Dynamics Characteristic of Solitons for Davey-Stewartson System

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§1 Equation

$$iu_t + \triangle u + E_1(u)u + |u|^{p-1}u = 0 \tag{1}$$

$$u = u(t,x) : [0,T) \times \mathbb{R}^N \to \mathbb{C}, 0 < T \leq \infty$$

$$1 < p < \begin{cases} \infty, & N = 1, 2, \\ \frac{N+2}{N-2}, & N \geq 3. \end{cases}$$

$$E_1(|u|^2) = \mathcal{F}^{-1}[\frac{\xi_1^2}{|\xi|^2}\mathcal{F}[|u|^2]]$$

- **Background**

Davey, A. and Stewartson, K. (1974)

• **Known Studies**:

Ghidaglia, J.-M. and Saut, J.C.(1990): Local well-posedness for $N = 2$

Guo, B.L. and Wang, B.X.(1999): Local well-posedness for $N = 3$

Cipolatti, R.(1992,1993), Ohta, M.(1994,1995): Existence and stabilities of ground state

Papanicolaou, G.C., Sulem, C. and Sulem, P-L.(1994), Ozawa, T.(1992): Existence of blow-up solutions

Richards, G.(2009): Concentration properties for $N = 2$, $p = 3$

Zhang, Gan, Li and Zhu(2006 $\sim$): Sharp threshold, blow-up dynamic, global dynamic.

Impose the initial data

$$u(0, x) = u_0, \qquad x \in \mathbb{R}^N. \tag{2}$$

Problems:

- Sharp criteria for the blow-up and global existence
- Global dynamic: stability of solitons, asymptotic behavior
- Blow-up dynamic: blow-up rate, concentration, limiting profile

Arguments:

- Variational methods:

  Weinstein, 1986; Tsutsumi, 1990; Merle, 1990$\sim$; Zhang, 1999$\sim$
- Profile decomposition:

  Gérard, 1998$\sim$, Hmidi and Keraani, 2005$\sim$

§3 Sharp Criteria

♦ Local Existence

$$H(v) := \frac{1}{2}\int |\nabla v(t,x)|^2 dx - \frac{1}{p+1}\int |v(t,x)|^{p+1} dx - \frac{1}{4}\int E_1(|v|^2)|v|^2 dx.$$

Lemma 1 *(Ghidaglia-Saut, 1990, Nonlinearity; Guo-Wang, 1999, CPAM)*

Let $N \in \{2,3\}$ *and* $u_0 \in H^1(\mathbb{R}^N)$. $\Rightarrow \exists!$ *a solution* $u(t,x)$ *of the Cauchy problem (1)–(2) in* $C([0,T);H^1)$ *for some* $T \in (0,\infty]$ *(maximal existence time), either* $T=\infty$ *(global existence) or else* $T<\infty$ *and* $\lim_{t\to T}||u||_{H^1}=\infty$ *(blow-up). Furthermore, for all* $t \in [0,T)$, $u(t,x)$ *satisfies*

$$\|u(t)\|_2 = \|u_0\|_2,$$

$$H(u(t)) = H(u_0).$$

♦ Cross-Constrained Variational Method (Zhang, 1999)

Let $N = 2, 3,\ \omega \in \mathbb{R}, 1 < p < \frac{N+2}{(N-2)^+}$ and $v \in H^1(\mathbb{R}^N)$. Define

$$I(v) := \int \frac{1}{2}|\nabla v|^2 + \frac{\omega}{2}|v|^2 - \frac{1}{p+1}|v|^{p+1} - \frac{1}{4}E_1(|v|^2)|v|^2 dx$$

$$S(v) := \int |\nabla v|^2 + \omega|v|^2 - |v|^{p+1} - E_1(|v|^2)|v|^2 dx$$

$$Q(v) := \int |\nabla v|^2 - \frac{N(p-1)}{2(p+1)}|v|^{p+1} - \frac{N}{4}E_1(|v|^2)|v|^2 dx$$

$$M := \{v \in H^1, S(v) < 0, Q(v) = 0\}$$

$$d_M := \inf_M I(v)$$

$$d_S := \inf_{\{v \in H^1 \setminus \{0\}, S(v)=0\}} I(v)$$

$$d := \min\{d_S, d_M\}$$

Invariant Sets

$$K := \{v \in H^1, I(v) < d,\ S(v) < 0,\ Q(v) < 0\}$$

$$K_+ := \{v \in H^1,\ I(v) < d,\ S(v) < 0,\ Q(v) > 0\}$$

$$D_+ := \{v \in H^1,\ I(v) < d,\ S(v) > 0\}$$

- **Sharp Criteria**

Theorem 1 *(Gan-Zhang, 2008, CMP)*

Let $N \in \{2,3\}$ *and* $1 + \frac{4}{N} \leq p < \frac{N+2}{(N-2)^+}$. *Then we have that*
(i) if $u_0 \in K_+ \cup D_+$, *then the solution* $u(t,x)$ *of (1)-(2) exists globally,*

