Strategic Departure Decisions and Correlation in Dynamic Congestion Games

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- Assume without loss $r_i(d_i) = -d_i$.

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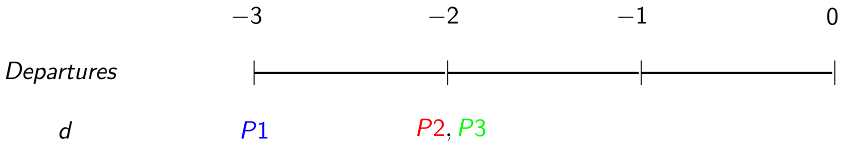
- IMPORTANT: We consider **fixed** n and assume C is large w.r.t. n.

A 3-player Example



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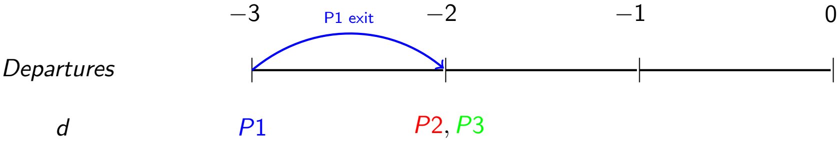

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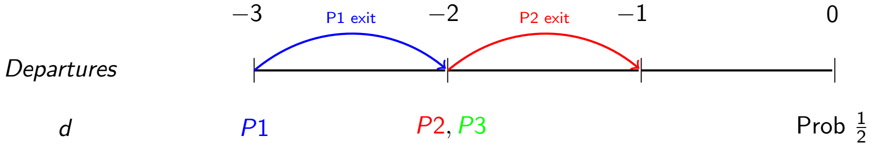

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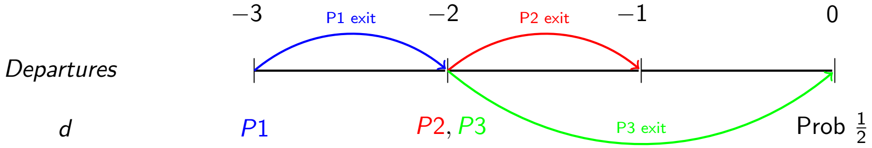

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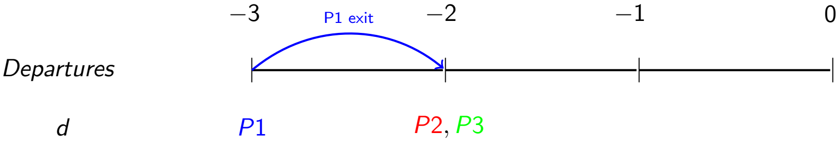

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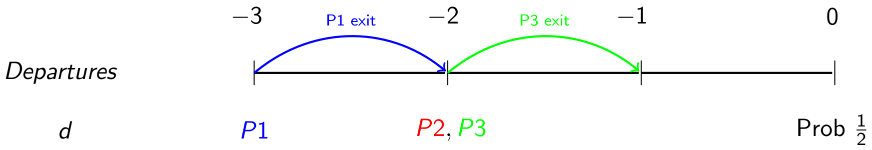

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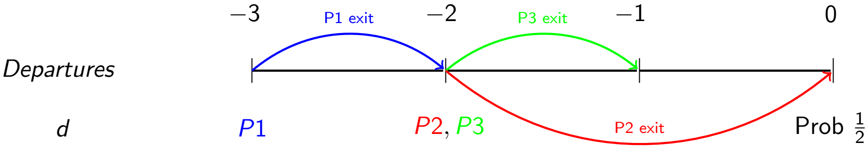

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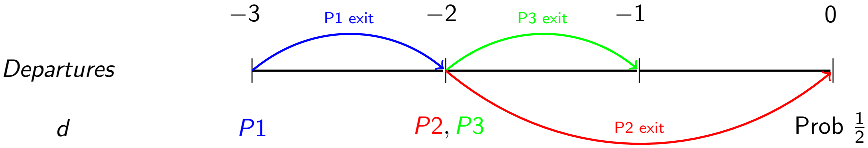

