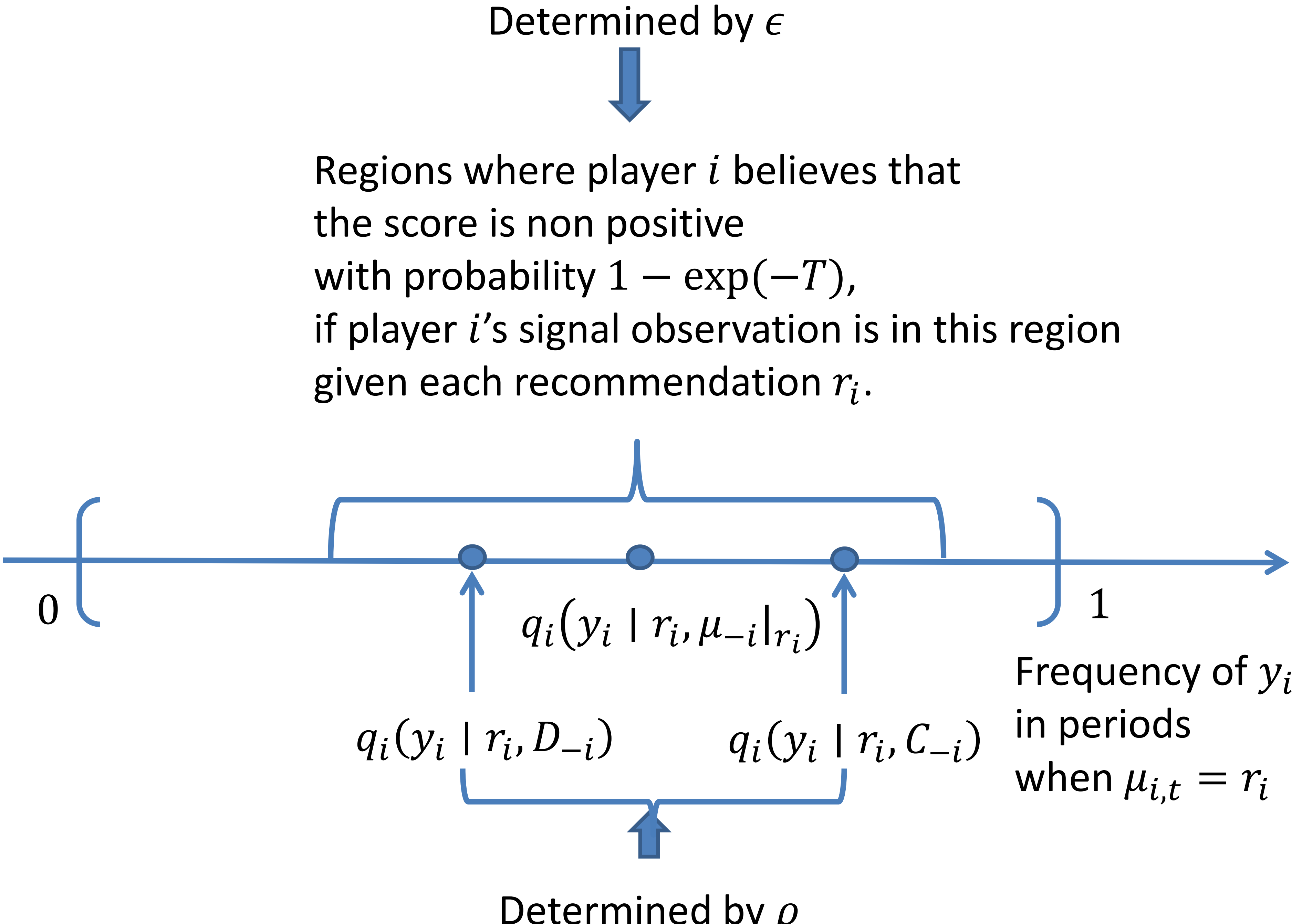
0
1
$q_i(y_i \mid r_i, \mu_{-i}|_{r_i})$
$q_i(y_i \mid r_i, D_{-i})$
$q_i(y_i \mid r_i, C_{-i})$
Frequency of y_i in periods when $\mu_{i,t} = r_i$