(ii) if $|x|u_0 \in L^2(\mathbb{R}^N)$ *satisfies* $u_0 \in K$, *then the solution* $u(t,x)$ *of (1)-(2) blows up in a finite time* $0 < T < \infty$.

♦ Existence of Solitons

• Profile Decomposition (Hmidi-Keraani, 2005, IMRN)

Proposition 1 *Let* $N \geq 2$ *and* $\{v_n\}_{n=1}^{\infty}$ *be a bounded sequence in* $H^1(\mathbb{R}^N)$. *Then there is a subsequence of* $\{v_n\}_{n=1}^{\infty}$ *(still denoted by* $\{v_n\}_{n=1}^{\infty}$ *) and a sequence* $\{V^j\}_{j=1}^{\infty}$ *in* $H^1(\mathbb{R}^N)$ *and a family of* $\{x_n^j\}_{j=1}^{\infty} \subset \mathbb{R}^N$ *such that*

(i) for every $j \neq k$, $|x_n^j - x_n^k| \to \infty$ *as* $n \to \infty$;

(ii) for every $l \geq 1$ *and every* $x \in \mathbb{R}^N$

$$v_n(x) = \sum_{j=1}^{l} V^j(x - x_n^j) + v_n^l(x) \ \text{ with } \ \lim_{l\to\infty} \limsup_{n\to\infty} \|v_n^l\|_r \to 0$$

for every $r \in (2, \frac{2N}{(N-2)^+})$.

Moreover, we have, as $n \to \infty$,

$$\|v_n\|_2^2 = \sum_{j=1}^{l} \|V^j\|_2^2 + \|v_n^l\|_2^2 + o(1)$$

and

$$\|\nabla v_n\|_2^2 = \sum_{j=1}^{l} \|\nabla V^j\|_2^2 + \|\nabla v_n^l\|_2^2 + o(1).$$

- Consider the scalar field equation in $\mathbb{R}^2$

$$\triangle R - R + E_1(|R|^2)R = 0, \quad R \in H^1(\mathbb{R}^2). \tag{3}$$

Proposition 2 (Zhu-Zhang-Yang, 2012, preprint) Let $v \in H^1(\mathbb{R}^2)$. Then

$$\int_{\mathbb{R}^2} E_1(|v|^2)|v|^2 dx \leq \frac{2}{\|R\|_2^2}\|\nabla v\|_2^2\|v\|_2^2,$$

where R is the ground state of (3).

- Consider the scalar field equation in $\mathbb{R}^2$

$$\triangle Q - Q + |Q|^2 Q + E_1(|Q|^2)Q = 0, \ Q \in H^1(\mathbb{R}^2). \tag{4}$$

Proposition 3 (Papanicolaou et al., 1994, Physica D) Let $v \in H^1(\mathbb{R}^2)$. Then

$$\int |v|^4 + E_1(|v|^2)|v|^2 dx \leq \frac{2}{\|Q\|_2^2}\|\nabla v\|_2^2 \ \|v\|_2^2,$$

where Q is the ground state of (4).

- Consider the scalar field equation in $\mathbb{R}^3$

$$\frac{3}{2}\triangle W - \frac{1}{2}W + |W|^2 W + E_1(|W|^2)W = 0,\ W \in H^1(\mathbb{R}^3). \tag{5}$$

Proposition 4 (Zhang-Zhu, 2011, DPDE) Let $v \in H^1(\mathbb{R}^3)$. Then

$$\int |v|^4 + E_1(|v|^2)|v|^2 dx \leq \frac{2}{\|W\|_2^2} \|\nabla v\|_2^3 \, \|v\|_2,$$

where W is the ground state of (5).