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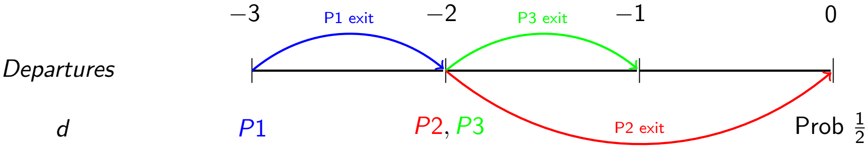

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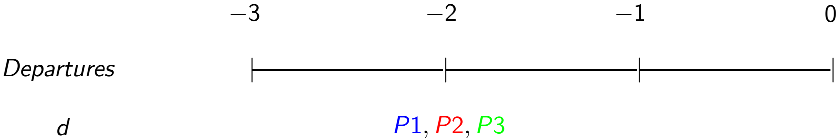

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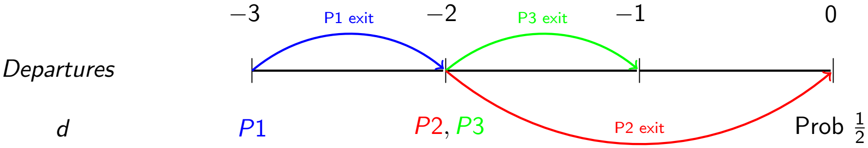

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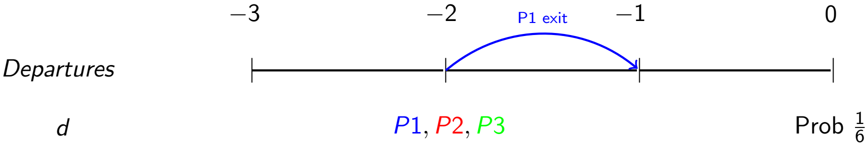

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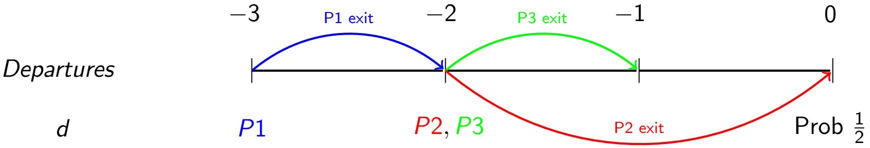

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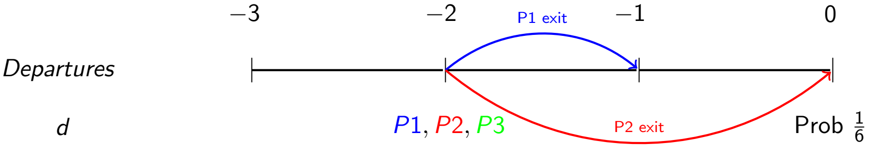

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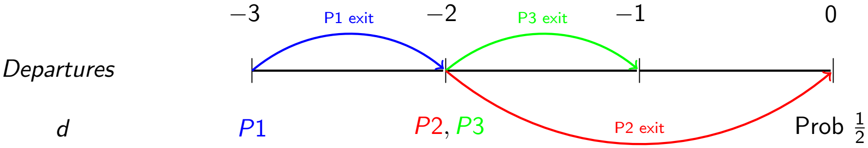

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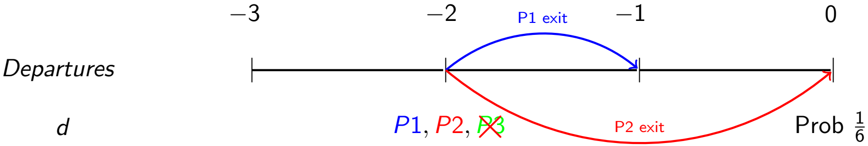

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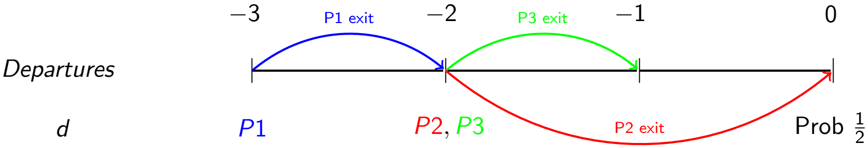

