“Efficiency in Chip Strategies”

joint work with Mikhail Safronov

1 Introduction

- Influential papers from the early 1990s naturally moved the emphasis of the repeated-game literature to characterizing equilibrium payoffs.

- However, exploring the properties of behavior that players exhibit in repeated interactions seems as important as the payoffs that this behavior generates.

- Given the richness of stage games, monitoring and information structures, any general results are unlikely to be possible.

- However, equilibrium behavior can be fruitfully studied in more specific settings.

- We focus on what is often called chip strategies, and which were introduced in the context of favor exchange model by Mobius (2001).

- According to these strategies: a player takes an individually suboptimal action if that action creates a “gain” for the opponent larger than the player’s “loss” from taking it.

- In exchange the player implicitly obtains from the opponent a chip that implicitly entitles him or her to receiving this kind of favor at some future date.

- A player who issued a limit number of chips is no longer entitled to receiving favors until provides a favor to the opponent, in which case the player receives one chip back.

2 Related Literature

- Mobius (2001) introduced the favor exchange model, in which favor opportunities arrive according to a Poisson process.
- Mobius studied somewhat different, less efficient chip strategies in which cooperation breaks down when a player issues a limit number of chips.
- Hauser and Hopenhayn (2008) introduced somewhat more general and somewhat more efficient chip strategies.
- Abdukadiroglu and Bagwell (2012, 2013) introduced a discrete-time version of Mobius' model, one that will be studied by us.
- In their 2012 paper, they analyze chip strategies in the same form as we do here. They identify the optimal limit number of chips given a discount factor, and compare this optimal chip mechanism with a more sophisticated favor-exchange relationship studied in their 2013 paper.
- None of these papers shows (explicitly or implicitly) that any kind of chip strategies attain efficiency when the discount factor converges to 1.
- Repeated Spulber's duopoly (more generally oligopoly) with i.i.d. types was studied in Athey and Bagwell (2001), Athey, Bagwell and Sanchirico (2004), Hörner and Jamison (2007), and with more general Markov types by Athey and Bagwell (2008) and Escobar and Toikka (2010).
- The focus of these papers is not on chip strategies, although some strategy profiles studied by these authors contain some elements that are similar.
- Chip strategies are remarkably simple compared to the strategies used by these authors, even for *i.i.d.* types, to show that efficiency can be attained when the discount factor converges to 1.

3 Discrete-time model of favor exchange

3.1 Setting and Result

- We study the model of Abdulkadiroglu and Bagwell (2012), which is a discrete version of Mobiüs (2001) model of favors.

- In the stage game, either player 1 is given an income of \$1, player 2 is given an income of \$1, or neither player is given an income. The former two states each occur with probability $p \in (0, 1/2)$ and the latter event thus occurs with probability $1 - 2p$.

- Each player is privately informed about her or his income.

- If a player receives income, then that player may send the income to the other player. The transferred income becomes $\gamma > 1$. The player may consume its income itself.

- This game is played repeatedly, states are *i.i.d.*, and players have a common discount factor δ.

- Players cannot store income, that is, income must be either transferred or consumed in the period it is received.

- The efficient total payoff of the two players is $v = 2p\gamma$.

We obtain the following result:

Theorem 1 *For every $\lambda > 0$, there is a $\underline{\delta}$ such that for every $\delta > \underline{\delta}$, there is a chip-strategy equilibrium of the repeated games in which players' discount factor is δ such that the ex ante payoff of each player in this equilibrium exceeds $v/2 - \lambda$.*

3.2 The description of chip strategies

- At the beginning of each period, each player i holds $k_i \in \{0, ..., n\}$ chips, where $k_1 + k_2 = n$.
- If player i obtains an income of \$1, and $k_i < n$, player i gives the income to player j, and j gives (implicitly) i one chip in return.
- If $k_i = n$, i.e., i already holds all the chips, then i consumes the \$1 itself. No chips are transferred in this case.
- At the beginning of period 1, each player has $n/2$ (or fifty-fifty over $(n-1)/2$ and $(n+1)/2$) chips.

3.3 Payoff efficiency

- The strategies induce a stochastic Markov chain over states $k = 0, ..., n$. (Say, $k = k_1$, but it can be k_2.)

- Obviously, each state is reached from any other state in n periods with positive probability.

- Thus, by the Ergodic Theorem there exists a probability distribution over states $\{\pi_k : k = 0, ..., n\}$ such that the probability of being in state k after a sufficiently large number of periods is arbitrarily close to π_k, independent of the initial state.

- This probability distribution is an eigenvector of the transition matrix corresponding to eigenvalue 1.

- It is easy to check that this eigenvector must have coordinates equal to $1/(n + 1)$.

- Since our strategies are inefficient only in states 0 and n, as δ is sufficiently close to 1, the inefficiency is approximately proportional to $2/(n + 1)$, the sum of the ergodic probabilities of states 0 and n.

- Therefore it disappears when n diverges to infinity.

3.4 Incentive Constraints

- Let V_k be the continuation payoff, contingent on being in state k, and let $\Delta_k := V_k - V_{k-1}$.

- Then:

$$\Delta_k = p\delta\Delta_{k-1} + p\delta\Delta_{k+1} + (1-2p)\delta\Delta_k$$

for $k = 2, ..., n-1$;

$$\Delta_1 = p(1-\delta)\gamma + p\delta\Delta_2 + (1-2p)\delta\Delta_1$$

and

$$\Delta_n = p(1-\delta) + p\delta\Delta_{n-1} + (1-2p)\delta\Delta_n.$$

- For $\delta = 1$, this system of linear equations is satisfied by all Δ's being equal to 0.