- Consider the scalar field equation in $\mathbb{R}^3$

$$\frac{3}{2}\triangle U - \frac{1}{2}U + E_1(|U|^2)U = 0,\ \ U \in H^1(\mathbb{R}^3). \tag{6}$$

Proposition 5 (Zhang-Zhu, 2011, DPDE) Let $v \in H^1(\mathbb{R}^3)$. Then

$$\int E_1(|v|^2)|v|^2 dx \leq \frac{2}{\|U\|_2^2} \|\nabla v\|_2^3 \, \|v\|_2,$$

where U is the ground state of (6).

• Consider the scalar field equation in $\mathbb{R}^N$ $(N = 2, 3)$

$$\frac{(p-1)N}{4}\triangle V - [1 - \frac{(N-2)(p-1)}{4}]V + |V|^{p-1}V = 0,\ V \in H^1(\mathbb{R}^N). \tag{7}$$

Proposition 6 (Weinstein, 1983, CMP) Let $v \in H^1(\mathbb{R}^N)$ and $1 < p < \frac{N+2}{(N-2)^+}$. Then

$$\|v\|_{p+1}^{p+1} \le \frac{p+1}{2\|V\|_2^{p-1}} \|\nabla v\|_2^{\frac{(2-N)p+2+N}{2}} \|v\|_2^{\frac{(p-1)N}{2}},$$

where V is the ground state of (7).

- **Sharp Criteria for $N = 2$ and $p = 3$**

Theorem 2 *(Papanicolaou et al., 1994, Physica D; Li, Zhang et al., 2011, JDE)*
Let $N = 2$, $p = 3$ and $u_0 \in H^1(\mathbb{R}^2)$. Then we have that

(i) if $\|u_0\|_2 < \|Q\|_2$, then the corresponding solution $u(t,x)$ of (1)-(2) exists globally,

(ii) if $\|u_0\|_2 \geq \|Q\|_2$, then the corresponding solution $u(t,x)$ of (1)-(2) may blow up in finite time $0 < T < \infty$, where Q is the ground state of (4).

Remark: Pseudo-conformal invariance

$$u(t,x) = \frac{1}{T-t}\exp\left(\frac{-i|x|^2}{4(T-t)} + \frac{i}{T-t}\right)Q\left(\frac{x}{T-t}\right).$$

is a finite-time blow-up solution to (1) with critical mass $\|u_0\|_2 = \|Q\|_2$.

• **Sharp Criteria for $N = 2$ and $3 < p < +\infty$**

Theorem 3 *(Zhu-Zhang-Yang, 2012, preprint)*

Let $N = 2$, $3 < p < +\infty$ and $B = \left(\frac{2\|V\|_2^{p-1}(\|R\|_2^2 - \|u_0\|_2^2)}{(p-1)\|R\|_2^2\|u_0\|_2^2}\right)^{\frac{1}{p-3}}$, where V is the ground state of (7) and R is the ground state of (3). Assume that the initial data $u_0 \in H^1$ satisfies

$$\|u_0\|_2 < \|R\|_2 \ \text{ and } \ H(u_0) < \frac{(p-3)(\|R\|_2^2 - \|u_0\|_2^2)}{2(p-1)\|R\|_2^2} B^2.$$

(i) If

$$\|\nabla u_0\|_2 < B,$$

then the solution $u(t, x)$ of (1)-(2) exists globally. Moreover, $u(t, x)$ satisfies

$$\|\nabla u(t, x)\|_2 < B.$$

(ii) If $|x|u_0 \in L^2$ and

$$\|\nabla u_0\|_2 > B,$$

then the solution $u(t, x)$ of (1)-(2) blows up in finite time $0 < T < +\infty$.

- **Sharp Criteria for $N = 3$ and $p = 3$**

Theorem 4 *(Zhang-Zhu, 2011, DPDE) Let $N = 3$ and $p = 3$. Suppose $u_0 \in H^1(\mathbb{R}^3)$ satisfies*

$$H(u_0)\|u_0\|_2^2 < \frac{2}{27}\|W\|_2^4,$$

where W is the ground state of (5). Then we have that

(i) if

$$\|\nabla u_0\|_2\|u_0\|_2 < \frac{2}{3}\|W\|_2^2,$$

then the solution $u(t,x)$ of (1)-(2) globally exists. Moreover, $u(t,x)$ satisfies

$$\|\nabla u(t,x)\|_2\|u(t,x)\|_2 < \frac{2}{3}\|W\|_2^2,$$

(ii) if $|x|u_0 \in L^2$ and

$$\|\nabla u_0\|_2\|u_0\|_2 > \frac{2}{3}\|W\|_2^2,$$

then the solution $u(t,x)$ of (1)-(2) blows up in finite time $T < +\infty$.