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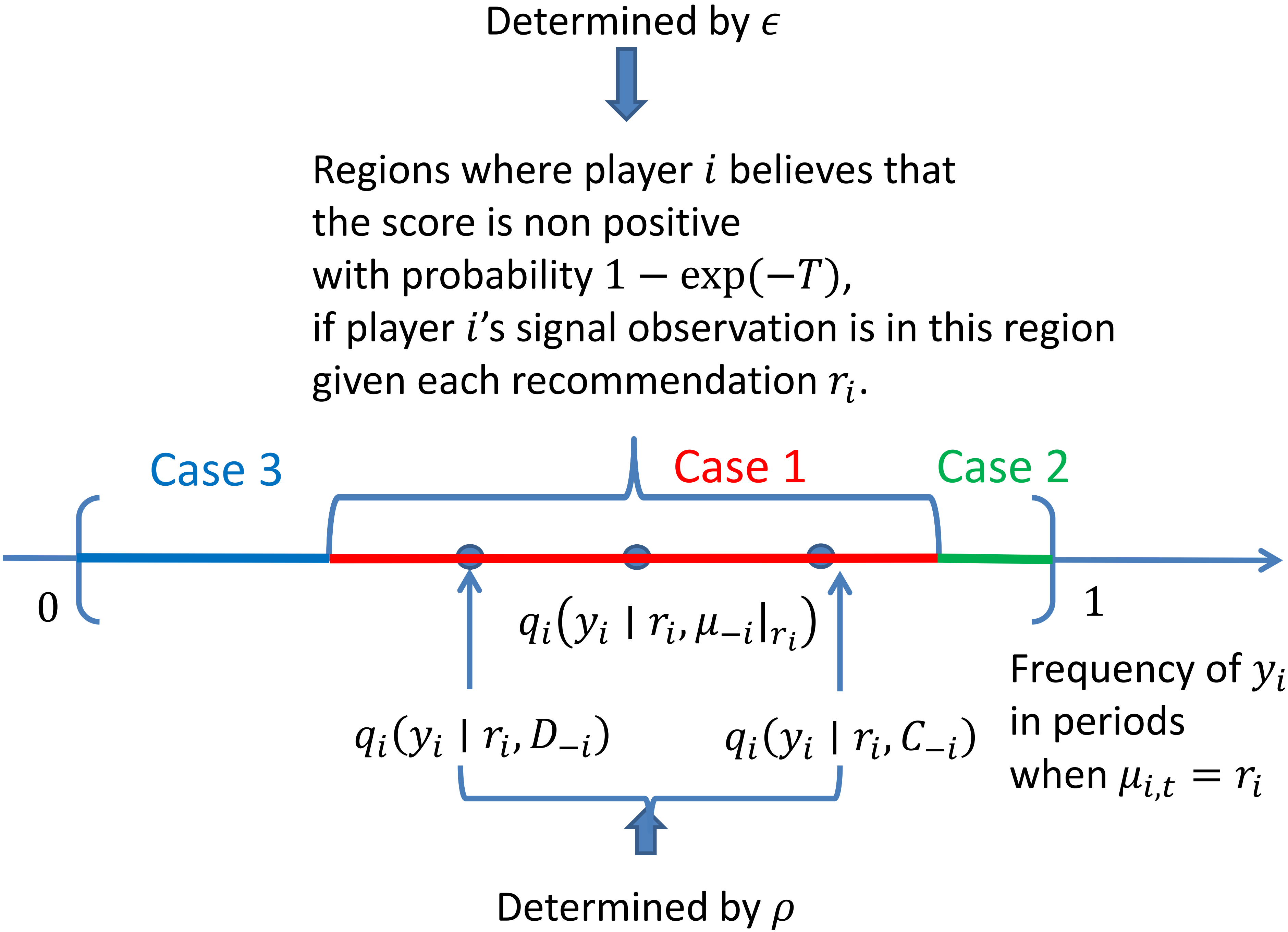
Frequency of y_i in periods when $\mu_{i,t} = r_i$
$q_i(y_i \mid r_i, D_{-i})$
$q_i(y_i \mid r_i, C_{-i})$
Determined by ρ