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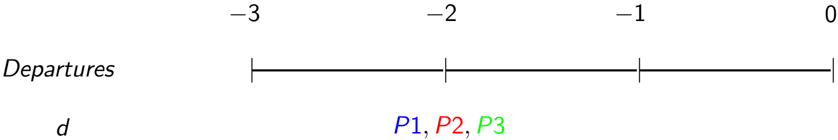

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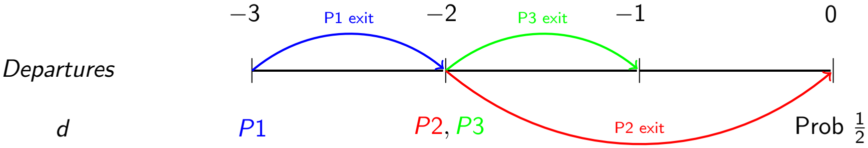

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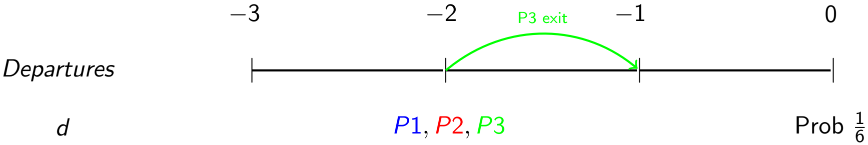

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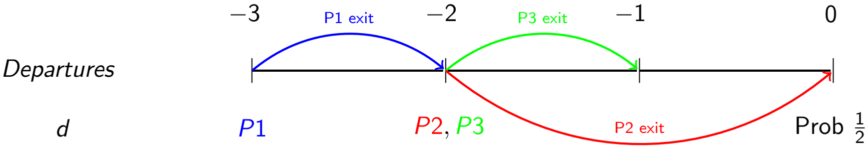

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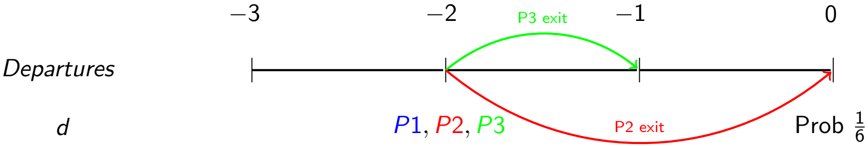

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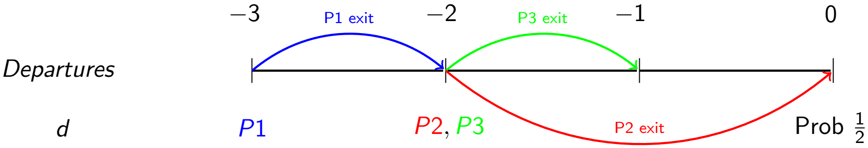

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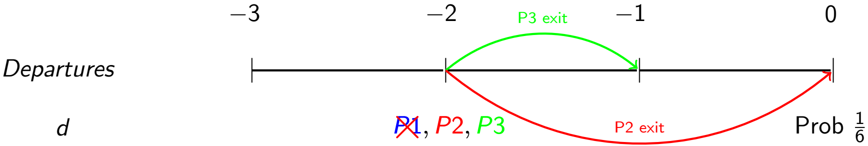

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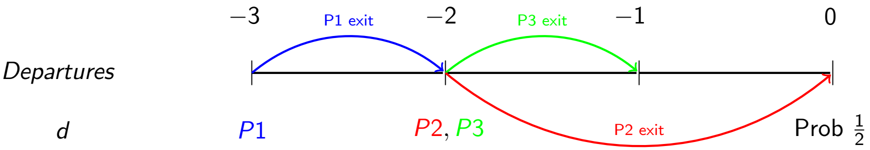

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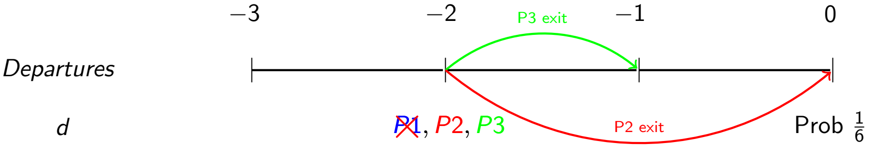