- **Sharp Criteria for $N = 3$ and $p = 1 + \frac{4}{3}$**

Theorem 5 *(Zhang-Zhu, 2011, DPDE) Let $N = 3$ and $p = 1+\frac{4}{3}$. Suppose $u_0 \in H^1(\mathbb{R}^3)$ satisfies*

$$\|u_0\|_2 < \|V\|_2 \ \text{ and } \ H(u_0) < \frac{2\|U\|_2^4(\|V\|_2^{\frac{4}{3}} - \|u_0\|_2^{\frac{4}{3}})^3}{27\|u_0\|_2^2\|V\|_2^4},$$

where U is the ground state of (6) and V is the ground state of (7). Then we have that

(i) if

$$\|\nabla u_0\|_2\|u_0\|_2 < \frac{2}{3}\frac{\|U\|_2^2(\|V\|_2^{\frac{4}{3}} - \|u_0\|_2^{\frac{4}{3}})}{\|V\|_2^{\frac{4}{3}}},$$

then the solution $u(t,x)$ of (1)-(2) exists globally. Moreover, $u(t,x)$ satisfies

$$\|\nabla u(t,x)\|_2\|u(t,x)\|_2 < \frac{2}{3}\frac{\|U\|_2^2(\|V\|_2^{\frac{4}{3}} - \|u_0\|_2^{\frac{4}{3}})}{\|V\|_2^{\frac{4}{3}}},$$

(ii) if $|x|u_0 \in L^2$ *and*

$$\|\nabla u_0\|_2 \|u_0\|_2 > \frac{2}{3} \frac{\|U\|_2^2 (\|V\|_2^{\frac{4}{3}} - \|u_0\|_2^{\frac{4}{3}})}{\|V\|_2^{\frac{4}{3}}},$$

then the solution $u(t,x)$ *of (1)-(2) blows up in finite time* $T < +\infty$.

Let us define a function $g(y)$ on $[0, +\infty)$

$$g(y) = 1 - \frac{3(p-1)\|u_0\|_2^{\frac{5-p}{2}}}{4\|V\|_2^{p-1}} y^{\frac{3(p-1)}{2}-2} - \frac{3\|u_0\|_2}{2\|V\|_2^2} y, \tag{8}$$

where V is the ground state of (7). We claim that there exists an unique positive solution y_0 for the equation $g(y) = 0$.

- **Sharp Criteria for** $N = 3$ **and** $1 + \frac{4}{3} < p < 3$

Theorem 6 *(Zhang-Zhu, 2011, DPDE) Let* $N = 3$ *and* $1 + \frac{4}{3} < p < 3$*. Suppose* $u_0 \in H^1(\mathbb{R}^3)$ *satisfies*

$$0 < H(u_0) < \frac{3p-7}{6(p-1)} y_0^2,$$

where y_0 *is the unique positive solution of the equation* $g(y) = 0$*. Then we have that*

(i) if

$$\|\nabla u_0\|_2 < y_0,$$

then the solution $u(t,x)$ *of (1)-(2) exists globally. Moreover,* $u(t,x)$ *satisfies*

$$\|\nabla u(t,x)\|_2 < y_0,$$

(ii) if $|x|u_0 \in L^2$ *and*

$$\|\nabla u_0\|_2 > y_0,$$

then the solution $u(t,x)$ *of (1)-(2) blows up in finite time* $T < +\infty$*.*

- **Sharp Criteria for $N = 3$ and $3 < p < 5$**

Theorem 7 *(Zhang-Zhu, 2011, DPDE) Let $N = 3$ and $3 < p < 5$. Suppose $u_0 \in H^1(\mathbb{R}^3)$ satisfies*

$$0 < H(u_0) < \frac{1}{6} y_0^2,$$

where y_0 is the unique positive solution of the equation $g(y) = 0$ and $g(y)$ is defined in (8). Then we have that

(i) if

$$\|\nabla u_0\|_2 < y_0,$$

then the solution $u(t, x)$ of (1)-(2) exists globally. Moreover, for all time t, $u(t, x)$ satisfies

$$\|\nabla u(t, x)\|_2 < y_0,$$

(ii) if $|x|u_0 \in L^2$ and

$$\|\nabla u_0\|_2 > y_0,$$

then the solution $u(t, x)$ of (1)-(2) blows up in finite time $T < +\infty$.