Determined by ϵ
Regions where player i believes that the score is non positive with probability $1 - \exp(-T)$, if player i's signal observation is in this region given each recommendation r_i.
Case 3
Case 1
Case 2
0
1
$q_i(y_i \mid r_i, \mu_{-i}|_{r_i})$
$q_i(y_i \mid r_i, D_{-i})$
$q_i(y_i \mid r_i, C_{-i})$
Frequency of y_i in periods when $\mu_{i,t} = r_i$
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Classifying player i's histories

- There exists r_i such that player i's signal frequency during periods when player i took r_i is in Case 2 or Case 3.

- For each r_i, player i's signal frequency during periods when player i took r_i is in Case 1.

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- For each r_i, player i's signal frequency during periods when player i took r_i is in Case 1.
- Player i believes that the score is non-positive.

What If $|Y_i| > |A_{-i}|$?

- If player i's signal observation given r_i is not close to $\mathrm{aff}\left(\{q_i(y_i|r_i, a_{-i})\}_{a_{-i} \in A_{-i}}\right)$ for some r_i, the mediator subtracts a large constant from players $-i$'s score, so that $\sum_{i \in I} x(\{r_t, y_t\}_t)$ is non positive.
- This does not affect players $-i$'s incentive.

The Proof of $\lim_{\delta\to 1} E(\delta) = Q$: General Case

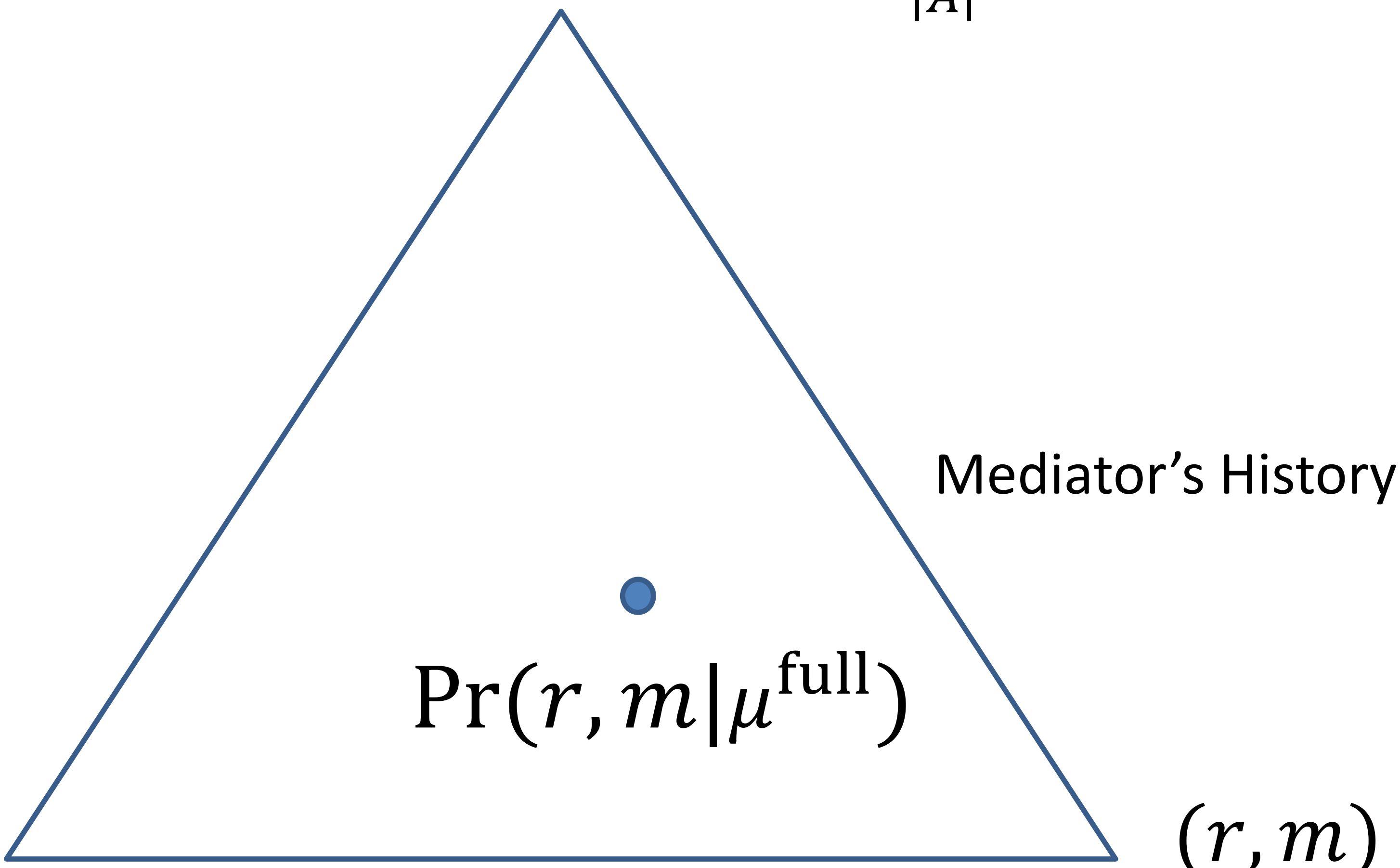
- We explain how to approximately support $v \in \mathrm{argmax}_{v' \in Q} \lambda \cdot v'$ for $\lambda = (1, \ldots, 1)$.
- There exist μ and $x(r, y)$ such that