- For $\delta < 1$, the system is harder to solve, so we will evaluate Δ's in approximation by referring to the Implicit Function Theorem.

- By this theorem, one can compute Δ's for δ's close to 1 in approximation by differentiating the system of equations for Δ's.

- This yields

$$\begin{bmatrix} 2 & -1 & 0 & . & . & . & 0 \\ -1 & 2 & -1 & . & & & . \\ 0 & . & . & . & . & & . \\ . & . & . & . & . & . & . \\ . & & . & . & . & . & 0 \\ . & & & . & -1 & 2 & -1 \\ 0 & . & . & . & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} \partial\Delta_n/\partial\delta \\ . \\ \\ . \\ \\ . \\ \partial\Delta_1/\partial\delta \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ . \\ . \\ . \\ 0 \\ -\gamma \end{bmatrix}.$$

- This system of linear equations can be easily solved by the Gauss-Jordan elimination method; the unique solution is given by

$$\frac{\partial\Delta_k}{\partial\delta} = -\frac{(n-k+1)\gamma + k}{n+1} < -1$$

for all k, which means that $\Delta_k > 1$ for $k = 1, ..., n$ and δ close to 1.

4 Repeated Spulber's duopoly

4.1 Setting and Result

- Two firms meet in periods $t = 1, 2...$ Each firm's privately known constant marginal production cost takes value $c = \underline{c}$ or $\overline{c}$.

- Each firm's cost follows a first-order Markov process. If the cost in a certain period is $\underline{c}$, then it will be $\underline{c}$ with probability $p \in [1/2, 1)$ and $\overline{c}$ with the remaining probability in the following period. Similarly, if the cost in a certain period is $\overline{c}$, then it will be $\overline{c}$ with probability p and $\underline{c}$ with the remaining probability in the following period.

- In every period t of the stochastic game, firms simultaneously select prices.

- A single consumer is willing to pay up to r dollars for one unit of the good, and buys from the firm that offers the lower price, and from each firm with the fifty-fifty chance if the two prices are equal.

- Firms are expected profit maximizers and discount future payoffs by a common discount factor $\delta < 1$.

- In period 0, the probability distribution over the cost of each firm is fifty-fifty.

- The efficient, or most collusive, total payoff of the two firms is $v = r - 3\underline{c}/4 - \overline{c}/4$.

We obtain the following result:

Theorem 2 *For every $\lambda > 0$, there is a $\underline{\delta}$ such that for every $\delta > \underline{\delta}$, there is an equilibrium of the repeated games in chip strategies in which firms' discount factor is δ such that the ex ante payoff of each firm in this equilibrium exceeds $v/2 - \lambda$.*

4.2 The description of chip strategies

- On the equilibrium path, the strategy profile has $4(2n+1)-6$ states. Each state is described by a number $k \in \{-n, ..., -1, 0, 1...n\}$, and the profile of firms' costs $c = (c_1, c_2) \in \{\underline{c}, \overline{c}\}^2$ in the previous period, with the exception that $k = -n$ implies that $c = (\underline{c}, \overline{c})$ and $k = n$ implies that $c = (\overline{c}, \underline{c})$.

(A positive number k is interpreted as firm 1 having k chips, and a negative number k is interpreted as firm 2 having k chips.)

- In all states except when $k = -n$ or n, if the cost of a firm is $\overline{c}$, the firm is supposed to select price r. If the cost is $\underline{c}$, the firm is supposed to select a price which is slightly lower than r, say, $r - \rho$, where ρ is an "infinitesimal" number.

- If both firms select the same price, the value of k does not change. If firm 1 selects $r - \rho$, and firm 2 selects r, then k is replaced with $k + 1$ for the next-period state; and if firm 1 selects r, and firm 2 selects $r - \rho$, then k is replaced with $k - 1$ for the next-period state.

- If the current state has $k = -n$, then firm 1 is supposed to charge $r - \rho$, and firm 2 is supposed to charge r, independent of its cost; and $k = -n$ is replaced with $k = -n + 1$ at the end of the current period.

- If the current state has $k = n$, then firm 1 is supposed to charge r, and firm 2 is supposed to charge $r - \rho$, independent of their costs; and $k = n$ is replaced with $k = n - 1$.

(Notice that state $k = -n$ can be reached only when firms reported $c = (\underline{c}, \overline{c})$ in the previous period, and state $k = n$ can be reached only when firms reported $c = (\overline{c}, \underline{c})$ in the previous period.)

- Off the equilibrium path firms play a bad equilibrium, for example, the worst carrot and stick equilibrium from Athey and Bagwell (2008).

4.3 Payoff efficiency

- An analogous argument to that from the favor exchange model, referring to the ergodic theorem.
- In this case, we were unable to compute the eigenvector; in particular it does not have all coordinates equal.
- However, we show that the limit chance of being in an inefficient state converges to zero as n diverges to infinity.

4.4 Incentive Constraints

- An analogous argument to that from the favor exchange model, referring to the implicit function theorem.
- In this case, the computation is more involved.
- However, we are able to show that all $\partial\Delta/\partial\delta$'s (and so Δ's) have the form of the difference between $r/2$ and a weighted average of $\overline{c}/2$ and $\underline{c}/2$.
- This shows incentives to "report" truthfully in any state, because a firm loses $(r-\underline{c})/2$ by reporting $\overline{c}$ when its cost is $\underline{c}$, and gains $(r-\overline{c})/2$ by reporting $\underline{c}$ when its cost is $\overline{c}$.