§4 Dynamics of Blow-up Solutions

- **Concentration**

Theorem 8 *(Li, Zhang et al., 2011, JDE) Let* $N = 2$*,* $p = 3$ *and* $u_0 \in H^1(\mathbb{R}^2)$*. If* $u(t,x)$ *is the blow-up solution of (1)-(2), then there exists* $x(t) \in \mathbb{R}^2$ *such that for all* $r > 0$

$$\liminf_{t\to T} \int_{|x-x(t)|\le r} |u(t,x)|^2 dx \ge \int |Q|^2 dx,$$

where Q *is the ground state of (4).*

Theorem 9 *(Zhang-Zhu-Yang, 2012, preprint) Let* $u_0 \in H^1$*,* $N = 3$*,* $1 < p \le 3$ *and* $u(t,x)$ *be the radial blow-up solution of (1)-(2). Then either there is a constant* $C_1 > 1$ *such that*

$$\int_{|x|\le C_1^2/\|\nabla u(t)\|_2^2} |u(t,x)|^3 dx \ge \frac{1}{C_1} \text{ as } t \to T,$$

or there exist $C_2 > 0$ *and a sequence of times* $t_n \to T$ *as* $n \to \infty$ *such that*

$$\int_{|x|\le C_2\|u_0\|_2^{\frac{3}{2}}/\|\nabla u(t_n)\|_2^{\frac{1}{2}}} |u(t_n,x)|^3 dx \to \infty \text{ as } n \to \infty.$$

• Limit Profile of Blow-up Solutions

Theorem 10 *(Li, Zhang et al., 2011, JDE)*
Let $N = 2$, $p = 3$ *and* $\|u_0\|_2 = \|Q\|_2$. *If* $u(t,x) \in C([0,T); H^1)$ *is the blow-up solution of Cauchy problem (1)-(2) such that* $\lim\limits_{t\to T} \|\nabla u(t)\|_2 = \infty$, *then we have that*
(i) there is $x_0 \in \mathbb{R}^2$ *such that*

$$|u(t,x)|^2 \to \|Q\|_2^2 \, \delta_{x=x_0} \quad \text{as} \ \ t \to T,$$

(ii) there is a constant $C > 0$ *such that*

$$\|\nabla u(t,x)\|_2 \geq \frac{C}{T-t}, \quad \forall\, t \in [0,T),$$

where Q *is the ground state of (4).*

§4 Stability of Standing Waves

• Orbital Stability

Let $N = 2$, $1 < p < 3$ and $0 < M_0 < \|R\|_2^2$,

$$d_{M_0} := \inf_{\{v \in H^1 |\ \|v\|_2^2 = M_0\}} H(v) \tag{9}$$

where R is the ground state solution of (3). Define

$$S_{M_0} := \{v \in H^1 |\ v \text{ is the minimizer of variational problem } (9)\}. \tag{10}$$

Theorem 11 *(Zhang-Zhu, 2014, preprint) Let* $N = 2$, $1 < p < 3$ *and* $0 < M_0 < \|R\|_2^2$*. Then for arbitrary* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that for any* $u_0 \in H^1$*,*

$$\inf_{v \in S_{M_0}} \|u_0(\cdot) - v(\cdot)\|_{H^1} < \delta,$$

then the corresponding solution $u(t, x)$ *of the Cauchy problem (1)-(2) satisfies*

$$\inf_{v \in S_{M_0}} \|u(t, \cdot) - v(\cdot)\|_{H^1} < \varepsilon$$

for all $t > 0$*, where* S_{M_0} *is defined in (10).*

- **Strongly Instability**

Let $N \in \{2,3\}$ and $1 + \frac{4}{N} \leq p < \frac{N+2}{(N-2)^+}$. Define

$$d_S := \inf_{\{v \in H^1 |\ S(v)=0\}} I(v). \tag{11}$$

Theorem 12 *(Gan-Zhang, 2008, CMP) Let* $N \in \{2,3\}$ *and* $1 + \frac{4}{N} \leq p < \frac{N+2}{(N-2)^+}$. *If* ω *satisfies* $d_M \geq d_S$, *then for the minimizer* v *of (11) and any* $\delta > 0$, *there exists* $u_0 \in H^1$ *with*

$$\|u_0(\cdot) - v(\cdot)\|_{H^1} < \delta$$

such that the solution $u(t,x)$ *of the Cauchy problem (1)-(2) blows up in finite time.*

Thank You For Your Attentions!