1. Promise Keeping:
$$v = u(\mu) + \mathbb{E}[x(r, y)|\mu].$$
2. Incentive compatibility: For each i and σ_i,
$$\mathbb{E}[u(r) + x(r, y)|\mu] \geq \mathbb{E}[u(a_i, r_{-i}) + x(r, m_i, y_{-i})|\sigma_i, \mu].$$
3. Ex-ante self generation:
$$\mathbb{E}[\lambda \cdot x(r, y)|\mu] \leq 0.$$

Perturbation

- Perturb μ so that μ^{full} has full support:

$$\mu^{\text{full}} = (1-\varepsilon)\mu + \varepsilon \sum_{a\in \mathrm{A}} \frac{1}{|A|}.$$



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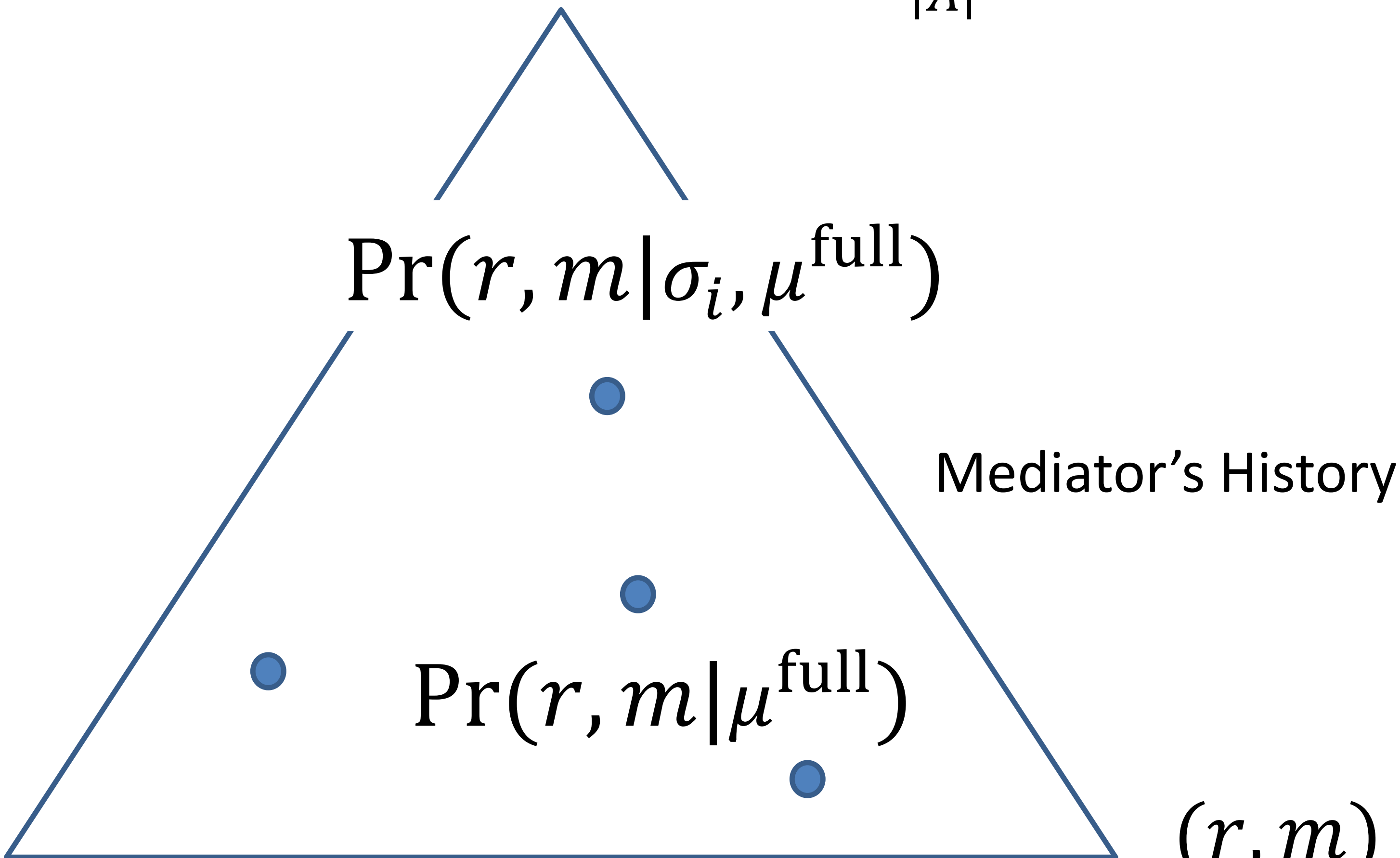
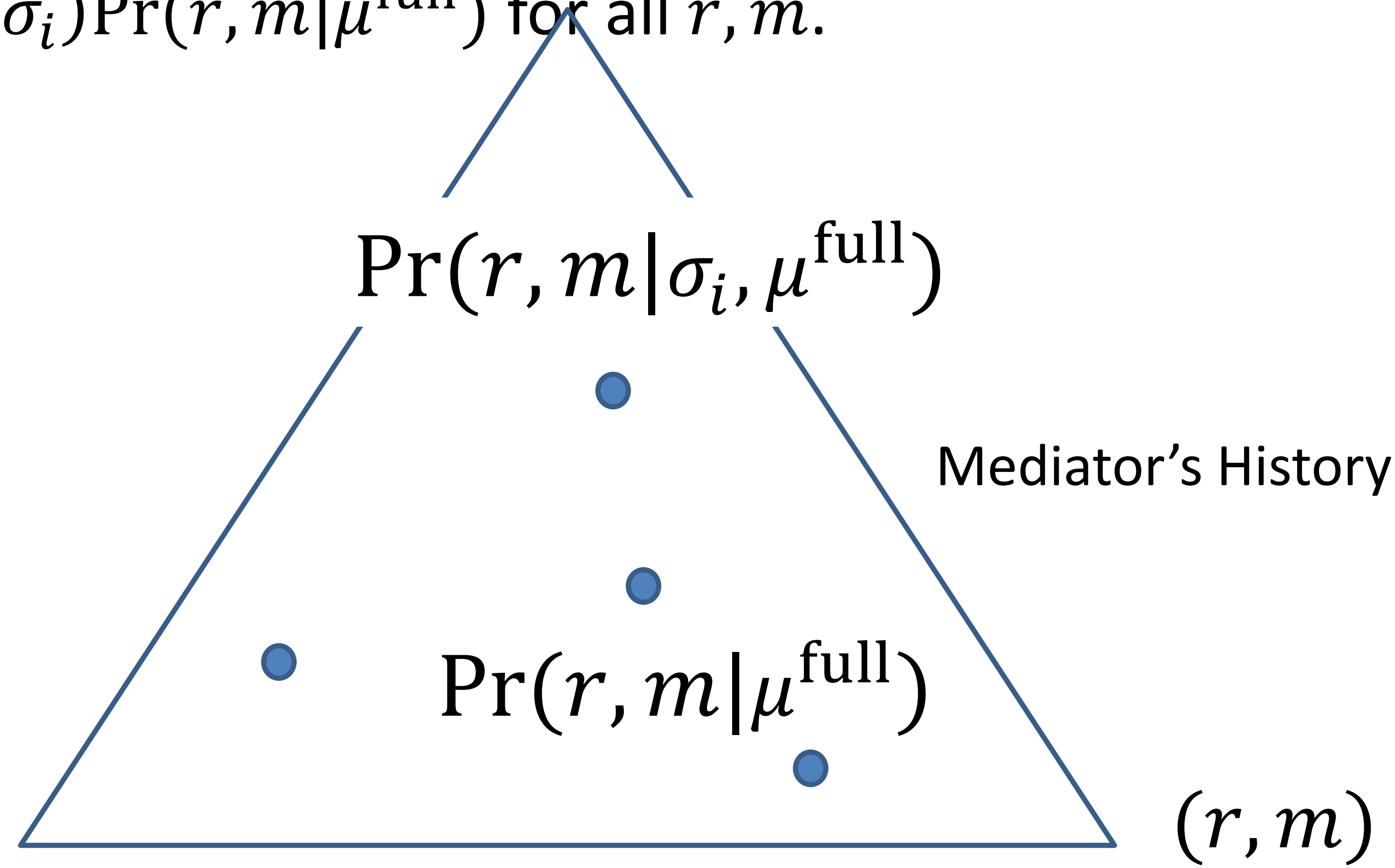