- Each player is late with prob $\frac{1}{3}$ $\implies$ $R_1 = R_2 = R_3 = 2 + \frac{1}{3} \cdot C$

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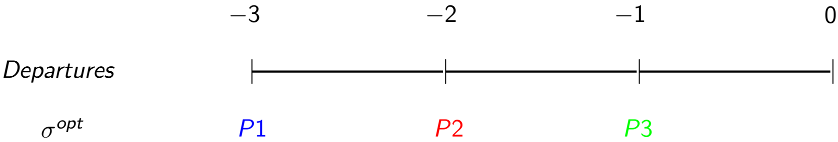

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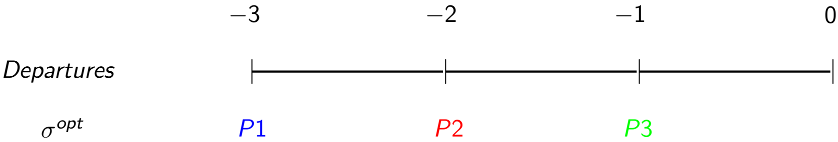

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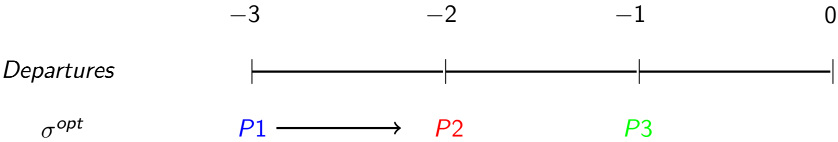

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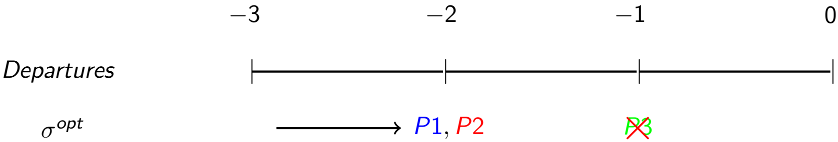

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  - P3 is late, but P1 and P2 are not.

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- (Sketch of Proof):
  - Time $-n$ is a *safe* time so $R_i(\sigma) \leq n$ in any equilibrium σ.
  - σ^{wst} is a NE that gives each player a payoff of exactly n.

The Price of Anarchy

- The social planner wants to minimize the sum of equilibrium payoffs.
- Define sum of costs as $SC(\sigma) := \sum_{i \in I} R_i(\sigma)$.

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- **Conclusion:** The worst equilibrium costs are roughly *twice* the optimum.

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- (Sketch of Proof):
- If no player departs at time $-n$ then at least one player is late for sure.
- As $C \to \infty$ the risk of being late becomes too large so there is a deviation to $-n$.
- Hence for large C, there exists $i \in I$ such that $-n \in \textbf{supp}\,(\sigma_i^{bst})$.

The Price of Stability

- **Result 3:** There exists $\bar{C} \in (n, n^2)$ such that for all $C > \bar{C}$ the *best* equilibrium payoffs are obtained by σ^{bst} with

  $\mathbf{supp}\,(\sigma_i^{bst}) = \{-n\}$ and $\mathbf{supp}\,(\sigma_j^{bst}) = \{-(n-1), -(n-2)\}$ for all $j \neq i$.

- (Sketch of Proof):
- If no player departs at time $-n$ then at least one player is late for sure.
- As $C \to \infty$ the risk of being late becomes too large so there is a deviation to $-n$.
- Hence for large C, there exists $i \in I$ such that $-n \in \mathbf{supp}\,(\sigma_i^{bst})$.
- But then, at least $n-1$ players must mix over time $-(n-1)$.

The Price of Stability

- **Corollary [Price of Stability]:** There exists $\bar{C} \in (n, n^2]$ such that for all $C > \bar{C}$

$$PoS := \frac{SC(\sigma^{bst})}{SC(\sigma^{opt})} = \frac{n + (n-1)^2}{\frac{n(n+1)}{2}} = 2 + \frac{2}{n(n+1)} - \frac{4}{n+1}$$

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- <u>**Conclusion**</u>: The *best* Nash equilibrium cost is **also** roughly twice the social optimum.