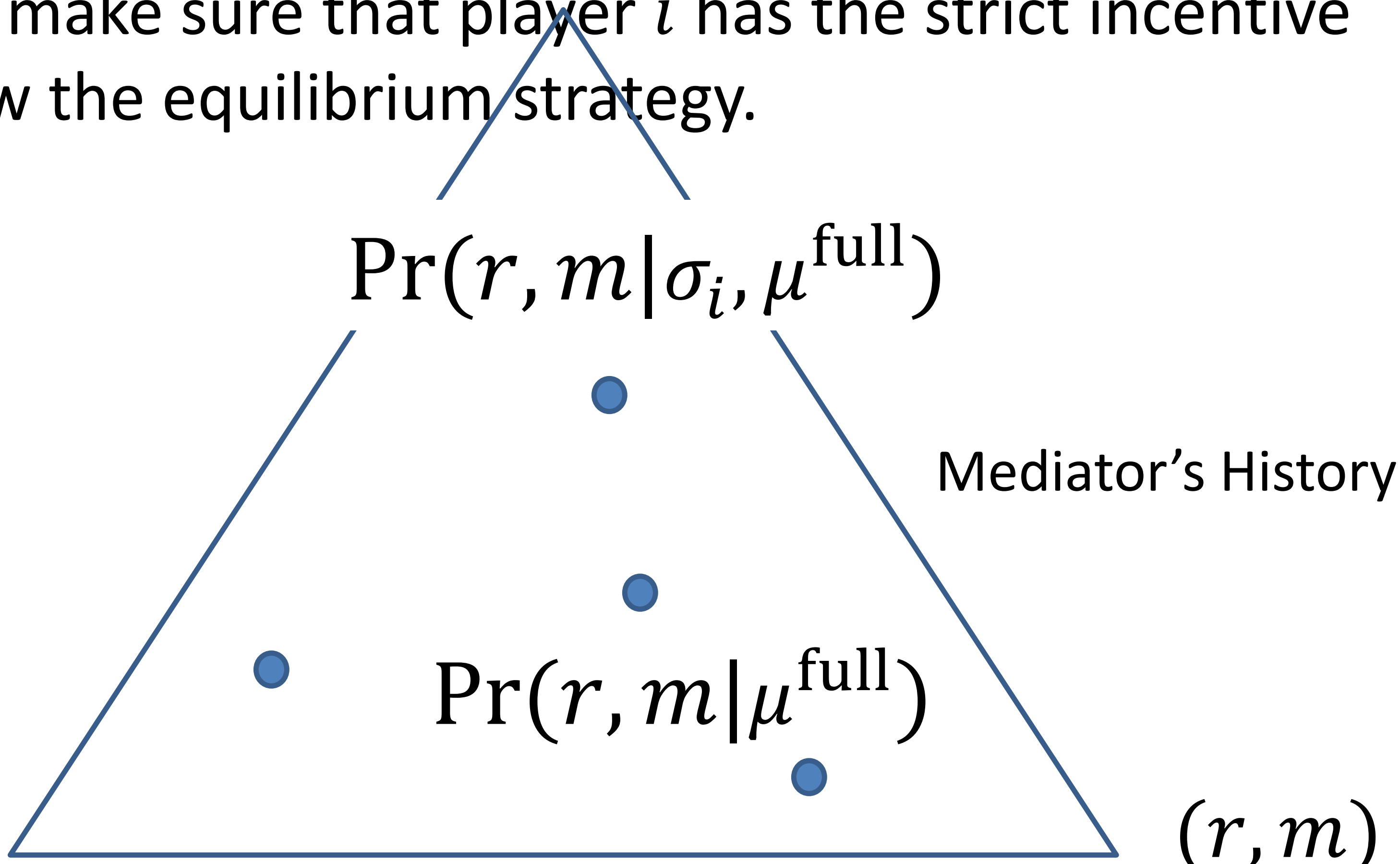
Perturbation

- Instead of μ^{full}, the mediator allows player i to take σ_i with probability $\Pr(\sigma_i)$ such that $\Pr\left(r, m \middle| \mu^{\text{full}}\right) = \sum_{\sigma_i} \Pr(\sigma_i) \Pr(r, m | \mu^{\text{full}})$ for all r, m.



Perturbation

- We take σ_i such that $\Pr(r, m|\sigma_i, \mu^{\text{full}})$ is an extreme point.
- We can make sure that player i has the strict incentive to follow the equilibrium strategy.



Given this Operation, …

There exist μ and $x(r, y)$ such that

1. Promise Keeping:
$$v = u(\mu) + \mathbb{E}[x(r, y)|\mu].$$
2. Strict incentive compatibility: For each i and non faithful σ_i,
$$\mathbb{E}[u_i(r) + x_i(r, y)|\mu] > \mathbb{E}[u_i(a_i, r_{-i}) + x_i(r, m_i, y_{-i})|\sigma_i, \mu].$$
3. For each $r_i \in \mathrm{supp}(\mu_i)$, the affine hull of $\Pr(y_i|r_i, r_{-i})$ with respect to $r_{-i} \in \mathrm{supp}(\mu_{-i}|_{r_i})$ is equal to the affine hull of $\Pr(y_i|r_i, r_{-i})$ with respect to $r_{-i} \in A_{-i}$.
4. Ex-ante self generation:
$$\mathbb{E}[\lambda \cdot x(r, y)|\mu] \leq 0.$$