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- **Conclusion**: The *best* Nash equilibrium cost is **also** roughly twice the social optimum.

- **Question:** Is there any way to coordinate the players actions to obtain an outcome closer to the social optimum?

Correlated Equilibrium Example

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- If playing s_i is optimal for each $i \in I$ and s_i in the support of Q
  - Given beliefs about s_{-i} formed using $s \sim Q$.
- Then Q is a correlated equilibrium.

Example: 4 players, C=20.

	-4	-3	-2	-1
Departures				
$s' :=$	$P1$	$P2$	$P3$	$P4$
$s'' :=$	$P1$	$P2$	$P3, P4$	
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Example: 4 players, C=20.

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- $Q^\star$ is a CE that yields the best SC:

$$SC(\sigma^{bst}) = 13 \qquad SC(\sigma^{opt}) = 10 \qquad SC(Q^\star) = 10.81$$

Characterizing Best Correlated Equilibrium

- $S = \mathbb{Z}_-^n$, we look for CE $Q \in \Delta(S)$ that minimize
$$SC(Q) := \sum_{s \in S} Q(s) SC(s)$$
- Only interested in $Q \in \Delta(S^Y)$: set of outcomes where no player is late.
- Enforcing strategies: $s \in S$ enforces time k for player i if when i is told to depart at time k, she is late with positive probability when departing at time $k-1$ instead, when others play s_{-i}.
- $Z^{i,k}$ set of strategies that enforce k for player i.
- $S^{i,k} = \{s \in S : s_i = k\}$.

- **Lemma:** *$Q \in \Delta(S^Y)$ is a correlated equilibrium of SD game with penalty C if and only if for all $i \in I$*

$$\sum_{s \in Z^{i,k}} Q(s) \geq \frac{k}{C} \left[\sum_{s \in S^{i,k}} Q(s) \right] \qquad \text{for } k = 2, ..., n$$

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- So player i, being told to depart at $-k$ does not want to deviate to $-(k-1)$ only if

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$$\mathbb{P}(s \in Z^{i,k} | s_i = -k) = \frac{\sum_{s \in Z^{i,k}} Q(s)}{\sum_{s \in S^{i,k}} Q(s)}$$

From Strategies to Outcomes

$$s = (4, 3, 3, 3) \to y^s = (1, 3, 0, 0)$$

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- If $y = (1, 3, 0, 0)$ then

$$S(y) = \{(4, 3, 3, 3), (3, 4, 3, 3), (3, 3, 4, 3), (3, 3, 3, 4)\}$$

- We show it is without loss to restrict attention to distributions over outcomes with this implementation.

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Theorem: There exists $\bar{C}$ such that for all $C > \bar{C}$, the best correlated equilibrium payoff is generated by $Q^\star \in \Delta(S^Y)$:

$$Q^\star(s) = \frac{1}{|S(y^s)|}\hat{Q}^o(y^s)$$

and $\hat{Q}^o(y) \in \Delta(Y)$ satisfies

$$\hat{Q}^o(y^k) = \frac{k}{C}[k\hat{Q}^o(y^{k+1}) + \sum_{j=2}^{k} \hat{Q}^o(y^j)] \quad \text{for } k = 3, ..., n$$

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  Corollary: As $C \to \infty$, $Q^\star(\sigma^{opt}) \to 1$.

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- Consider the following *toll pricing mechanism* $\mathcal{M}_\tau$: Any player exiting the road after time 0 pays a large toll of τ.

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- Proof: $\mathcal{M}_\tau$ effectively increases $C \to C + \tau$.

- **Correlated Price of Stability:**

$$CPoS := \frac{SC(Q^\star)}{SC(\sigma^{opt})} = 1 + \delta(C)$$

- where $\delta(C) \to 0$ as $C \to \infty$.

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  Corollary There exists $\bar{C} \in \mathbb{R}$ such that for all $C > \bar{C}$ our results regarding the PoA, PoS, and CPoS are robust to changes in C and to the specification of $f(a_i, C)$.

Thank you!

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